Obstacle problems and isotonicity

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Abstract

For variational inequalities of an abstract obstacle type, a comparison principle for the dependence of solutions on the constraint set $\mathcal{K}$ is used to provide a continuity estimate.

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1 Introduction

Our goal is to consider some aspects of the dependence on $\mathcal{K}$ of the solution u of a variational inequality

$$[\mathbf{F} + \partial I_{\mathcal{K}}]\,(u) \ni z, \tag{1.1}$$

in the context of partially ordered spaces. Here $\mathbf{F} : \mathcal{X} \to \mathcal{X}_0^*$ is a (possibly nonlinear) monotone operator[1] and $I_{\mathcal{K}}$ is the lsc *indicator function* for an

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[1]It is convenient here, in looking at the dependence of solutions on the constraint set $\mathcal{K}$, to introduce the larger space $\mathcal{X} \supset \mathcal{X}_0$ on which $\mathbf{F}$ is to be defined, but we then assume, as a consistency condition on $\mathbf{F}, \mathcal{K}$, that

$$u, v \in \mathcal{K} \quad \Rightarrow \quad (v - u) \in \mathcal{X}_0. \tag{1.2}$$

For a single such variational inequality we could arbitrarily select some $u_* \in \mathcal{K}$ and then consider $\hat{\mathcal{K}} = \mathcal{K} - u_*$ and $\hat{\mathbf{F}}(u) := \mathbf{F}(u_* + u)$ for $u \in \mathcal{X}_0$. This gives the more usual admissible subset $\mathcal{K} \subset \mathcal{X}$ (i.e., $I_\mathcal{K}(x) = 0$ if $x \in \mathcal{K}$ and $= +\infty$ if $x \notin \mathcal{K}$). [For simplicity we are taking $\mathbf{F}(x)$ to be defined and single valued for each $x \in \mathcal{K}$.] We interpret (1.1) in the weak form

$$u \in \mathcal{K}; \qquad \text{if } v \in \mathcal{K}, \text{ then } \langle v - u, \mathbf{F}(u) - z\rangle \geq 0. \tag{1.3}$$

Taking $\mathcal{X}$ and $\mathbf{F}$ as implicitly given, we refer to (1.3) as 'VI$(\mathcal{K}, z)$'. The simplest version of this, in the context of a space $\mathcal{X}$ of functions on a set Ω, would be an *obstacle problem*, by which we mean a variational inequality for which the constraint set $\mathcal{K}$ has the form[2]

$$\mathcal{K} = \{u \in \mathcal{X} : \varphi_- \leq u(s) \leq \varphi_+(s) \text{ for } s \in \Omega\} \tag{1.4}$$

for some given 'obstacle functions' $\varphi_\pm(\cdot)$, obviously with $\varphi_- \leq \varphi_+$ pointwise.

In the context of a partially ordered Banach space $\mathcal{X}$, our first result is the isotonicity, under suitable hypotheses, of the solution dependence — not only on z, but on $\mathcal{K}$. I.e., for corresponding solutions u, u' of VI$(\mathcal{K}, z)$ and VI$(\mathcal{K}', z')$ one has Theorem 3.1:

$$\text{if } \ \mathcal{K} \prec \mathcal{K}' \ \text{ (and } z \leq z'), \qquad \text{then } u \leq u'. \tag{1.5}$$

[For $\mathcal{K}, \mathcal{K}'$ of the obstacle form (1.4), $\mathcal{K} \prec \mathcal{K}'$ would just mean that $\varphi_- \leq \varphi'_-$ and $\varphi_+ \leq \varphi'_+$.] The relevant background material and definitions related to the partial ordering and its relevance for the constraint set $\mathcal{K}$ and the operator $\mathbf{F}$ will be developed in the next section; the comparison theorem and proof are then in section 3. with some examples in section 4.

From Theorem 3.1 we then obtain, also in section 3., a continuity estimate — based on the partial order and so independent of (and rather different from) the kinds of pointwise estimates for dependence on $z, \mathcal{K}$ which have been obtained, e.g., via an embedding: $H^s(\Omega) \hookrightarrow \mathcal{C}(\bar{\Omega})$.

2 Preliminaries

Remark 2.1. We begin by noting that, without further specificity, we assume throughout that existence of solutions is not an issue for the variational

$\hat{\mathbf{F}} : \mathcal{X}_0 \to \mathcal{X}_0^*$ and we have VI$(\hat{\mathcal{K}}, z)$ using $\hat{\mathbf{F}}$ equivalent (with a solution shifted by u_*) to the original VI$(\mathcal{K}, z)$ using $\mathbf{F}$.

[2] The case of a single obstacle: $\mathcal{K} = \{u \in \mathcal{X} : u(s) \geq \varphi(s) \text{ for } s \in \Omega\}$ simply takes $\varphi_+ \equiv +\infty$.

inequalities under consideration: i.e., implicit in our consideration of variational inequalities is the imposition of conditions on $\mathcal{X}, \mathbf{F}, \mathcal{K}$ of monotonicity, continuity, etc., (cf., e.g., [1], [4], [8]) ensuring that we will have existence of solutions as needed. See Section 4 for some examples.

It is generally expositionally convenient to assume $\mathbf{F}$ everywhere defined: $\mathcal{X} \to \mathcal{X}_0^*$, but in considering variational inequalities of evolution we will take the domain to be of the form $\mathcal{Y} = L^2([0,T] \to \mathcal{X})$ with codomain $\mathcal{Y}_0^* = L^2([0,T] \to \mathcal{X}_0^*)$ and $\mathbf{F}_0(t,\cdot) : \mathcal{X} \to \mathcal{X}_0^*$ monotone for each $t \in [0,T]$, assuming such regularity in t as to ensure that

$$\mathbf{F} : \ \mathcal{Y} \to \mathcal{Y}_0^* : \quad y \mapsto \mathbf{F}_0(\cdot, y(\cdot)) \tag{2.1}$$

is consistent with the implicit requirements for existence. ■

We take the Banach space $\mathcal{X}$ to be partially ordered by a convex closed positive cone $\mathcal{P}$ (so $x \le y$ means $(y-x) \in \mathcal{P}$) with the assumption that $\mathcal{P} \cap \mathcal{X}_0$ is similarly a positive cone for $\mathcal{X}_0$. It is then standard that the dual $\mathcal{X}_0^*$ is also partially ordered with $0 \le \xi \in \mathcal{X}_0^*$ when $\langle x, \xi \rangle \ge 0$ for all $x \in \mathcal{P} \cap \mathcal{X}_0$. We make the key assumption that 'max' is well-defined in $\mathcal{X}$ — i.e., for each $x, y \in \mathcal{X}$ there is an element $\max\{x,y\} = x \vee y$ in $\mathcal{X}$ characterized by

$$x \vee y \ge x, y \qquad \text{and} \qquad z \ge x, y \ \ \Rightarrow \ \ z \ge x \vee y.$$

We note a few consequent properties:

- if $\mathcal{P}$ is pointed[3], then $x \vee y$ is unique;
- automatically, $x \wedge y = \min\{x,y\}$ is also defined (as $-[-x \vee -y]$) so
$$x \wedge y \le x, y \qquad \text{and} \qquad z \le x, y \ \ \Rightarrow \ \ z \le x \vee y;$$
- $(x \vee y) + z = (x+z) \vee (y+z)$ and $(x \wedge y) + z = (x+z) \wedge (y+z)$;
- setting $x_+ := x \vee 0$ and $x_- := x \wedge 0$, we have $x_+ + x_- = x$ and $x_+ \wedge (-x_-) = 0$.

[3]We need not necessarily assume that $\mathcal{P}$ is pointed ($x \ge 0$ and $x \le 0$ only for $x = 0$), although this is the case for all our examples. If $\mathcal{P}$ is not pointed, then $x \vee y$ would be a coset of the subspace $\mathcal{P}_0 = [-\mathcal{P}] \cap \mathcal{P}$.

We will assume that the constraint sets $\mathcal{K}$ under consideration are consistent with the partial order on $\mathcal{X}$. In particular, we say that $\mathcal{K} \subset \mathcal{X}$ is *admissible* if it is closed, nonempty, and satisfies both (1.2) and

$$u, v \in \mathcal{K} \qquad \Rightarrow \qquad u \wedge v,\ u \vee v \in \mathcal{K}. \tag{2.2}$$

The collection of all admissible subsets of $\mathcal{X}$ will be denoted by $\mathcal{A} = \mathcal{A}(\mathcal{X})$.

An obstacle problem is a variational inequality in which $\mathcal{K} \subset \mathcal{X}$ is a *generalized order interval*, meaning that

$$u, v \in \mathcal{K} \qquad \Rightarrow \qquad [u \wedge v, u \vee v] \subset \mathcal{K} \tag{2.3}$$

where $[a, b]$ (with $a, b \in \mathcal{X},\ a \leq b$) here denotes the order interval:

$$[a, b] = \{x \in \mathcal{X} : a \leq x \leq b\}.$$

It is easily seen that ordinary order intervals $[a, b]$ are generalized order intervals and that nonempty generalized order intervals are admissible sets.

In the typical cases where $\mathcal{X}$ is a function space with pointwise ordering, a constraint set such as (1.4) is obviously a generalized order interval, but need not be an order interval since the 'obstacle functions' $\varphi_{\pm}(\cdot)$ need not be in $\mathcal{X}$ or even be proper functions at all — e.g., we simply set $\varphi_+ = +\infty$ on $\Omega' \subset \Omega$ to indicate that $u \in \mathcal{K}$ is unconstrained above on Ω'; we never let φ_- take the value $+\infty$ or let φ_+ take the value $-\infty$. We then employ the usual algebra for these obstacle functions and, in particular, if $a(\cdot)$ is any ordinary function on Ω we write $\varphi \leq \varphi' + a$ if this holds pointwise where φ, φ' are both finite, otherwise taking $\pm\infty + \text{finite} = \pm\infty$ and $-\infty \leq \text{finite} \leq +\infty$; we will have $\varphi_+ - \varphi_- \leq a$ if $\varphi_+ \leq \varphi_- + a$. [One easily sees that (1.4) will satisfy (2.3) and so (2.2) provided $\varphi_- \leq \varphi_+$, but must verify separately that such a $\mathcal{K}$ would be nonempty. On the other hand, as we may see by (4.5), a generalized order interval in this context is more general than the form (1.4).]

We next define a partial ordering on $\mathcal{A}$ by writing $\mathcal{K} \prec \mathcal{K}'$ when

$$u \in \mathcal{K},\ v' \in \mathcal{K}' \qquad \Rightarrow \qquad (u \wedge v') \in \mathcal{K},\ (u \vee v') \in \mathcal{K}' \tag{2.4}$$

It is easy to see that '$\prec$' is, indeed, a partial order on $\mathcal{A}$ and we write $[\mathcal{K}, \mathcal{K}']$ for the order interval $\{\hat{\mathcal{K}} \in \mathcal{A} : \mathcal{K} \prec \hat{\mathcal{K}} \prec \mathcal{K}'\}$. Note that, for admissible $\mathcal{K}$ and any $a \in \mathcal{X}$ the set $\mathcal{K} + a := \{u + a : u \in \mathcal{K}\}$ is also admissible, but $a \geq 0$ does not imply $\mathcal{K} \prec (\mathcal{K} + a)$. For $\mathcal{K}, \mathcal{K}'$ of the form (1.4) we note that $\mathcal{K} \prec \mathcal{K}'$ if $\varphi_- \leq \varphi'_-$ and $\varphi_+ \leq \varphi'_+$.

Lemma 2.2. *Let $\mathcal{K}, \mathcal{K}'$ be generalized order intervals. Then*

1. $\mathcal{K}' \subset \mathcal{K} + \mathcal{P},\ \mathcal{K} \subset \mathcal{K}' - \mathcal{P}$ *is equivalent to* $\mathcal{K} \prec \mathcal{K}'$*;*
2. *For* $a \geq 0$ *one has* $(\mathcal{K} - a) \prec \mathcal{K}$ *and* $\mathcal{K} \prec (\mathcal{K} + a)$*.*

Proof: The condition: $\mathcal{K}' \subset \mathcal{K} + \mathcal{P},\ \mathcal{K} \subset \mathcal{K}' - \mathcal{P}$ just means that: for $u' \in \mathcal{K}'$ there exists $u \in \mathcal{K}$ with $u' \geq u$ and for $v \in \mathcal{K}$ there exists $v' \in \mathcal{K}'$ with $v \leq v'$ — which is immediate from (2.4). For the converse, we suppose $u \in \mathcal{K}$, $v' \in \mathcal{K}'$. Then the condition gives $u \leq u'$ for some $u' \in \mathcal{K}'$ so $v' \leq (u \vee v') \leq (u' \vee v')$. As $\mathcal{K}'$ is a generalized order interval, we then have $(u \vee v') \in [v', u' \vee v'] \subset \mathcal{K}'$. Similarly, $(u \wedge v') \in \mathcal{K}$. This shows part *1.* of the Lemma and *2.* follows immediately. ■

Returning to the operator $\mathbf{F} : \mathcal{X} \to \mathcal{X}^*$, we define an important relation between the operator and the partial order. For a family of admissible sets $\mathcal{A}_0 \subset \mathcal{A}$, we will say that the pair $[\mathbf{F}, \mathcal{A}_0]$ is *compatible* if

$$\begin{array}{c} \mathcal{K}, \mathcal{K}' \in \mathcal{A}_0, \quad \mathcal{K} \prec \mathcal{K}' \quad \text{and} \quad x \in \mathcal{K},\ (x+h) \in \mathcal{K}' \\ \Rightarrow \quad \left\{ \begin{array}{c} h_+ \in \mathcal{X}_0 \quad \text{and} \\ \langle h_+, \mathbf{F}(x+h) - \mathbf{F}(x) \rangle \leq 0 \quad \Rightarrow \quad h \leq 0. \end{array} \right. \end{array} \tag{2.5}$$

Note that this automatically gives (1.2) for $\mathcal{K} \in \mathcal{A}_0$: if $u, v \in \mathcal{K}$, then (2.5) gives $(v - u)_+ \in \mathcal{X}_0$, noting that $\mathcal{K} \prec \mathcal{K}$, and also $(u - v)_+ \in \mathcal{X}_0$ whence $v - u = (v - u)_+ - (u - v)_+$ is in $\mathcal{X}_0$. Note also that the conclusion $h \leq 0$ of the final implication in (2.5) is equivalent to having $h_+ \leq 0$ — which, in turn, means $h_+ = 0$ if $\mathcal{P}$ is pointed.

3 Principal results

We are now ready for our first result:

Theorem 3.1. *In addition to the standing existence hypotheses of Remark 2.1, assume compatibility* (2.5) *of the pair* $[\mathbf{F}, \mathcal{A}_0]$ *for some* $\mathcal{A}_0 \subset \mathcal{A}$*. Then the solution* u *of* (1.3) *depends isotonically on* z *and on* $\mathcal{K} \in \mathcal{A}_0$*. More precisely:*

For sets $\mathcal{K}, \mathcal{K}' \in \mathcal{A}_0$ *and for* $z, z' \in \mathcal{X}_0^*$*, let* u, u' *be, respectively, solutions of*

$$\begin{array}{lll} u \in \mathcal{K}; & \textit{if } v \in \mathcal{K}, \textit{ then} & \langle v - u, \mathbf{F}(u) - z \rangle \geq 0, \\ u' \in \mathcal{K}'; & \textit{if } v' \in \mathcal{K}', \textit{ then} & \langle v' - u', \mathbf{F}(u') - z' \rangle \geq 0. \end{array} \tag{3.1}$$

Then we have the isotonicity (1.5): *If $\mathcal{K} \prec \mathcal{K}'$ and $z \leq z'$ in $\mathcal{X}_0^*$, then $u \leq u'$.*

PROOF: Let $v = u' \wedge u$ and $v' = u' \vee u$. With $u \in \mathcal{K}$, $u' \in \mathcal{K}'$, the inequality definition (2.4) ensures that $v \in \mathcal{K}$, $v' \in \mathcal{K}'$. We now set $h = u - u'$ and note that $v - u = -h_+$ and $v' - u' = h_+$ by the distributivity of addition over max. With these choices, (3.1) becomes

$$-\langle h_+, \mathbf{F}(u) - z\rangle \geq 0, \qquad \langle h_+, \mathbf{F}(u') - z'\rangle \geq 0$$

and adding these we have

$$\langle h_+, \mathbf{F}(u' + h) - \mathbf{F}(u')\rangle = \langle h_+, \mathbf{F}(u) - \mathbf{F}(u')\rangle \leq \langle h_+, z - z'\rangle \leq 0$$

since $z \leq z'$ in $\mathcal{X}_0^*$. By (2.5) this gives $h \leq 0$, i.e., $u \leq u'$ as asserted. ■

Compare[4] [1, Proposition 1.9], [3]; Theorem 3.1 differs in the distinction between $\mathcal{X}$ and $\mathcal{X}_0$ and in the formulation of the abstract compatibility condition (2.5).

[While we had not previously assumed that solutions of $\mathrm{VI}(\mathcal{K}, z)$ are necessarily unique, we now observe that one necessarily has uniqueness for $\mathrm{VI}(\mathcal{K}, z)$ if $\mathcal{K} \in \mathcal{A}_0$ when $\mathcal{P}$ is pointed and $[\mathbf{F}, \mathcal{A}_0]$ is compatible: if there were two solutions u, u' we could take $\mathcal{K}' = \mathcal{K}$ and apply Theorem 3.1 to see that $u' \geq u$; similarly, $u \geq u'$ so $u = u'$.]

We next give our principal result, an estimate for the solution map: $\mathcal{K}, z \mapsto u$ of $\mathrm{VI}(\cdot, \cdot)$ for obstacle problems, i.e., when $\mathcal{K}$ is a generalized order interval. In this generality the result is a somewhat clumsy corollary to Theorem 3.1; its value may become more apparent through the examples in the next section.

Theorem 3.2. *Given $\mathbf{F} : \mathcal{X} \to \mathcal{X}_0^*$, a generalized order interval $\mathcal{K} \subset \mathcal{X}$, and $z \in \mathcal{X}_0^*$, let u be the solution of $\mathrm{VI}(\mathcal{K}, z)$. Suppose that, for some positive $\omega, \omega' \in \mathcal{X}$ we have*

$$\zeta = \mathbf{F}(u) - \mathbf{F}(u - \omega) \geq 0, \qquad \zeta' = \mathbf{F}(u + \omega') - \mathbf{F}(u) \geq 0. \tag{3.2}$$

Further, suppose the pair $[\mathbf{F}, \mathcal{A}_0]$ is compatible for some $\mathcal{A}_0 \subset \mathcal{A}$ containing $(\mathcal{K} - \omega)$, $\mathcal{K}$, $(\mathcal{K} + \omega')$. Then $\mathcal{K} - \omega \prec \mathcal{K} \prec \mathcal{K} + \omega'$] and the solution $\hat{u}$ of $\mathrm{VI}(\hat{\mathcal{K}}, \hat{z})$ will then be in the order interval $[u - \omega,\ u + \omega']$, i.e.,

$$-\omega \leq \hat{u} - u \leq \omega' \tag{3.3}$$

[4] We are indebted to the referee for pointing out the reference to [1]. Brezis notes there that the argument follows [3].

for any $\hat{z}$ such that $-\zeta \le \hat{z} - z \le \zeta'$ and any $\hat{\mathcal{K}} \in \mathcal{A}_0 \cap [\mathcal{K} - \omega,\, \mathcal{K} + \omega']$ — where $[\mathcal{K} - \omega, \mathcal{K} + \omega']$ is here the order interval in $\mathcal{A}$.

PROOF: Setting $\mathcal{K}' = \mathcal{K} + \omega'$ and $z' = z + \zeta'$, we first note that, as $\mathcal{K}$ is a generalized order interval and $\omega' \ge 0$ by assumption, Lemma 2.2 gives $\mathcal{K} \prec \mathcal{K}'$; similarly, $\mathcal{K} - \omega \prec \mathcal{K}$. From (1.3) we note that $u' = u + \omega'$ is the solution of $\mathrm{VI}(\mathcal{K} + \omega', z + \zeta')$ — indeed, this is precisely equivalent to having u as a solution of $\mathrm{VI}(\mathcal{K}, z)$. As $\hat{\mathcal{K}} \prec \mathcal{K}'$ and $\hat{z} \le z + \zeta'$ by assumption, Theorem 3.1 then applies to show $\hat{u} \le u + \omega'$. Similarly, $u - \omega$ is the solution of $\mathrm{VI}(\mathcal{K} - \omega, z - \zeta)$ and Theorem 3.1 gives $u - \omega \le \hat{u}$. Combining these gives (3.3) as desired. ■

4 Examples and remarks

Example 4.1. As a first example, we let $\mathcal{X} = \mathcal{X}_0 = \mathbb{R}^d$ with the usual entrywise partial order and consider symmetric M-matrices $\mathbf{A}$ so the matrix entries A_{ij} satisfy

$$A_{ij} = A_{ji} \qquad A_{ij} \le 0 \text{ for } i \ne j \qquad \sum_j A_{ij} > 0. \tag{4.1}$$

We first note that symmetry gives real eigenvalues and the Gershgorin Lemma then ensures that these are positive so (4.1) means A must be positive definite. We next claim the compatibility of $[\mathbf{F}, \mathcal{A}]$ for $\mathbf{F}(x) = Ax$. By linearity, we are claiming that $\langle h_+, Ah \rangle \le 0$ only if $h \le 0$, i.e., that the set $\mathcal{S} = \{j : h_j > 0\}$ is empty. To see this, note that when $i \in \mathcal{S}$ we have $A_{ij} h_j \ge 0$ for $j \notin \mathcal{S}$ so $\langle h_+, Ah \rangle \ge \langle h_+, Ah_+ \rangle$. Using the positive definiteness, we then have (2.5). [One consequence of this is the well-known fact that (4.1), here with symmetry, implies the positivity of A^{-1} in the sense of preserving the partial order.]

Somewhat more generally, we observe that we have (2.5) for any $\mathcal{C}^1$ nonlinear function $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ whose derivative satisfies (4.1) pointwise. Since

$$\langle h_+, \mathbf{F}(x+h) - \mathbf{F}(x) \rangle = \int_0^1 \langle h_+, \mathbf{F}'(x + th)h \rangle \, dt,$$

we could only have $\langle h_+, \mathbf{F}(x+h) - \mathbf{F}(x)\rangle \le 0$ if $\langle h_+, \mathbf{F}'(x+th)h\rangle \le 0$ for some t whence, as above, we would have $h_+ = 0$.

For *box variational inequalities*, with $\mathcal{K}$ an order interval $[a,b]$, Theorem 3.1 just means that increasing a and/or b will increase (or at least not decrease) the solution of $\mathrm{VI}(\mathcal{K}, z)$ for such $\mathbf{F}$. Theorem 3.2 is easiest to interpret in the linear case $\mathbf{F} = A$ (with z fixed so we are looking only at the dependence of solutions on $\mathcal{K}$): for the solutions u of $\mathrm{VI}([a,b],z)$ and u' of $\mathrm{VI}([a',b'],z)$ Theorem 3.2 gives

$$-\omega \le \; a'-a,\, b'-b \; \le \omega' \qquad \Rightarrow \qquad -\omega \le u'-u \le \omega' \tag{4.2}$$

provided ω, ω' and $A\omega$, $A\omega'$ are each positive.

Example 4.2. As a second example, we let $\mathcal{X}$ be the Hilbert space $H^1(\Omega)$ for some bounded region $\Omega \subset \mathbb{R}^d$, taken with the usual pointwise partial order: $x \ge 0$ in $\mathcal{X}$ if $x(s) \ge 0$ for almost all $s \in \Omega$. It is well-known (cf., e.g., [7]) that the Nemytski operator: $u \mapsto \psi \circ u$ with $\psi(r) = \{r \text{ for } r \ge 0; 0 \text{ for } r \le 0\}$ (so $\psi(u) = u_+$) is continuous on $\mathcal{X}$ with, pointwise ae on Ω,

$$\nabla u_+ = \begin{cases} \nabla u & \text{where } \nabla u_+ \ne 0, \text{ where } u > 0 \\ 0 & \text{else} \end{cases} \tag{4.3}$$

We then let $\mathcal{X}_0 = H_0^1(\Omega)$ so $\mathcal{X}_0^* = H^{-1}(\Omega)$ and let $\mathbf{F}$ be the Laplace operator $-\Delta$ on Ω with unspecified boundary conditions. An application of the Divergence Theorem gives

$$\langle v-u, \mathbf{F}(v) - \mathbf{F}(u)\rangle = \int_\Omega |\nabla(v-u)|^2 \; - \int_{\partial\Omega} (v-u)\frac{\partial(v-u)}{\partial\nu} \tag{4.4}$$

where $\langle\cdot,\cdot\rangle$ is the $\mathcal{X}_0$–$\mathcal{X}_0^*$ duality pivoting on $L^2(\Omega)$. We will take $\mathcal{A}_0$ to consist of constraint sets imposing Dirichlet boundary conditions so the boundary integral vanishes (compare the comment following (1.2)) and, noting the Poincaré Inequality, we then have strong monotonicity and existence of solutions is not an issue; cf., e.g., [1] or [8]. Thus, we are considering $\mathcal{A}_0$ to consist of $\mathcal{K} \subset \mathcal{X}$ having the form

$$\mathcal{K} = \{u \in \mathcal{X} : \; \varphi_- \le u \le \varphi_+ \text{ on } \Omega \text{ with } u = g \text{ on } \partial\Omega\} \tag{4.5}$$

where φ_-, φ_+ are specified functions on Ω — allowing, e.g., $\varphi_+ \equiv +\infty$ to consider a single obstacle problem; we assume consistency to ensure $\mathcal{K} \ne \emptyset$

and so require that $\varphi_- \leq \varphi_+$ on Ω (with suitable regularity where one might have $\varphi_- = \varphi_+$) and that g is given in $H^{1/2}(\partial\Omega)$ with $\varphi_- \leq g \leq \varphi_+$ on $\partial\Omega$. [Note that we cannot write (4.5) as $\mathcal{K} = \{u \in [\varphi_-, \varphi_+] : u\Big|_{\partial\Omega} = g\}$ since $\varphi_\pm$ need not be in $\mathcal{X}$ so there would be no order interval $[\varphi_-, \varphi_+]$.]

Clearly each $\mathcal{K}$ as above is a generalized order interval in $\mathcal{X}$ so $\mathcal{A}_0 \subset \mathcal{A}$ and we next claim (2.5) with respect to this $\mathcal{A}_0$. First, for $\mathcal{K}, \mathcal{K}'$ of the form (4.5) with $\mathcal{K} \prec \mathcal{K}'$, we must have $g \leq g'$ in $H^{1/2}(\partial\Omega)$ so, for $x \in \mathcal{K}$, $x+h \in \mathcal{K}'$, we have $h = g - g' \leq 0$ on $\partial\Omega$ giving $h_+\Big|_{\partial\Omega} = 0$ whence $h_+ \in \mathcal{X}_0$ and the boundary term of (4.4) will vanish. Then, using (4.3),

$$\langle h_+, \mathbf{F}(x+h) - \mathbf{F}(x)\rangle = \int_\Omega \nabla h_+ \cdot \nabla h = \int_\Omega |\nabla h_+|^2$$

giving the implication (2.5).

Theorem 4.3. *Let $\mathcal{K}, \mathcal{K}'$ be of the form (4.5). Assume $(\varphi_+ - \varphi'_+)$ and $(\varphi_- - \varphi'_-)$ are in $L^\infty(\Omega)$ and $(g - g') \in L^\infty(\partial\Omega)$. Then the solutions u, u' of VI($\mathcal{K}, z$), VI($\mathcal{K}', z$) satisfy the L^∞ continuity estimate*

$$\begin{aligned} &\|u - u'\|_{L^\infty(\Omega)} \\ &\qquad \leq \max\{\|\varphi_- - \varphi'_-\|_{L^\infty(\Omega)}, \|\varphi_+ - \varphi'_+\|_{L^\infty(\Omega)}, \|g - g'\|_{L^\infty(\partial\Omega)}\}. \end{aligned} \tag{4.6}$$

[Note that having $(\varphi_+ - \varphi'_+) \in L^\infty(\Omega)$ still permits φ_+, φ'_+ to be simultaneously infinite on a subset of Ω, and similarly for φ_-, φ'_-.]

Proof: This is now a corollary to Theorem 3.2. Noting (4.4) in interpreting (1.3), $u \in \mathcal{K}$ is the solution of

$$\|\nabla(v - u)\|^2_{L^2(\Omega)} + \langle v - u, z\rangle \geq 0 \qquad \text{for } v \in \mathcal{K}$$

and correspondingly for $u' \in \mathcal{K}'$. Note that any constant $\omega = \omega' \equiv \varepsilon$ satisfies $\mathbf{F}(u \pm \omega) - \mathbf{F}u = 0 = \zeta$, etc. — and, as we are keeping z fixed so we are looking only at the dependences on $\varphi_\pm$ and on g, we have $-\zeta \leq z - z \leq \zeta$ as in Theorem 3.2. If we take ε to be the right hand side of (4.6), then one easily sees that our assumptions give $\mathcal{K} - \omega \prec \mathcal{K}' \prec \mathcal{K} + \omega$ so Theorem 3.2 gives $-\varepsilon \leq u' - u \leq \varepsilon$ pointwise on Ω, which is just (4.6). ∎

Everything above holds if we take $\mathbf{F}(u) = -\nabla \cdot A\nabla u$ where the matrix $A = A(x)$ is bounded and is positive definite ae. Some modification of (4.6) would be needed if we also add to $\mathbf{F}$ a Lipschitzian Nemytsky operator and/or wish to consider variation of z. It is possible, albeit a bit more complicated, to apply the comparison ideas of Theorems 3.1 and 3.2 in the context of Neumann boundary conditions. Finally, in view of (4.3) we note that

$$|\nabla(u \wedge v)|,\ |\nabla(u \vee v)| \leq \max\{|\nabla u|,\ |\nabla v|\},$$

so we could include in the specification of $\mathcal{K}$ a pointwise bound on the gradient: $|\nabla u| \leq a(\cdot)$. Etc.

Example 4.4. As a final example we consider Moreau's *sweeping process* (cf., e.g., [2]) for the motion of a heavy particle in $\mathbb{R}^d$ constrained to remain in a moving convex set $\mathcal{C}(t)$: the particle is stationary when in the interior $C^o(t)$ and is pushed frictionlessly normal to the wall, as necessary, when in contact with $\partial\mathcal{C}(t)$. Thus, we have

$$-\dot{x} \in N_{\mathcal{C}(t)}(x) \qquad\qquad x(0) = x_0 \in \mathcal{C}(0) \tag{4.7}$$

where $N_{\mathcal{C}}(x)$ is the outward normal cone to $\mathcal{C} = \mathcal{C}(t)$ at $x = x(t)$. The problem (4.7) can then be formulated as a variational inequality (1.1) with $\mathcal{X} = Lip(0,T)$, $\mathbf{F} = d/dt$, and $\mathcal{K} = \{x \in \mathcal{X} : x(0) = x_0,\ x(t) \in \mathcal{C}(t)\}$, and $z = 0$. The choice of space here comes from the existence result [2, Theorem 2]):

> *If the set-function $\mathcal{C}(\cdot)$ is Lipschitz continuous (with respect to the Hausdorff metric) and $x_0 \in \mathcal{C}(0)$, then (4.7) has a (unique) Lipschitz solution. Further, one has the well-posedness estimate:*
>
> $$|x(t) - x'(t)|^2 \leq |x_0 - x_0'|^2 + 2(L + L') \int_0^t \Delta(s)\, ds \tag{4.8}$$
>
> *where L, L' are the respective Lipschitz constants for $\mathcal{C}(\cdot), \mathcal{C}'(\cdot)$ and $\Delta(t)$ is their Hausdorff distance.*

In order to apply Theorems 3.1 and 3.2, we use the usual entrywise partial order on $\mathbb{R}^d$ to induce a partial order on $\mathcal{X}$) and restrict each $\mathcal{C}(t)$ to be a box so $\mathcal{K}$ would have the form

$$\mathcal{K} = \{x \in \mathcal{X} : x(0) = x_0;\ a(t) \leq x(t) \leq b(t) \text{ for } 0 \leq t \leq T\}. \tag{4.9}$$

We will take $\mathcal{A}_0$ to consist of $\mathcal{K} \subset \mathcal{X}$ of the form (4.9) with the assumptions that: the functions $a, b : [0,T] \to \mathbb{R}^d$ are Lipschitzian; $a(t) \leq b(t)$ on $[0,T]$, and $a(0) \leq x_0 \leq b(0)$.

For (1.1), we have $\mathbf{F} = d/dt$ and $z = 0$; in view of the causality of the problem — we can restrict to any $[0,T'] \subset [0,T]$ — the weak form of (4.7) is then

$$x \in \mathcal{K}, \qquad \int_0^{T'} (y - x)\,\dot{x}\,dt \;\geq 0 \;\text{ for all } y(\cdot) \in \mathcal{K},\; \text{all } T' \in [0,T]. \tag{4.10}$$

To see that the pair $[d/dt, \mathcal{A}_0]$ is compatible for the specified $\mathcal{A}_0$, note that $\mathcal{K} \prec \mathcal{K}'$ for $\mathcal{K}, \mathcal{K}' \in \mathcal{A}_0$ means: $x_0 \leq x_0'$ with $a \leq a'$, $b \leq b'$ pointwise on $[0,T]$ and (2.5) means here that

$$0 \geq \int_0^T h_+\,\dot{h}\,dt \qquad \Rightarrow \qquad h_+ \equiv 0. \tag{4.11}$$

As in (4.3), we have $h_+\,\dot{h} = h_+\,\dot{h}_+ = [\frac{1}{2}(h_+)^2]^{\,\cdot}$ so, as $\mathcal{K} \prec \mathcal{K}'$ gives $x_0 \leq x_0'$ and so $h_+(0) = (x_0 - x_0')_+ = 0$, the inequality on the left in (4.11) just gives $h_+(T) = 0$. By causality, a solution on $[0,T]$ is also a solution on each $[0,T'] \subset [0,T]$ so we actually have $h_+(T') = 0$ for each $0 \leq T' \leq T$ — i.e., $h_+ \equiv 0$ on $[0,T]$ as asserted by (4.11).

We now see that — at least for constraints of this box form — we can extend the notion of solution from Lipschitzian to continuous constraints with an improved continuity estimate for the solution map — compare (4.12) below to (4.8). We may also note Proposition III.2.5 of [9], following [5].

Theorem 4.5. *Assume $a, b : [0,T] \to \mathbb{R}^d$ are continuous with $a(t) \leq b(t)$ on $[0,T]$ and $a(0) \leq x_0 \leq b(0)$; let $\mathcal{K}$ have the form* (4.9) *with $\mathcal{X} = C[0,T]$. Then there is a unique mild solution $x \in \mathcal{X}$ of the variational inequality* (4.10)*. We have, componentwise, the well-posedness estimate for the difference of two solutions:*

$$|x_k(t) - x_k'(t)| \leq \max\left\{ |(x_0)_k - (x_0')_k|,\; \max_{0 \leq s \leq t}\{\Delta_k(s)\} \right\} \tag{4.12}$$

where $\Delta_k = \max\{|a_k - a_k'|,\; |b_k - b_k'|\}$.

Proof: Initially restricting attention to Lipschitzian $a(\cdot)$, $b(\cdot)$ to take advantage of [2, Theorem 2], we prove (4.12) much as (4.6) above. Again we have $\mathbf{F}(x+\omega') - \mathbf{F}(x) \equiv 0 = \zeta$ for any constant ω', etc., when considering the problem on the interval $[0,t]$. Taking the constant to be, componentwise, the right hand side of (4.12), we see from Theorem 3.2 that $x' \leq x + \omega'$ on $[0,t]$. Similarly, we get $x' \geq x - \omega$ so we have (4.12).

This gives Lipschitz continuity of the map: $a(\cdot), b(\cdot) \mapsto x(\cdot)$ in max-norm for Lipschitzian a, b, x and this map therefore extends by continuity to continuous a, b, x from that dense set — of course retaining the well-posedness estimate (4.12). We refer to the solutions in this extended sense as *mild solutions* of the variational inequality — even though these new mild solutions need not satisfy (4.10) or (4.7) in any more direct sense, since such a solution $x(\cdot)$ need not even be the integral of its derivative. ■

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