The spectral radius in partially ordered algebras

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Dedicated to F.L. Bauer on the occasion of his 80th birthday in gratitude for his seminal contributions to the field of linear algebra.

Abstract

We prove theorems of Perron-Frobenius type for positive elements in partially ordered topological algebras satisfying certain hypotheses. We show how some of our results relate to known results on Banach algebras. We give examples and state some open questions.

Key words: Partially ordered algebras, positive cone, spectrum, spectral radius, Perron-Frobenius, semimonotone norms, completeness.

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1 Introduction

Perron-Frobenius theory — and such related results as, e.g., comparison theorems for operator splittings: [7], [6] and Theorem 7.1 — rely essentially on the notion of 'positive operators', usually formulated (cf., e.g., [7], [3]) in terms of preservation of a positive cone in the underlying space $\mathcal{X}$ on which

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the operators act. As with other spectral considerations (compare [2]), it seems of interest to treat some of this in the context of the *operator algebra* $\mathcal{A}$ — in which $\mathcal{X}$ has no direct relevance: indeed, we consider the algebra $\mathcal{A}$ abstractly, with no suggestion that its elements act at all as operators. We then reformulate as much as possible of the results in that context, now emphasizing the order-theoretic aspects of the situation.

Thus our concern will be with real *partially ordered algebras* and our objective in this paper is to show that this reformulation can be done with some success so that we can generalize some known results for operators acting on Banach spaces. Our chief hypotheses are found in this introduction. In Section 2 we introduce our definitions of *spectrum* and *spectral radius* and we focus on a set of elements called *tame* which play a role comparable to bounded operators on a Banach space. In the Section 3 we examine in more detail the relation of our present concepts to some known results in Banach algebras. Some preliminary results are in Section 4 and our principal results are in Section 5. In particular, our form of Perron-Frobenius is stated as Theorem 5.7. Our final Sections 6 and 7 contain examples, comments, and open questions.

The partially ordered algebras we consider are characterized algebraically by the hypothesis:

[H1] *$\mathcal{A}$ is an algebra over the reals with a multiplicative identity, denoted by 1 (notationally we identify multiples of this with scalars). $\mathcal{A}$ is partially ordered by a positive cone $\mathcal{P}$. If $x, y \geq 0$, $\lambda > 0$, then also $(x+y), xy, \lambda x \geq 0$; $1 \in \mathcal{P}$. Finally, $\mathcal{P}$ is pointed, i.e., $\mathcal{P} \cap [-\mathcal{P}] = \{0\}$.*

We will equivalently write "$x \leq y$" or "$(y - x) \in \mathcal{P}$". We will also often have occasion to write "$-u \leq x \leq u$" (necessarily with $u \geq 0$) or, equivalently, "$\pm x \leq u$" or "$x = p - q$ with $p, q \geq 0$" (with $p + q = u$ on taking $p = [u + x]/2$, $q = [u - x]/2$).

We have included, for convenience, the assumption that the positive cone $\mathcal{P}$ is pointed, but have not included a possible complementary requirement that $\mathcal{P} + [-\mathcal{P}] = \mathcal{X}$, since this has been unnecessary for formulating our results.

It will later be necessary to supplement the algebraic completeness of [H1] by some form of topological topological completeness, much as in the usual distinction of the field of reals from the rationals. We begin with the hypothesis:

[H2] *The topology on $\mathcal{A}$ is Hausdorff. The positive cone $\mathcal{P}$ is closed in $\mathcal{A}$. The algebraic operations of addition and multiplication by scalars are continuous; multiplication is continuous if the factors are each constrained to an order interval $[-u, u]$.*

We can then introduce a completeness condition with respect to the order. We will say that the partially ordered algebra $\mathcal{A}$ is *'$\mathcal{P}$-complete'* if

> *Given $c_N \searrow 0$ in $\mathcal{P}$, if "$\pm [x_j - x_k] \leq c_N$ for all $j, k \geq N$," then there is some $\bar{x} \in \mathcal{A}$ such that $x_k \to \bar{x}$.*

2 The spectrum and spectral radius

We begin by noting that for a real matrix x or, more generally, a bounded linear operator x on a real normed space, it is standard to define the *spectral radius* as the radius of the smallest disk centered at 0 in $\mathbb{C}$ containing the spectrum, i.e.,

$$\rho_*(x) = \max\{|\lambda| : \lambda \in \sigma(x)\}. \tag{2.1}$$

Since we are considering only *real* algebras and wish to work only with real scalars, we must be careful in discussing the spectrum for elements of $\mathcal{A}$. To this end, we introduce the polynomial

$$q(\zeta) = q(\zeta; \lambda) = \left[|\lambda|^2 - (\lambda + \bar{\lambda})\zeta + \zeta^2\right] = (\lambda - \zeta)(\bar{\lambda} - \zeta) \tag{2.2}$$

and, as usual, define $q(x) = q(x; \lambda)$ by the substitution $\zeta \hookleftarrow x$.

> In making the substitution $\zeta \hookleftarrow x$, we are identifying λ with that multiple of the identity, etc., and taking the first equality in (2.2) as definition; this involves only real operations in $\mathcal{A}$, even for complex λ, so $q(x; \lambda)$ is always a well-defined element of $\mathcal{A}$. The final equality in (2.2) is included as motivation, but this is purely formal in connection with the substitution: we make no suggestion that $(\lambda - x)$ or $(\bar{\lambda} - x)$ have any independent meaning.

We then define the spectrum $\sigma(x)$ as the complement of the resolvent set $\sigma'(x)$:

$$\sigma(x) = \mathbb{C} \setminus \sigma'(x) \qquad \sigma'(x) = \{\lambda \in \mathbb{C} : q(x; \lambda) \text{ is invertible}\}, \tag{2.3}$$

noting that noninvertibility of $q(x; \lambda)$ is equivalent to the noninvertibility of $(\lambda - x)$ when λ is real. [The set of all bounded linear operators on a Banach space forms a Banach algebra so this is automatic in that context, but we note that in considering operators on an infinite dimensional space it is standard to take invertibility to mean existence of a *bounded* inverse.]

Both the definition (2.1) of the spectral bound and the definition (2.3) of the spectrum are meaningful for a general real Banach algebra. Our objective in this paper is to extend these ideas still further — to partially ordered topological algebras without any norm — and to demonstrate a generalization of the Perron-Frobenius Theorem (2.4) in that context. The classical Perron-Frobenius theory of nonnegative matrices (in the simplest form: matrices with each entry nonnegative), states that the spectral radius ρ_* is itself an element of the spectrum:

$$\rho_*(x) \in \sigma(x) \quad \text{if } x \geq 0. \tag{2.4}$$

This has been extended to operators on partially ordered Banach spaces in many places; for expositions and references see [3] (e.g., [3, Theorem 8.1]) and the Appendix of [5].

At this point we recall (cf., e.g., [4]) that the spectral radius ρ_* of (2.1) for a bounded linear operator x on a normed space is computable as

$$\rho = \rho(x) = \lim_{k\to\infty} \left\{ \left\| x^k \right\|^{1/k} \right\}$$

which may equivalently be formulated as

$$\rho(x) = \inf \left\{ 1/\alpha : \begin{array}{cc} \alpha > 0, & \text{and} \\ \left\{ [\alpha x]^k : k = 1, 2, \ldots \right\} & \text{is a bounded set} \end{array} \right\}. \tag{2.5}$$

To provide a suitable notion — in terms of order-theoretic ideas — of "bounded set" in a partially ordered algebra, a set $S \subset \mathcal{A}$ will be called *'$\mathcal{P}$-bounded'* if it is contained in some order interval $[-u, u]$ — i.e., if there is some $u \in \mathcal{P}$ such that $\pm x \leq u$ for each $x \in S$. We can then use (2.5) as the *definition* of 'spectral bound' in our present context by interpreting 'bounded' to mean $\mathcal{P}$-bounded. Thus, given an element $x \in \mathcal{A}$, we set

$$\mathcal{S}(x) := \left\{ \alpha > 0 : \begin{array}{cc} \text{there is some } u = u_\alpha \in \mathcal{P} \text{ for which} & \\ -u \leq [\alpha x]^k \leq u & \text{for } k = 1, 2, \ldots \end{array} . \right\} \tag{2.6}$$

The definition (2.5) becomes

$$\rho(x) := \frac{1}{\sup \mathcal{S}(x)} = \inf\{1/\alpha : \alpha \in \mathcal{S}(x)\} \tag{2.7}$$

if $\mathcal{S}(x)$ is nonempty, setting $\rho(x) = \infty$ otherwise.

We refer to $\rho(x)$, defined in this manner, as the 'spectral bound' since we will indeed show as Theorem 5.2— under appropriate conditions — that $\rho(x)$ does bound the spectrum $\sigma(x)$ of (2.3) and so, noting our version of Perron-Frobenius in Theorem 5.7, must coincide with $\rho_*(x)$ at least for $x \in \mathcal{P}$. This is, in some sense, our principal result.

We let $\mathcal{B}$ denote the set of those elements of $\mathcal{A}$ for which $\mathcal{S}(x)$ is nonempty and refer to elements $x \in \mathcal{B}$ as being *tame*. We further say that a set $\mathcal{U} \subset \mathcal{A}$ is *uniformly tame* if there is some $u \in \mathcal{P}$ and some $\alpha > 0$ such that

$$-u \leq [\alpha x]^k \leq u \qquad \text{for } k = 1, 2, \dots \text{ and all } x \in \mathcal{U}. \tag{2.8}$$

Note that ρ is bounded on any uniformly tame set $\mathcal{U}$.

The set $\mathcal{B}$ of tame elements will play a role in $\mathcal{A}$ quite comparable to the set of *bounded operators* on a Banach space. Thus, when we discuss the spectrum $\sigma(x)$ for an element x it should be noted that we now will interpret the invertibility in (2.3) to mean existence of a *tame* inverse so, e.g., we require existence of $(\lambda - x)^{-1} \in \mathcal{B}$ for $\lambda \in \mathbb{R}$ to be in the resolvent set $\sigma'(x)$. [In general, not all $x \in \mathcal{A}$ will be tame (cf., Remark 6.3-*5*) and, unlike the situation for 'bounded operators', $\mathcal{B}$ need not itself be an algebra.]

3 Normed algebras

A Banach algebra is called a *'partially ordered Banach algebra'*, if it is furnished with a *positive cone* $\mathcal{P}$ (assumed closed and pointed with $1 \in \mathcal{P}$) which is closed under addition and multiplication (hence, convex) so $x \geq y$ means $(x - y) \in \mathcal{P}$. The hypotheses [H1], [H2] are then almost immediate.

In this section we will primarily be concerned with the relation between the usual completeness condition for a Banach algebra and our condition of $\mathcal{P}$-completeness. We will also be concerned with the relation between the norm and the order, recalling that the norm of a partially ordered normed space (or normed algebra) is called *semimonotone* with respect to the order if there is some real $a > 0$ such that:

$$\text{If } 0 \leq u \leq v, \text{ then } \|u\| \leq a\|v\|. \tag{3.1}$$

For information on semimonotone norms in partially ordered Banach spaces see [3, Section 4.1].

Theorem 3.1. *, Every partially ordered Banach algebra $\mathcal{A}$ satisfies* [H1], [H2]; *$\mathcal{A}$ is $\mathcal{P}$-complete if and only if its norm is semimonotone with respect to the positive cone $\mathcal{P}$.*

PROOF: The hypotheses [H1], [H2] follow immediately from the assumption that $\mathcal{A}$ is a partially ordered Banach algebra.

Now suppose the norm is semimonotone. For the $\mathcal{P}$-completeness condition, the criterion means that $0 \leq c_N \pm (x_j - x_k) \leq 2c_N$ — giving

$$\|c_N \pm (x_j - x_k)\| \leq 2a\,\|c_N\| \ \to \ 0$$

whence $\|x_j - x_k\| \ \to \ 0$. This is the usual Cauchy criterion for convergence in the Banach algebra and so ensures the convergence required for $\mathcal{P}$-completeness.

Conversely, suppose one had $\mathcal{P}$-completeness but not semimonotonicity, i.e., (3.1) fails so there would exist pairs $[u_i, v_i]$ with $0 \leq u_i \leq v_i$ and $\|v_i\| \leq 2^{-i}$, $\|u_i\| \geq 2^i$. Letting $x_k = u_1 + \cdots + u_k$, we would then have $-c_N \leq x_j - x_k \leq c_N$ $(j, k \geq N)$ with $c_N = \sum_{N+1}^{\infty} v_i$ (absolutely convergent in the Banach algebra) so

$$\|c_N\| \leq \sum_{N+1}^{\infty} \|v_i\| \leq 2^{-N} \to 0$$

yet, with $\|u_i\| \not\to 0$ it would be impossible to have convergence of x_k — a contradiction to $\mathcal{P}$-completeness. ■

The classic context for the theories we are discussing is the example:

Example 3.2. *Let $\mathcal{A} = \mathcal{M}_n$ be the set of all $n \times n$ real matrices (usual matrix operations), calling a matrix x nonnegative ($x \in \mathcal{P}$) if each of its entries is nonnegative. We topologize $\mathcal{M}_n$ as $\mathbb{R}^{n^2}$.*

Remark 3.3. One easily verifies that $\mathcal{M}_n$ satisfies [H1], [H2]. This topology is equivalently induced by the matrix norm $\|x\| = \max\{|x_{jk}|\}$ and we always have $\pm x \leq y$ where y is the $n \times n$ matrix with all entries $\|x\|$ so $\|y\| = \|x\|$; the $\mathcal{P}$-completeness criterion is equivalent to completeness for this norm, so $\mathcal{M}_n$ is $\mathcal{P}$-complete. In this example one easily sees that $\mathcal{P}$-boundedness of a set is equivalent to norm boundedness, that all elements are tame (so we can ignore the tameness requirement in (2.3)), and that (2.1) and (2.7) are equivalent. It is well-known here that $\rho(x), \sigma(x)$ depend continuously on x and, using local compactness, it is convenient at this point to anticipate Theorem 5.5 by observing that in $\mathcal{M}_n$:

$$\text{For any } \alpha < \rho(x), \text{ there is a neighborhood } \mathcal{U} \text{ of } x \text{ such that } \alpha \in \mathcal{S}(y) \text{ for } y \in \mathcal{U} \text{ with } u_\alpha \text{ in (2.6) taken constant on } \mathcal{U}. \tag{3.2}$$

Before we turn to the next example we provide a lemma:

Lemma 3.4. *Let $\mathcal{X}$ be a Banach space partially ordered by a closed, pointed positive cone $\mathcal{P}_0$. Suppose the $\mathcal{X}$-norm is semimonotone with respect to $\mathcal{P}_0$ (i.e.,* (3.1) *holds: there is some a such that, if $\xi, \eta - \xi \in \mathcal{P}_0$, then $|\xi| \le a|\eta|$) and $\mathcal{X} = \mathcal{P}_0 - \mathcal{P}_0$ (i.e., each element of $\mathcal{X}$ is a difference of positive elements — equivalently: for $\xi \in \mathcal{X}$ there exists $\omega \in \mathcal{P}_0$ with $\pm\xi \le \omega$). Then, with a as above and some $b > 0$, one has the apparently stronger conditions:*

$$\begin{array}{ll} (a) & \text{If } \pm\xi \le \omega, \text{ then } |\xi| \le 2a|\omega|, \\ (b) & \text{For each } \xi \in \mathcal{X} \text{ there is some } \omega \in \mathcal{P}_0 \text{ with} \\ & \quad |\omega| \le b|\xi|, \qquad \text{and} \qquad \pm\xi \le \omega. \end{array} \tag{3.3}$$

Proof: Suppose $\pm\xi \le \omega$. Then one notes that $0 \le \omega \pm \xi \le 2\omega$ gives $|\omega \pm \xi| \le 2a|\omega|$ so $2|\xi| = |(\omega + \xi) - (\omega - \xi)| \le 4a|\omega|$, and we have (3.3-$a$). That the condition $\mathcal{X} = \mathcal{P}_0 - \mathcal{P}_0$ implies (3.3-b) is essentially Theorem 1.5 of [3]. ■

Example 3.5. *Let $\mathcal{X}$ be a Banach space, partially ordered by a closed pointed positive cone $\mathcal{P}_0$ such that $\mathcal{X} = \mathcal{P}_0 - \mathcal{P}_0$; suppose the $\mathcal{X}$-norm $|\cdot|$ is semimonotone with respect to $\mathcal{P}_0$. We then let $\mathcal{A} = \mathcal{L}(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$ with the induced norm and the induced partial order given by $\mathcal{P} = \{x \in \mathcal{A} : \xi \in \mathcal{P}_0 \Rightarrow x\xi \in \mathcal{P}_0\}$.*

Remark 3.6. $\mathcal{A}$ is here a partially ordered Banach algebra so, noting Theorem 3.1, we have [H1] (clearly $\mathcal{P}$ is pointed) and [H2]. To see that it is $\mathcal{P}$-complete, we first note that the induced operator norm in $\mathcal{A} = \mathcal{L}(\mathcal{X})$ is semimonotone with respect to $\mathcal{P}$.

By Lemma 3.4 we have (3.3) for $\mathcal{X}$. Now consider $x, y \in \mathcal{A}$ with $0 \leq x \leq y$. For arbitrary $\xi \in \mathcal{X}$, we note that (3.3-b) gives existence of some $\omega \geq 0$ with $|\omega| \leq b|\xi|$ and $\pm\xi \leq \omega$. Then $0 \leq x \leq y$ gives $\pm x\xi \leq x\omega \leq y\omega$ so, by (3.3-a),

$$|x\xi| \leq 2a|y\omega| \leq 2a\|y\|\,|\omega| \leq 2a\|y\|\,b|\xi|$$

whence $\|x\| \leq (2ab)\|y\|$.

Thus, $\mathcal{A} = \mathcal{L}(\mathcal{X})$ is $\mathcal{P}$-complete by Theorem 3.1.

It is not yet clear whether necessarily $\mathcal{A} = \mathcal{P} - \mathcal{P}$ for Example 3.5 and it is precisely for this reason that we have not included that property in our definition of a partially ordered algebra.

We place the following lemma here, as particularly related to Banach algebras, — despite the fact that its proof uses Lemma 4.2 and Theorem 5.2 and so might well have been deferred.

Lemma 3.7. *Let $\mathcal{A}$ be any $\mathcal{P}$-complete partially ordered Banach algebra with a semimonotone norm. If $\pm x \leq u$ for some u, necessarily in $\mathcal{P}$, then x is tame with $\rho(x) \leq \|u\|$. In particular, every $u \geq 0$ is tame with $\rho(u) \leq \|u\|$. If we have the property* (3.3-a) *for $\mathcal{A}$, then every element $x \in \mathcal{A}$ is tame with $\rho(x) \leq \|x\|$.*

Proof: Given $u \geq 0$, choose any $\alpha > 0$ for which $\alpha\|u\| = r < 1$. Then $\|(\alpha u)^k\| \leq r^k$ and the Neumann series $\sum_k (\alpha u)^k$ converges absolutely to some $z \geq 0$. As in Theorem 5.2, we thus have $(\alpha u)^k \leq z$ for each k. This shows $\alpha \in \mathcal{S}(u)$ for all $\alpha < 1/\|u|$ so u is tame with $\rho(u) \leq \|u\|$. If $\pm x \leq u$, then Lemma 4.2 inductively gives $\pm(\alpha x)^k \leq (\alpha u)^k \leq z$ for each k so this α is also in $\mathcal{S}(x)$, showing $\rho(x) \leq \rho(u) \leq \|u\|$.

Given (3.3-a) for $\mathcal{A}$ — e.g., if $\mathcal{A} = \mathcal{P} - \mathcal{P}$ by Lemma 3.4 — then the final assertion follows similarly on noting that, for $\alpha\|x\| = r < 1$, we can let $\pm(\alpha x)^k \leq c_k$ and then set $u = \sum c_k$. ■

As noted above, our hypotheses have not required in general that $\mathcal{A} = \mathcal{P} - \mathcal{P}$ so we are not assured that any such u exists for arbitrary x in the Banach algebra and without (3.3-a) we do not know tameness for all elements of $\mathcal{A}$.

We note that a sequence (even of powers) can be bounded in norm without being bounded in order. For an example, consider the Banach algebra $\mathcal{A} =$

ℓ^1 with convolution as multiplication and the usual componentwise partial order. Letting $e_{(m)}$ be the element with all components 0 except for a 1 in the $k = m$ place, one has here $\|e_{(m)}\| = 1$ and $\left(e_{(1)}\right)^k = e^k$ for $k = 1, 2, \ldots$ Thus, to have a bound in the sense of the order — $c = (c_j) \geq \left(e_{(1)}\right)^k$ for all k — one would need each $c_k \geq 1$, which is impossible with $c \in \ell^1 = \mathcal{A}$. In this example, however, one easily sees that all elements are tame; e.g., although we have just seen that $1 \notin \mathcal{S}(e_{(1)})$, we do have $\rho(e_{(1)}) = 1$.

For an example of a $\mathcal{P}$-complete partially ordered Banach algebra in which not all elements are tame, consider the algebra $C_B(\mathbb{R})$ consisting of bounded functions on $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}\}$. Ordering this by the positive cone

$$\mathcal{P} = \{u \in \mathcal{A} : u(t) \geq 0 \,\text{on}\, \mathbb{R} \text{ with } u(t) \to 0 \text{ as } t \to \pm\infty\},$$

we easily verify that $\mathcal{A} = C_B(\mathbb{R})$ becomes a partially ordered algebra, satisfying [H1], [H2], and is $\mathcal{P}$-complete. However, an element x will be tame only if $x(t) \to 0$ as $t \to \pm\infty$. Note that we do not have $\mathcal{A} = \mathcal{P} - \mathcal{P}$ for this example.

On the other hand, when tameness is universal in a Banach algebra, the spectrum as we have defined it here clearly coincides exactly with the usual Banach algebra spectrum.

Theorem 3.8. *Let $\mathcal{A}$ be a partially ordered normed algebra,satisfying* [H1], [H2] *and such that $\mathcal{A} = \mathcal{P} - \mathcal{P}$. Assume also that $\mathcal{A}$ is $\mathcal{P}$-complete so its norm is semimonotone. Then every element of $\mathcal{A}$ is tame and $\rho(\cdot)$, as defined by* (2.6), (2.7)*, coincides with the standard Banach algebra definition*

$$\rho_B(x) := \lim_{k\to\infty} \left\|x^k\right\|^{1/k}. \tag{3.4}$$

[This is often defined as a lim sup, but it is standard that the limit always exists.]

Proof: Using the semimonotonicity of the norm and $\mathcal{A} = \mathcal{P} - \mathcal{P}$, we may apply Lemma 3.4 to $\mathcal{A}$ as a vector space to obtain (3.3).

Suppose, first, that $1/\alpha > \rho(x)$ so, as in (2.7), we have $\pm[\alpha x]^k \leq u$ for some $u \geq 0$ and all k. Then $\alpha^k\|x^k\| \leq 2a\|u\|$ for all k so $\|x^k\| \leq 2a\|u\|\alpha^{-k}$. Thus, using the definition (3.4), we have $\rho_B(x) \leq \lim_{k\to\infty}[a\|u\|\alpha^{-k}]^{1/k} = 1/\alpha$; this shows that $\rho(x) \geq \rho_B(x)$ for all x.

Conversely, if $1/\alpha > \rho_B(x)$, we can choose $\beta > \alpha$ such that $1/\beta > \rho_B(x)$ so, for $k > K$, we have $\|x^k\|^{1/k} < 1/\beta$ and $\|[\alpha x]^k\| \leq (\alpha/\beta)^k$ then. Using (3.3), for every k we have $\pm[\alpha x]^k \leq u_k$. Since $\|u_k\| \leq b\|[\alpha x]^k\| \leq b(\alpha/\beta)^k$

for large k, the series $\sum_k u_k$ is convergent in $\mathcal{A}$ to some u — clearly with $u \geq u_k$ for each k. Then $\pm[\alpha x]^k \leq u_k \leq u$ for each k gives $\alpha \in S(x)$ whence $1/\alpha > \rho(x)$. This shows $\rho(x) \leq \rho_B(x)$ for all x, completing the proof. ■

4 Preliminary results

We begin with some results for partially ordered algebras which use [H1], but which are independent of any reliance on [H2].

Lemma 4.1.

1. *If $\mathcal{U}$ is uniformly tame in $\mathcal{A}$, then $\{\rho(x) : x \in \mathcal{U}\}$ is bounded in $\mathbb{R}$.*
2. *A finite union $\mathcal{U} = \bigcup_j \mathcal{U}_j$ of uniformly tame sets is uniformly tame.*

PROOF: The definition (2.8) of uniform tameness fixes some $0 < \alpha \in \mathcal{S}(x)$ — so, by (2.7), $\rho(x) \leq 1/\alpha$ — for each $x \in \mathcal{U}$. Given α_j, u_j such that $\pm[\alpha x]^k \leq u_j$ for $0 < \alpha \leq \alpha_j$ and all $x \in \mathcal{U}_j$, one can take $\alpha_* = \min_j\{\alpha_j\}$ and $u_* = \sum_j u_j$ to have $\pm[\alpha x]^k \leq u_*$ for $0 < \alpha \leq \alpha_*$ and all $x \in \mathcal{U}$. ■

Lemma 4.2. *If $\pm x \leq a$ and $\pm y \leq b$, then $\pm x^j y^k \leq a^j b^k$ for $j, k = 1, 2, \ldots$; in particular, $\pm x^j \leq a^j$ for $j = 1, 2, \ldots$*

PROOF: By assumption we have $(a-x), (a+x), (b-y), (b+y) \in \mathcal{P}$ so the products $(a-x)(b+y) = (ab-xy) + (ay-xb)$ and $(a+x)(b-y) = (ab-xy) - (ay-xb)$ are also in $\mathcal{P}$ by [H1]; adding these shows $2(ab-xy) \in \mathcal{P}$ so $xy \leq ab$. Similarly, the positivity of $(a+x)(b+y)$ and $(a-x)(b-y)$ shows $-xy \leq ab$ and we have shown the result for $j = k = 1$.

Using this with y, b inductively replaced by x^j, a^j gives $\pm x^j \leq a^j$. Replacing x, a, y, b by x^j, a^j, y^k, b^k then gives the desired general result. ■

Lemma 4.3. *Suppose x and y are tame. Then*

1. *Either*

- *both xy and yx are tame with $\rho(yx) = \rho(xy)$ or*
- *neither xy nor yx is tame.*

2. *If $xy = yx$, the second alternative of 1. cannot occur and we have $\rho(xy) = \rho(yx) \leq \rho(x)\rho(y)$.*

Compare this with Lemma 4.4-3. where it is not given that x, y are tame.

PROOF: By assumption we have $\pm x \leq a$, $\pm y \leq b$. Now, recalling that $0 < \alpha < 1/\rho(x)$ implies $\alpha \in \mathcal{S}(x)$, suppose xy is tame so for $\alpha \in \mathcal{S}(xy)$ we have $\pm(\alpha xy)^k \leq u = u_\alpha$ for each $k = 1, 2, \ldots$ Using Lemma 4.2 we then have

$$\pm(\alpha yx)^k = \pm\alpha y(\alpha xy)^{k-1}x \leq |\alpha| a u_\alpha b =: u'$$

for $k = 0, 1, \ldots$ so $\alpha \in \mathcal{S}(yx)$. This shows tameness of yx and that $\mathcal{S}(yx) \supset \mathcal{S}(xy)$ so $\rho(yx) \leq \rho(xy)$. Symmetrically, we obtain the reverse inequality so the spectral bounds are equal as asserted. This shows that one cannot have one of xy, yx tame without the other.

We have, by assumption, $\pm[\alpha x]^k \leq u$ and $\pm[\beta y]^k \leq v$. If $xy = yx$, we have $\pm[\alpha\beta xy]^k = \pm[\alpha x]^k[\beta y]^k \leq uv$ by Lemma 4.2 so $\alpha\beta \in \mathcal{S}(xy)$ when $\alpha \in \mathcal{S}(x)$, $\beta \in \mathcal{S}(y)$. ■

Remark 6.3-*4* below shows that the second alternative above is possible when x, y do not commute.

Lemma 4.4. *Suppose $u, v \in \mathcal{B}_+$, i.e., tame and nonnegative.*

1. *If $uv = vu$, then $uv \in \mathcal{B}_+$ with $\rho(uv) \leq \rho(u)\rho(v)$.*
2. *If $\pm x \leq u$, then $x \in \mathcal{B}$ with $\rho(x) \leq \rho(u)$.*
3. *If $\pm x \leq u$ and $\pm y \leq v$ with $xy = yx$, then xy is tame; if also $uv = vu$, then $\rho(xy) \leq \rho(uv) \leq \rho(u)\rho(v)$.*

PROOF: For *1.*, we need only note that $(\gamma uv)^k = (\alpha u)^k(\beta v)^k$ if $uv = vu$ and $\gamma = \alpha\beta$; the result follows by letting $\alpha \to \rho(u)$, $\beta \to \rho(v)$. For *2.*, we have $\pm x^k \leq u^k$ by Lemma 4.2 and the result follows. Finally, for *3.* if $uv = vu$ we again set $\gamma = \alpha\beta$ and note that

$$\pm(\gamma xy)^k = \pm(\alpha x)^k(\beta y)^k \leq (\alpha u)^k(\beta v)^k = (\gamma uv)^k$$

for each k, which gives $\rho(xy) \leq \rho(uv)$ and we can then apply *1.*; if u, v need not commute, replace u, v by $w = u + v$ to get the tameness. ■

Considering $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in $\mathcal{M}_2$ shows that the commutativity requirements for *1., 3.* are not merely artifacts of the proof and cannot be omitted.

Lemma 4.5. *Suppose $x, y \in \mathcal{B}$.*

1. *For any real λ one has*

$$(\lambda x) \in \mathcal{B} \text{ with } \rho(\lambda x) = |\lambda|\rho(x). \tag{4.1}$$

2. *If $xy = yx$, then $(x + y) \in \mathcal{B}$ with*

$$\rho(x + y) \leq \rho(x) + \rho(y). \tag{4.2}$$

PROOF: To see *1.*, note that $\alpha \in \mathcal{S}(x)$ if and only if $(\alpha/|\lambda|) \in \mathcal{S}(\lambda x)$ and $\mathcal{S}(-x) = \mathcal{S}(x)$.

For *2.*, let $\rho(x) =: \xi$, $\rho(y) =: \eta$ and let $\zeta = \xi + \eta$. For any $0 < r < 1$ we set $\alpha = r/\xi$, $\beta = r/\eta$, $\gamma = r/\zeta$ so $\zeta\gamma(x + y) = (\xi\alpha x + \eta\beta y)$. Since $\alpha \in \mathcal{S}(x)$, $\beta \in \mathcal{S}(y)$, there exist u, v such that $\pm(\alpha x)^k \leq u$ and $\pm(\beta y)^k \leq v$ for each k. [Note that we need not have $uv = vu$, but the assumed $xy = yx$ ensures applicability below of the Binomial Theorem.] We now have

$$\begin{aligned}
\pm(\gamma[x + y])^k &= \pm\zeta^{-k}(\xi\alpha x + \eta\beta y)^k \\
&= \pm\zeta^{-k}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\xi^j(\alpha x)^j\eta^{k-j}(\beta y)^{k-j} \\
&\leq \zeta^{-k}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\xi^j u\,\eta^{k-j}v \\
&= \zeta^{-k}(\xi + \eta)^k uv \quad = uv
\end{aligned}$$

for each k so $\gamma \in \mathcal{S}(x + y)$. This for each $r < 1$ gives (4.2). ■

Again, taking $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in $\mathcal{M}_2$ shows that the commutativity requirement here cannot be omitted for (4.2) even if $x, y \geq 0$. Indeed, we will see later (in Remark 6.3-*4*) that $\mathcal{B}$ need not even be closed under addition.

Lemma 4.6. *If $x, y \in \mathcal{B}$ and $w \in \mathcal{B}_+$ with x, y, w commuting, then*

$$-w \leq x - y \leq w \qquad \textit{implies} \qquad |\rho(x) - \rho(y)| \leq \rho(w). \tag{4.3}$$

Proof: Assume $\pm(x - y) \leq w$. Then from *2.* of Lemma 4.4 we have $\rho(\pm[x - y]) \leq \rho(w)$. From *2.* of Lemma 4.5 we have

$$\begin{aligned} \rho(x) &\leq \rho(y) + \rho(x - y) \leq \rho(y) + \rho(w), \\ \rho(y) &\leq \rho(x) + \rho(y - x) \leq \rho(x) + \rho(w) \end{aligned}$$

and the result follows. ∎

The remaining lemmas of this section assume [H2] in addition to [H1].

Lemma 4.7. *In a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$, suppose $\pm[x_k - \bar{x}] \leq u_k$ (i.e., $-u_k \leq x_k - \bar{x} \leq u_k$) with $u_k \to 0$. Further, either assume that $\{u_k\}$ is monotone (each $u_{k+1} \leq u_k$) or that $\mathcal{P}$ has nonempty interior. Then $x_k \to \bar{x}$.*

Proof: Setting $z_k = x_k - \bar{x}$, we seek to prove that $z_k \to 0$.

Under the monotonicity assumption, $\pm[z_j - z_k] \leq u_j + u_k \leq 2u_N$ for $j, k \geq N$ so $\mathcal{P}$-completeness implies convergence $z_k \to z$. Since we have $[u_k - z_k] \in \mathcal{P}$ and $\mathcal{P}$ is closed, it follows that $[0 - z] \in \mathcal{P}$ in the limit; similarly, $[0 + z] \in \mathcal{P}$. Thus, since $\mathcal{P}$ is pointed, we have $z = 0$ so $z_k \to 0$.

In the second case, fix some v in the interior of $\mathcal{P}$ so the set $v - \mathcal{P}$ contains an open set $\mathcal{U}_1$ containing 0. We then set $\mathcal{U} = \mathcal{U}_1 \bigcap [-\mathcal{U}_1]$ so $\pm x \leq v$ for any x in the open set $\mathcal{U}$. For any subsequence $\{u_{k(j)}\}$ we can recursively select a subsubsequence $\{u_{k(j(i))}\}$ with $0 \leq u_{k(j(i))} \leq 2^{-i}v$.

To see this, note that $\pm x \leq 2^{-i}v$ for $x \in 2^{-i}\mathcal{U}$ and that $2^{-i}\mathcal{U}$ is an open set containing 0 so convergence to 0 ensures that $u_{k(j)} \in 2^{-i}\mathcal{U}$ for some $j = j(i)$ (asking also that $j(i) > j(i-1)$) whence $0 \leq u_{k(j(i))} \leq 2^{-i}v$ by construction.

For this subsubsequence we have $\pm z_{k(j(i))} \leq u_{k(j(i))} \leq 2^{-i}v$. Since $2^{-i}v \searrow 0$ as $i \to \infty$, we may apply the first case result to see that $z_{k(j(i))} \to 0$. The standard 'Subsubsequence Lemma' then gives convergence $z_k \to 0$ for the full sequence. ■

Lemma 4.8. *Let $\mathcal{A}$ be a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$.*

1. *Sandwich Lemma: Let $x_k \leq y_k \leq z_k$ with x_k, z_k convergent. Assume $(z_k - x_k) \searrow 0$ so the limits are the same. Then also y_k converges to this same limit.*
2. *If $\pm x_k \leq u$ in $\mathcal{A}$ and $\alpha_k \to 0$ in $\mathbb{R}$, then $\alpha_k x_k \to 0$.*

PROOF: For *1.*, let $\lim_k x_k = \lim_k z_k = y$. We have $0 \leq y_k - x_k \leq z_k - x_k \searrow 0$ so the condition in the definition of $\mathcal{P}$-completeness holds whence $u_k = (y_k - x_k)$ converges to some $u \in \mathcal{A}$. Since each $u_k \in \mathcal{P}$ and $\mathcal{P}$ is closed, $u \in \mathcal{P}$. By the continuity of addition, $v_k = (z_k - y_k) = (z_k - x_k) - u_k \to -u$ so, as each $v_k \in \mathcal{P}$, we have $-u \in \mathcal{P}$ — showing $u = 0$. Then $y_k = u_k + x_k \to u + y = y$.

Remark 6.6-*1.* shows the importance here of the $\mathcal{P}$-completeness assumption.

For *2.*, given any subsequence, we may extract a subsubsequence such that $\alpha_k \searrow 0$ (or that $-\alpha_k \searrow 0$). We then have $-\alpha_k u \leq \alpha_k x_k \leq \alpha_k u \searrow 0$ whence, by *1.*, we have $\alpha_k x_k \to 0$ for the subsubsequence. Thus, by the usual Subsubsequence Lemma, we have $\alpha_k x_k \to 0$ for the full sequence. ■

Lemma 4.9. *Suppose $x_k \to \bar{x}$ and $\pm(x_k - \bar{x}) \leq u_k$ with $\{x_j, u_k\}$ all commuting and with $\rho(u_k) \to 0$ as $k \to \infty$. Then*

$$\rho(\bar{x}) = \lim_{k\to\infty} \rho(x_k). \tag{4.4}$$

In particular, $\bar{x}$ is tame if $\{\rho(x_k)\}$ is bounded.

PROOF: By [H1] we have $\bar{x}x_k = x_k\bar{x}$ on letting $j \to \infty$. Then by Lemma 4.6 we have $|\rho(\bar{x}) - \rho(x_k)| \leq \rho(u_k) \to 0$. ■

This is a restricted continuity result for the spectral bound $\rho(\cdot)$ with respect to the topology of $\mathcal{A}$. On the other hand, the spectral bound need not be $\mathcal{A}$-continuous in general (cf., Remark 6.3-*5*).

5 Results on σ and ρ

We begin with a spectral calculus: Given a fixed element x in the partially ordered algebra $\mathcal{A}$, we seek to define $f(x) \in \mathcal{A}$ for suitable functions f. The definition is obvious when f is a polynomial with real coefficients and we wish to extend this appropriately to a larger function class $\mathcal{F}$. Given some $R > 0$, we let $\Omega = \Omega_R = \{\zeta \in \mathbb{C} : |\zeta| < R\}$ and — compare Example 6.10 — will let $\mathcal{F}$ be the algebra:

$$\mathcal{F} = \mathcal{F}_R = \mathcal{F}(\Omega) = \left\{ f : \Omega_R \overset{\text{analytic}}{\longrightarrow} \mathbb{C} : f(\bar{\zeta}) = \overline{f(\zeta)} \text{ for } \zeta \in \Omega \right\}.$$

[Each $f \in \mathcal{F}$ has a power series expansion $\sum_k a_k \zeta^k$ convergent on Ω_R with real coefficients $\{a_k\}$.] We topologize $\mathcal{F}$ by uniform convergence on compact subsets of Ω.

Theorem 5.1. *Let the partially ordered algebra $\mathcal{A}$ be $\mathcal{P}$-complete and let $x \in \mathcal{A}$ be tame (so $\rho(x) < \infty$); choose $R > \rho(x)$. Now let $\mathcal{F} = \mathcal{F}(\Omega_R)$ as above and define a map $F = F_x : \mathcal{F} \to \mathcal{A} : f \mapsto f(x)$ by*

$$F = F_x : f \mapsto f(x) = \sum_{k=1}^{\infty} a_k x^k \quad \textit{for} \;\; f(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k \tag{5.1}$$

Then:

1. *The map F is well-defined on $\mathcal{F} = \mathcal{F}_R$ and provides a continuous algebra homomorphism: $\mathcal{F} \to \mathcal{A}$.*

2. *If $x \geq 0$ in $\mathcal{A}$ and each coefficient $a_k \geq 0$, then $f(x) \geq 0$ in $\mathcal{A}$. Thus, for each $f \in \mathcal{F}$ one has $\pm f(x) \leq f^+(x)$ where $f^+(\zeta) = \sum_k |a_k| \zeta^k$.*

3. *$f(x)$ is tame for each $f \in \mathcal{F}$ with*

$$\rho(f(x)) \leq f^+(\rho(x)) \quad \textit{where } f^+(\zeta) = \sum_{k=1}^{\infty} |a_k|\, \zeta^k. \tag{5.2}$$

4. *If $\mathcal{U}$ is uniformly tame, $R > \sup\{\rho(x) : x \in \mathcal{U}\}$, then $\{f(x) : x \in \mathcal{U}\}$ is again uniformly tame.*

5. *Each $f_* \in \mathcal{F}$ has a neighborhood $\mathcal{N} \subset \mathcal{F}$ such that $\mathcal{U} = \{f(x) : f \in \mathcal{N}\}$ is uniformly tame.*

PROOF: For $f \in \mathcal{F}$ giving $f(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k$, we note that each coefficient a_k is real and that $\sum_k a_k r^k$ converges absolutely in $\mathbb{R}$ if we choose r so $\rho(x) < r < R$. We now take the obvious definition of $f(x)$ when f is a polynomial and then interpret the infinite series defining $f(x)$ for more general f as resulting from convergence of the sequence of the polynomial partial sums $f^N(\zeta) = \sum_{k=1}^{N} a_k \zeta^k$.

With r as above and noting that the definition of $\rho(x)$ gives $\pm x^k \le r^k u$, we have the inequality

$$\pm \left[f^M(x) - f^N(x)\right] = \pm \left[\sum_{k=N+1}^{M} a_k x^k\right] \le \left[\sum_{k=N+1}^{\infty} |a_k|\, r^k\right] u \tag{5.3}$$

for each $M > N$. Note that $r < R$ ensures $\left[\sum_{k=N+1}^{\infty} |a_k|\, r^k\right] \to 0$ as $N \to \infty$ so the $\mathcal{P}$-completeness of $\mathcal{A}$ ensures convergence of the sequence $\left\{f^N(x)\right\}$ to some (unique) limit, which we now call $f(x)$. Thus, $F : \mathcal{F} \to \mathcal{A}$ is well-defined. It will be useful to note that

$$\pm \left[f(x) - f^N(x)\right] \le \left[\sum_{k=N+1}^{\infty} |a_k|\, r^k\right] u, \tag{5.4}$$

as follows, since $\mathcal{P}$ is closed, by letting $M \to \infty$ in (5.3). We also observe similarly that, uniformly in N, we have $\pm f^N(x) \le f^+(r)u$.

Continuity of $F = F_x$ means that convergence $f_n \to f$ in $\mathcal{F}$ (defined as uniform convergence on compact subsets of Ω for $f_n(\zeta) = \sum_k a_{n,k}\zeta^k$) should imply that $f_n(x) \to f(x)$ in $\mathcal{A}$.

To see this, note first that a standard complex analysis argument shows that $\sum_0^N |a_{n,k}|\, r^k \to \sum_0^N |a_k|\, r^k$ as $n \to \infty$ uniformly in N (and uniformly in $r \le \bar{r}$ if $\bar{r} < R$). We then note that, for any N, one has

$$\begin{aligned}&[f(x) - f_n(x)] \\ &\quad = \left(\left[f(x) - f^N(x)\right] - \left[f_n(x) - f_n^N(x)\right] + \left[\sum_0^N (a_k - a_{n,k})\, x^k\right]\right).\end{aligned} \tag{5.5}$$

Now, standard estimation of the three terms, using (5.3) for $f - f^N$ and for $f_n - f_n^N$ with the noted uniformity and the tameness of x, lets us conclude that $\pm [f(x) - f_n(x)] \le \varepsilon_n u$ with $\varepsilon_n \to 0$, so Lemma 4.7 applies to show $f_n(x) \to f(x)$.

Noting that the property *2.* is obvious for polynomials and the positive cone $\mathcal{P}$ is closed in $\mathcal{A}$ by [H2], it now extends to $\mathcal{F}$ by continuity.

We next wish to consider *3.* We have already observed that $\left[f^{+}(r)u \pm f^{N}(x)\right]$ are in P for each N and this holds in the limit $f^N \to f$ since $\mathcal{P}$ is closed. Applying this to powers $[f]^k$ then gives the tameness and the bound (5.2).

Given f and k, let $e(\zeta) = [f(\zeta)]^k$ and $e_+(\zeta) = [f^+(\zeta)]^k$. We then have $\pm[f(x)]^n = \pm e(x) \le e_+(r)u$ for $\rho(x) < r < R$. Noting that $e(r) \le e_+(r) = [f^+(r)]^k$ for any $r \ge 0$, we have shown that

$$\pm[\alpha f(x)]^k \le \left[\alpha f^+(r)\right]^k u. \tag{5.6}$$

This holds for each $k = 1, 2, \dots$ and is bounded by u if $\alpha \le 1/f^+(r)$. Thus $f(x)$ is tame and, letting $r \to \rho(x)$, one obtains (5.2) as desired for *2.*

We observe that this argument shows, further, that

$$\pm[\alpha f(x)]^k \le u \qquad \text{for all } 0 < \alpha < 1/g(\rho(x)),\ k = 1, 2, \dots \tag{5.7}$$

for any $g(\zeta) = \sum_{k=1}^{\infty} b_k \zeta^k$ with each $b_k \ge |a_k|$. Note that u in (5.7) is the same as in (2.6), defining tameness of x.

We immediately get *4.* since (5.6) holds uniformly in $x \in \mathcal{U}$. The extension (5.7) permits us to verify *5.*

For any $r < R$ (still with $r > \rho(x)$) and any $\beta > 0$, the set

$$\mathcal{N} = \{f \in \mathcal{F} : |f(\zeta) - f_*(\zeta)| < \beta \text{ for } |\zeta| \le r\}$$

is open in $\mathcal{F}$, so is a neighborhood of f_*. The Cauchy Integral Formula then gives

$$|a_k - a_{*k}| \le \beta r^{-k} \quad \text{for } k = 1, 2, \dots,\ f(\zeta) = \sum_k a_k \zeta^k \in \mathcal{N}$$

and, uniformly for $f \in \mathcal{N}$, we may take $g(\zeta) = f_*^+(\zeta) + \beta/(r - \zeta)$ in (5.7). This g is not in $\mathcal{F} = \mathcal{F}(\Omega_R)$, but is in the corresponding function algebra $\mathcal{F}(\Omega_r)$ so our results continue to apply. The inequality (5.7) then holds for all $f \in \mathcal{N}$ so $\mathcal{U} = \{f(x) : f \in \mathcal{N}\}$ is uniformly tame.

Finally, we verify that F is an algebra homomorphism — i.e., show that for $s = f + g$ and $p = fg$ one has $F[s] = F[f] + f[g]$ and $F[p] = F[f]\,F[g]$.

For the sum we need only note that $s^N(x) = f^N(x) + g^N(x)$ for each N so we have equality also in the limit. For the product we write $g(z) = \sum_k b_k z^k$ and get

$$p(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n \quad \text{with} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

(so $|c_n| \le \sum_0^N |a_k|\,|b_{n-k}|$) while

$$p_N(\zeta) = f^N(\zeta)\, g^N(\zeta) = \textstyle\sum_n c_n^N \zeta^n$$
$$\text{with} \quad c_n^N = \textstyle\sum \{a_k\, b_{n-k} : 0 \le k, n-k \le N\}\,.$$

Noting that $c_n^N = c_n$ for $N \ge n$ and $\left|c_n^N - c_n\right| \le \sum_0^N |a_k|\,|b_{n-k}|$, we have $p^N - p_N \to 0$ in $\mathcal{F}$ so $p_N(x) \to p(x)$ in $\mathcal{A}$. On the other hand, $f^N(x) \to f(x)$ and $g^N(x) \to g(x)$ with $\{f^N(x)\}, \{g^N(x)\}$ $\mathcal{P}$-bounded so, by the continuity of multiplication as assumed in [H2], we have $p_N(x) = f^N(x)g^N(x) \to f(x)g(x)$ whence $p(x) = f(x)g(x)$. ■

Theorem 5.2. *Let x be a tame element of a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$. Then*

1. *$\hat{q}(x;\mu) = (1 - [\mu + \bar{\mu}]\, x + |\mu|^2 x^2)$ is tamely invertible for $|\mu| < 1/\rho(x)$, [Note that $\hat{q}(\zeta;\mu) = q(\zeta;\lambda)/|\lambda|^2$ with $\lambda = 1/\mu$.]*

2. *the spectral bound $\rho(x)$ is, indeed, a bound on the spectrum:*

$$\lambda \in \sigma(x) \quad \Rightarrow \quad |\lambda| \le \rho(x),$$

   i.e., $q(x;\lambda)$, as in (2.3), always has a tame inverse when $|\lambda| > \rho(x)$,

3. *for $|\lambda| > \rho(x)$ we have the estimate*

$$\rho(q(x;\lambda)) \le \left(\frac{1}{|\lambda| - \rho(x)}\right)^2 . \tag{5.8}$$

Proof: Part *1.* follows from Theorem 5.1.

For $|\mu| < 1/\rho(x)$ we take $\rho(x) < R < 1/|\mu|$ and set $\hat{q}(\zeta) = \hat{q}(\zeta;\mu)$, etc. Note that

$$f(\zeta) = \frac{1}{\hat{q}(\zeta)} = \frac{1}{1 - [\mu + \bar{\mu}]\,\zeta + |\mu|^2\zeta^2} = \frac{1}{(1-\mu\zeta)} \cdot \frac{1}{(1-\bar{\mu}\zeta)} \tag{5.9}$$

is in $\mathcal{F}(\Omega_R)$: analytic on Ω_R and real for real ζ. Since $f(\zeta)\,\hat{q}(\zeta) \equiv 1$ on Ω_R, Theorem 5.1 gives $f(x) = [\hat{q}(x)]^{-1}$ so we have invertibility with $[\hat{q}(x)]^{-1}$ tame. The estimate

$$\rho(\hat{q}(x;\mu)) \le \left(\frac{1}{1 - |\mu|\rho(x)}\right)^2 \tag{5.10}$$

follows from (5.2) on computing the series for $f(\zeta)$ as the product of the power series (in ζ) for $(1-\mu\zeta)^{-1}$ and for $(1-\bar{\mu}\zeta)^{-1}$.

This can be simplified to a Neumann series if λ is real and we then note that for $0 < \mu = 1/\lambda < 1/\rho(x)$ Theorem 5.1-*2.* shows $(1-\mu x)^{-1} \ge 0$ for $x \ge 0$.

With $\mu = 1/\lambda$ for $|\lambda| > \rho(x)$, the invertibility of $\hat{q}(x)$ then gives invertibility of $q(x;\lambda) = |\lambda|^2\hat{q}(x)$, whence $\lambda \in \sigma'(x)$ as asserted and (5.8) then follows immediately from (5.10). ■

Specializing this, we have the following:

Lemma 5.3. *Let x be a tame element of a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$ and suppose α is real with $|\alpha| < 1/\rho(x)$. Then:*

1. *the Neumann series $1 + \alpha x + [\alpha x]^2 + \cdots$ converges to a tame element $y = y_\alpha = (1-\alpha x)^{-1}$; if $\alpha > 0$ and $x \in \mathcal{P}$, then $y_\alpha \in \mathcal{P}$*

2. *if $\alpha = 1/\lambda$ so $|\lambda| > \rho(x)$, then $(\lambda - x)^{-1} = \alpha y_\alpha$ and*

$$\rho\left((\lambda - x)^{-1}\right) = |\alpha|\,\rho\left((1-\alpha x)^{-1}\right) \leq \frac{1}{|\lambda| - \rho(x)}.$$ ■

Lemma 5.4. *Let $\{x_n\}$ be uniformly tame in a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$ and let $\alpha_n \to 0$ in $\mathbb{R}$. Then $\{y_n := (1-\alpha_n x_n)^{-1}\}$ is uniformly tame with $y_n \to 1$.*

PROOF: By Lemmas 4.1 and 5.3 we have y_n defined for large enough n. Further, $\{y_n\}$ is uniformly tame by Theorem 5.1-*3.* and a similar argument shows that $\{x_n y_n\}$ is also uniformly tame. Noting that $y_n - 1 = \alpha_n x_n y_n$, the desired result follows from Lemma 4.8. ■

Theorem 5.5. *For any tame element x in a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$:*

1. *the set $\sigma(x)$ is closed in $\mathbb{C}$, i.e., its complement $\sigma'(x)$ is open;*

2. *the $\mathcal{B}$-valued symmetrized resolvent map*

$$\lambda \mapsto r(\lambda) = [q(x;\lambda)]^{-1} \qquad [q(z;\lambda) = (\lambda - z)(\bar{\lambda} - z)] \tag{5.11}$$

   is continuous on $\sigma'(x)$;

3. *each $\lambda \in \sigma'(x)$ has a neighborhood $\mathcal{N}$ (actually a symmetric neighborhood of $\lambda, \bar{\lambda}$) such that $r(\mathcal{N}) = \{r(\lambda) : \lambda \in \mathcal{N}\}$ is uniformly tame.*

PROOF: Given λ_*, set $q_* = q(\zeta;\lambda_*)$ as in (2.2). For each given ζ, we set $\omega = 1/q_*(\zeta)$ and for $\lambda_* \in \sigma'(x)$, we then set $r_* = r(\lambda_*) = [q_*(x)]^{-1}$ — which, by the assumption, exists, is tame and, of course, commutes with x. Finally, setting $q = q(\zeta;\lambda)$ for more general λ, some manipulation provides the identity

$$q = q_* \left(1 - \left[(\lambda_* - \lambda)(\bar\lambda_* - \zeta) + (\bar\lambda_* - \bar\lambda)(\lambda_* - \zeta)\right]\omega + |\lambda_* - \lambda|^2\omega\right).$$

Substituting $x \hookleftarrow \zeta$ means also substituting $r_* \hookleftarrow \omega$ and we thus obtain the identity

$$\begin{aligned} q(x;\lambda) &= q_*(x)[1-y] \qquad \text{with } y = y(\lambda) = (\alpha r_* + \beta x r_*) \\ \text{where} &\quad \begin{cases} \alpha &= (\lambda_* - \lambda)\bar\lambda_* + (\bar\lambda_* - \bar\lambda)\lambda_* - |\lambda_* - \lambda|^2 \\ \beta &= 2\mathcal{R}e\{\lambda - \lambda_*\}\ . \end{cases} \end{aligned} \tag{5.12}$$

Using Lemmas 4.4 and 4.5, we see that

$$\rho(y) \le [|\alpha| + |\beta|\rho(x)]\,\rho(r_*)$$

so, as $\alpha, \beta = \mathcal{O}(|\lambda - \lambda_*|)$, it is clear that there is some $\varepsilon > 0$ such that

$$|\lambda - \lambda_*| < \varepsilon \quad \Rightarrow \quad \rho(y) < 1,$$

from which it follows by Theorem 5.2 that $q(x;\lambda)$ is also tamely invertible when $|\lambda - \lambda_*| < \varepsilon$ so we have *1.*

Clearly we have $y(\lambda) \to 0$ as $\lambda \to \lambda_*$ and, as in Lemma 5.4, we have $[1 - y(\lambda)]^{-1} \to 1$. Inverting in (5.12) we obtain the resolvent identity

$$r(x;\lambda) - r_* = \left([1 - y(\lambda)]^{-1} - 1\right) r_* \tag{5.13}$$

and, using Lemma 5.4, it follows that $r(\lambda) \to r_*$ so we have *2.* This argument is uniform over a neighborhood of λ_*, so we also have *3.* ■

Lemma 5.6. *Let $x \ge 0$ and suppose $(1 - \alpha x)$ is invertible for some $\alpha > 0$ with $y = y_\alpha = (1 - \alpha x)^{-1} \ge 0$. Then*

1. *$0 \le (\alpha x)^k \le y$ for each k so $\alpha \in \mathcal{S}(x)$ and x is tame.*

We may then use $u_\alpha = y$ in the definition (2.6) *of $\mathcal{S}(x)$; as y is a limit of polynomials in x, this necessarily commutes with every element which commutes with x.*

2. *if y_α is tame, then $\alpha < \bar{\alpha} = 1/\rho(x)$.*

PROOF: For each k we have $(1-\alpha x)\left[1+\cdots+(\alpha x)^k\right] = 1-(\alpha x)^{k+1}$ so

$$1+\cdots+(\alpha x)^k + (\alpha x)^{k+1}(1-\alpha x)^{-1} = (1-\alpha x)^{-1} = y. \tag{5.14}$$

Since each term on the left is nonnegative, we have $0 \le (\alpha x)^k \le y$ so $\alpha \in \mathcal{S}(x)$ and $x \in \mathcal{B}_+$.

In fact, we have shown not only that the individual powers $(\alpha x)^k$ are uniformly bounded, but that the partial sums of the Neumann series form a bounded monotone sequence. Nevertheless, if we do not already know that y is tame, it is not clear from this that we must necessarily have $(\alpha x)^k \to 0$, so we cannot conclude here from the identity (5.14) that $(1-\alpha x)^{-1}$ should be given by a convergent Neumann series.

To obtain the strict inequality $\alpha < \bar{\alpha}$ in *2.*, we first note that the product yx is tame by Lemma 4.4. We can then choose $0 < \varepsilon < 1/\rho(yx)$ so, by Theorem 5.2, there exists $(1-\varepsilon yx)^{-1} \ge 0$. Noting $1-[\alpha+\varepsilon]x = (1-\alpha x)(1-\varepsilon yx)$, we see that this is invertible with $(1-[\alpha+\varepsilon]x)^{-1} = (1-\varepsilon yx)^{-1}y \ge 0$. It follows, much as above, that $[\alpha+\varepsilon] \in \mathcal{S}(x)$ so $\alpha+\varepsilon \le \bar{\alpha}$ and $\alpha < \bar{\alpha}$. ■

We now note our principal result: that (2.4) holds in this context — compare this with the Krein-Bonsall-Karlin Theorem as cited in [3, Theorem 8.1] for operators on a partially ordered Banach space.

Theorem 5.7. *Let $\mathcal{A}$ be a $\mathcal{P}$-complete partially ordered algebra and let $x \ge 0$ be tame. Then $\rho(x) \in \sigma(x)$, i.e., $[\rho(x) - x]$ cannot be tamely invertible.*

PROOF: Suppose, to the contrary, that $[\rho(x)-x]$ were invertible so, with $\bar{\alpha} = 1/\rho(x)$, there would be a tame element y with $y(1-\bar{\alpha}x) = (1-\bar{\alpha}x)y = 1$. For any $0 < \alpha < 1/\rho(x) = \bar{\alpha}$, Theorem 5.2 gives existence of $y_\alpha = (1-\alpha x)^{-1} \ge 0$. Letting $\alpha \nearrow \bar{\alpha}$, we certainly have $(1-\alpha x) \to (1-\bar{\alpha}x)$ with $\{(1-\alpha x)\}$ $\mathcal{P}$-bounded. Hence, by the the continuity of the inversion map as asserted in [H2], we have $y_\alpha \to y_{\bar{\alpha}}$ so, as $\mathcal{P}$ is closed, we have $y_{\bar{\alpha}} \ge 0$ as well.

But with $y_{\bar{\alpha}}$ tame, Lemma 5.6 would give $\bar{\alpha} < 1/\rho(x)$ — a contradiction. ■

We note that this gives $\rho(x) = \rho_*(x)$ for $x \geq 0$.

For $\mathcal{M}_n = \mathcal{L}(\mathbb{R}^n)$ — and, somewhat more generally, for $\mathcal{L}(\mathcal{X})$ with $\mathcal{X}$ infinite dimensional, subject to a compactness condition — the usual theory supplements (2.4) by asserting existence of a positive eigenvector. Here that consideration is moot since our algebra's elements are not presented as operators, so there are no such things as eigenvectors in our context.

6 Some examples

We now introduce some further examples which will enable us to clarify the significance of our hypotheses and the sharpness of our results.

Example 6.1. *Let $\Omega \subset \mathbb{R}^k$ be an open set and then take $\mathcal{A} = \mathcal{C}(\Omega)$ to consist of all continuous functions on Ω with the topology of uniform convergence on compact subsets, pointwise operations, and the usual notion of positivity: $x \geq 0$ if $x(t) \geq 0$ in $\mathbb{R}$ for each $t \in \Omega$. Somewhat more generally, we may let these functions be matrix valued — say, $n \times n$ real matrices, ordered entrywise as in Example 3.2 above — so $\mathcal{A} = \mathcal{C}(\Omega \to \mathcal{M}_n)$.*

For Example 6.1 so $\mathcal{A} = \mathcal{C}(\Omega \to M_n)$, one verifies immediately that this is a partially ordered algebra, satisfying [H1] and [H2]. It is also easy to verify the $\mathcal{P}$-completeness of $\mathcal{C}(\Omega \to M_n)$.

To see this, fix any compact $\Omega_* \subset \Omega$ and note that $c_n \to 0$ just means that there is a scalar sequence $\alpha_n \to 0+$ with $|c_n(t)| \leq \alpha_n$ on Ω_*. The uniform convergence on Ω_* then implies that $\{x_n\}$ is a Cauchy sequence in the complete metric space $\mathcal{C}(\Omega_* \to M_n)$ with its usual uniform metric; hence x_n converges uniformly on Ω_* to some continuous limit function x. Since this holds for each such Ω_*, we have convergence $x_n \to x \in \mathcal{A}$ in the sense of Example 6.1.

We now show how to compute $\rho(x)$, characterizing $\mathcal{B}$, and $\sigma(x)$ in terms of the functions $\rho(x(t)) = \rho_*(x(t))$ and the spectra $\sigma(x(t))$ — computed in M_n for each $t \in \Omega$.

Lemma 6.2. *For $x \in \mathcal{A} = \mathcal{C}(\Omega \to M_n)$ one has*

$$\begin{aligned} \rho(x) &= \sup\{\rho(x(t)) : t \in \Omega\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(x(t)), t \in \Omega\} \end{aligned} \tag{6.1}$$

so x is tame in $\mathcal{A}$ when $\rho(x(\cdot))$ is bounded as a function on Ω, and

$$\sigma(x) = \overline{\bigcup_{t\in\Omega} \sigma(x(t))}. \tag{6.2}$$

It follows that for Example 6.1 one has $\sigma(x)$ compact and also that $\rho(x) = \rho_*(x) := \max\{|\lambda| : \lambda \in \sigma(x)\}$ for all tame x.

Proof: If $\alpha < 1/\rho(x)$, then $\alpha \in \mathcal{S}(x)$ so for $k = 1, \dots$ one has $\pm[\alpha x]^k \le c_\alpha \in \mathcal{A}$ and equivalently $\pm[\alpha x(t)]^k \le c_\alpha(t)$, giving $\alpha \le 1/\rho(x(t))$, for each $t \in \Omega$. Thus, $\rho(x) \ge \sup\{\rho(x(t)) : t \in \Omega\}$. Conversely, if one would have $\alpha < 1/\sup\{\rho(x(t)) : t \in \Omega\}$, then for each $t \in \Omega$ there would be some $c_\alpha(t) \in \mathcal{M}_n$ such that $[\alpha x(t)]^k \le c_\alpha(t)$ $(k = 1, \dots)$ and this would give $\alpha \in \mathcal{S}(x)$ if $c_\alpha : [t \mapsto c_\alpha(t)]$ would be continuous and so a (positive) element of $\mathcal{A}$. The observation made following Example 3.2 shows that we can, indeed, choose $c_\alpha(\cdot)$ to be continuous.

> Using the continuity of $x(\cdot)$, we have open neighborhoods $\mathcal{U}(t) \subset \Omega$ on which we can take $c = c_*(t)$ constant. There is then a locally finite subcover $\{\mathcal{U}(t_j)\}$ of Ω and a corresponding partition of unity $\{\varphi_j\}$ with each φ_j continuous and supported on $\mathcal{U}(t_j)$. Then $c_\alpha(t) := \sum_j \varphi_j(t) c_*(t_j)$ defines a suitable element of $C(\Omega \to \mathcal{M}_n)$ — continuous and dominating each $[\alpha x(t)]^k$.

Thus $\rho(x) = \sup\{\rho(x(t)) : t \in \Omega\}$. Since $\rho \equiv \rho_*$ on $\mathcal{M}_n$, we have (6.1).

Note that in Example 6.1 one has $[x^{-1}](t) = [x(t)]^{-1}$ for each $t \in \Omega$. Thus, if $\lambda \in \mathbb{R}$ is in $\sigma(x(t_*))$ for some $t_* \in \Omega$, we could not have $[\lambda - x(t)]^{-1}$ defined at $t = t_*$ whence $[\lambda - x]^{-1}$ could not exist: $\lambda \in \sigma(x)$. The same consideration also applies to $\left[|\lambda|^2 - (\lambda + \bar\lambda)x + x^2\right]^{-1}$ for complex λ, so $\sigma(x) \supset \bigcup_{t\in\Omega} \sigma(x((t))$. On the other hand, if $\lambda_* \notin \bigcup_{t\in\Omega} \sigma(x((t))$, then (for real λ_*) the $\mathcal{M}_n$-valued function $[\lambda_* - x(\cdot)]^{-1}$ is defined on all of Ω and is easily seen to be continuous, hence in our algebra. Noting that the Spectral Mapping Theorem for $\mathcal{M}_n$ gives

$$\sigma([\lambda_* - x(t)]^{-1}) = \left\{ \frac{1}{\lambda_* - \lambda} : \lambda \in \sigma(x(t)) \right\},$$

we see from (6.1) that $[\lambda_* - x]^{-1}$ is then a tame element of $C(\Omega \to \mathcal{M}_n)$ precisely if λ_* is actually bounded away from $\bigcup_{t\in\Omega} \sigma(x((t))$. Since the same consideration also applies to complex λ_*, we have (6.2). Comparing with (6.1), we see that $\rho(x)$ is finite (i.e., x is tame) if and only if $\bigcup_{t\in\Omega} \sigma(x((t))$ is bounded so $\sigma(x)$ is compact and the final assertion of the lemma is then immediate. ■

Remark 6.3. We now observe some possibilities arising in Example 6.1, noting where these complement some of our results.

1. It is fairly standard that the topology of $\mathcal{C}(\Omega \to \mathcal{M}_n)$ is separable and metrizable, but cannot be given by a norm if Ω is not compact.

2. From Lemma 6.2 we see that one will always have both tame and non-tame elements in the algebra $\mathcal{A} = \mathcal{C}(\Omega \to \mathcal{M}_n)$. Indeed (see *5.* below) a tame element x may be algebraically invertible with x^{-1} existing in $\mathcal{A}$ but not itself tame.

3. It is not difficult to see that $\mathcal{P}$ has empty interior for this example.

   Ω is not compact, but we can find an increasing sequence $\{\Omega_n\}$ of compact sets with $\bigcup_n \Omega_n = \Omega$. Then, given any $x \in \mathcal{P}$, we can find $x_n \in \mathcal{A}$ coinciding with x on Ω_n but taking negative values somewhere outside that. Then $x_n \notin \mathcal{P}$ while, for any fixed compact set, $x_n \equiv x$ for large enough n so $x_n \to x$.

4. We see that the product or sum of positive tame elements $u, v \in \mathcal{B}_+$ need not be tame.

   To see this, consider $\mathcal{A} = \mathcal{C}(\mathbb{R} \to \mathcal{M}_2)$ and fix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2$. We now take $u(t) = f(t)M$ and $v(t) = f(t)M^*$ for a continuous positive scalar function f so $u, v \geq 0$ in $\mathcal{A}$. Clearly u, v are tame (with $\rho(u) = \rho(v) = 0$) since $[u^k](t) = [f(t)]^k M^k = 0$ for $k > 1$ and similarly for v. On the other hand $[uv](t) = f^2(t)P$ where P is the idempotent $P = MM^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ so $[\alpha uv]^k(t) = \alpha^k f^{2k}(t)P$. If we take f to be unbounded, then there can be no nonzero choice of α, independent of t, for which, $\alpha^k f^{2k}(t)P$ is bounded for each t. Thus uv is not tame. Note that $uv \neq vu$ since M, M^* do not commute. Essentially the same calculation shows that $u + v$ also cannot be tame.

   This shows that the hypothesis in Lemma 4.3 that one of the products is tame cannot be omitted and that the commutativity hypotheses in Lemmas 4.4 and 4.5 are necessary.

5. We emphasize the significance of including the requirement of tameness of the inverse in the definition of $\sigma(x)$: without that Theorem 5.7 would be incorrect.

   To see this, consider $x(t) = t^2/(1+t^2)$ in $C(\mathbb{R})$, giving $\rho(x) = 1$. We note that $(1-x)$ is here not accepted as invertible — while the algebraic inverse $(1-x)^{-1} = 1+t^2$ exists in $\mathcal{A}$, this is unbounded as a scalar function on $\mathbb{R}$ and so, by Lemma 6.2, is not tame.

   Changing the topology of Example 6.1 to the order topology (making bounded monotone sequences converge, say, by requiring existence of sup and inf), shows that this need not provide $\mathcal{P}$-completeness.

   Let $x_n(s) = \max\{0, \min\{ns, 1\}\}$ in $\mathcal{A} = \mathcal{C}(\mathbb{R})$ so $0 \le x_n \le 1$; each x_n is continuous so $x_n \in \mathcal{B}_+ \subset \mathcal{A}$. We easily verify that $x_n(s) \le x_{n+1}(s)$ for each s so we have a bounded monotone sequence in $\mathcal{A}$. On the other hand, the pointwise limit is 0 for $s \le 0$ and 1 for $0 < s$ — which does not correspond to any element of $\mathcal{A}$ — so the sequence cannot be convergent in $\mathcal{A}$.

   Complementing Lemma 4.9, this example shows that the spectral bound need not be continuous.

   To see this, choose $x_* \in \mathcal{C}(\mathbb{R})$ such that $x_*(t) = 0$ for $|t| \le 1$ and $x_*(t) = 1$ for $|t| \ge 2$; set $x_k(t) = x_*(t/k)$. Clearly $x_k \le 1$ and $x_k \to 0$ in the sense of Example 6.1, but $\rho(x_k) = 1$ for every k. Thus

$$\lim_k \rho(x_k) = 1 \ne 0 = \rho(0) = \rho(\lim_k x_k).$$

6. If we were to take Ω compact, then this would become a Banach algebra with the monotone norm $\|x\| = \max\{|x(t)| : t \in \Omega\}$.

We next consider a generalization of an example presented in [3], due to Stetsenko.

Example 6.4. *Let $\Omega \subset \mathbb{C}$ be a connected open set containing a nontrivial closed real interval $\mathcal{J}$. We then take $\mathcal{A} = A(\Omega, \mathcal{J})$ to be the collection of all complex-valued functions which are analytic on Ω and are real on $\mathcal{J}$, using pointwise operations. Take $\mathcal{P}$, in this case, to be the subset of functions in $\mathcal{A}$ which are nonnegative on $\mathcal{J}$. Finally, we take convergence in $\mathcal{A}$ to be uniform convergence on each compact subset of Ω.*

For Example 6.4 we first note that we can always take Ω symmetric across the reals with no loss of generality since $x(\bar{\zeta}) = \overline{x(\zeta)}$ for $\zeta \in \Omega,\ x \in \mathcal{A}$. Next, if $\pm x \in \mathcal{P}$, then $x \equiv 0$ on $\mathcal{J}$ so, by analyticity, $x \equiv 0$ on Ω — i.e., $\mathcal{P}$ is pointed. Further, for any $x \in \mathcal{A}$ we have $y = (1 + x^2)/2 \in \mathcal{P}$ and $\pm x \leq y$, from which it follows that $\mathcal{P} + (-\mathcal{P}) = \mathcal{A}$.

We now compute $\rho(x) = \max_{\mathcal{J}} |x(\cdot)|$ for Example 6.4.

Lemma 6.5. *Every $x \in \mathcal{A} = \mathcal{A}(\Omega, \mathcal{J})$ is tame and we have*

$$\rho(x) = \max\{|x(t)| : t \in \mathcal{J}\}, \qquad \sigma(x) = \{x(\zeta) : \zeta \in \Omega\}. \tag{6.3}$$

PROOF: Note that $x(\cdot)$ is continuous on the compact set $\mathcal{J} \subset \Omega$ so $c = \max\{|x(s)| : s \in \mathcal{J}\} = |x(s_*)|$ exists. For any $0 < \alpha < 1/c$ we have $\pm\alpha x(s) \leq 1$ for $s \in \mathcal{J}$ so $\pm[\alpha x]^k \leq 1(\cdot)$ in the partially ordered algebra $\mathcal{A}$ where $1(\cdot) \in \mathcal{P} \subset \mathcal{A}$ is the constant function 1, showing that $\rho(x) \leq c$. On the other hand, if u is any function in $\mathcal{A}$ and $\alpha > 1/c$, we have $\left|[\alpha x]^k(s_*)\right| = [\alpha c]^k \to \infty$ as $k \to \infty$ so, eventually, $\left|[\alpha x]^k(s_*)\right| > u(s_*)$ and it is impossible that $\pm[\alpha x]^k \leq u$ in $\mathcal{A}$. This also shows that each $x \in \mathcal{A}$ is tame: we have here $\mathcal{B} = \mathcal{A}$.

The identity of $\mathcal{A}$ is the constant function 1 and we have pointwise operations so, if y is invertible, $y^{-1}(\zeta) = 1/y(\zeta)$. If $y(\zeta_*) = 0$ for some $\zeta_* \in \Omega$ and some $y \in \mathcal{A}$, then $1/y(\cdot)$ cannot be analytic at ζ_*. Thus, $[\lambda - x]$ cannot be invertible in $\mathcal{A}$ if $x(\zeta_*) = \lambda$ for any $\zeta_* \in \Omega$. Similarly, for non-real λ, we note that if $x(\zeta_*) = \lambda$, then — setting $\alpha = 1/\lambda$ and

$$y = (\lambda - x)(\bar{\lambda} - x)/\lambda^2 = \left[1 - (\alpha + \bar{\alpha})x + |\alpha|^2 x^2\right] \in \mathcal{A}$$

— we have $y(\zeta_*) = 0$ so y is not invertible and $\lambda \in \sigma(x)$; of course we also have $\bar{\lambda} \in \sigma(x)$. ■

Remark 6.6. We now observe some possibilities arising in Example 6.4.

1. This example shows that a partially ordered algebra, satisfying [H1] and [H2], need not be $\mathcal{P}$-complete.

   To see this, let Ω be the unit disk $\{\zeta \in \mathbb{C} : |\zeta| < 2\}$ and $\mathcal{J} = [-1, 1] \subset \Omega$. If we consider the sequences $x_n(\zeta) = (1/n)\cos(n\zeta)$ and $c_n \equiv 1/n$ in $\mathcal{A} = \mathcal{A}(\Omega, \mathcal{J})$, then we can verify that $c_n \searrow 0$ uniformly on Ω (so

$c_n \to 0$ in $\mathcal{A}$) and that $\pm x_n \le c_n$ in $\mathcal{A}$ (i.e., pointwise on $\mathcal{J}$). On the other hand, the sequence of values $\{x_n(\zeta) : n = 1, \ldots\}$ is unbounded for almost all $\zeta \notin \mathbb{R}$ so it is not possible (even for a subsequence) to have convergence of (x_n) in $\mathcal{A}$ to any limit.

This also shows the importance of the $\mathcal{P}$-completeness assumption imposed for Lemma 4.8.

With x_n, c_n as above, we have $-c_n \le x_n \le c_n$ with $0 = \lim_n(-c_n) = \lim_n c_n$, but do not have $x_n \to 0$.

2. In Example 6.4 we will *never* have $\rho(x) = \rho_*(x)$ — indeed, $\rho(x)$ will not even be a bound on the spectrum $\sigma(x)$ — unless the function $x(\cdot)$ is a constant.

   This follows from Lemma 6.5: by the maximum principle for analytic functions, if $x(\cdot)$ is not a constant, then there must exist some $\zeta_* \in \Omega$ giving $\lambda = x(\zeta_*) \in \sigma(x)$ such that $|\lambda| > \max\{|x(s)| : s \in \mathcal{J}\} = \rho(x)$.

   This observation indicates the significance of the $\mathcal{P}$-completeness condition in Theorem 5.7.

3. We now note that Example 6.4 shows that our Theorem 5.7 does subsume [3, Theorem 4.1]. Theorem 3.1 and Example 3.5 show that Stetsenko's example cited in [3] (where the $\mathcal{X}$-norm is not semimonotone) would not provide a counterexample to Theorem 5.7, but does indicate the necessity for that theorem of our hypothesis of $\mathcal{P}$-completeness.

Example 6.7. *Let $\mathcal{A}_0 = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, topologized as a subset of $L^2(\mathbb{R})$. There is no unit in $\mathcal{A}_0$ so we adjoin one, letting $\mathcal{A} = \mathbb{R} \oplus \mathcal{A}_0$, viewed as the set of functions on $\mathbb{R}$ of the form $x(t) = a + x_0(t)$ with $x_0 \in \mathcal{A}_0$. The operations and positivity are taken in the usual sense of 'pointwise a.e.'*

Since these are bounded functions, products are again in $\mathcal{A}$ so we have [H1]. It is also clear that the positive cone is closed; on the other hand, it is easy to find counterexamples to the general continuity of multiplication. Nevertheless, we easily verify this continuity when the factors are constrained to order intervals and so satisfy uniform L^∞ bounds; thus we also have [H2].

Remark 6.8.

1. The algebra of Example 6.7 is $\mathcal{P}$-complete.

   Suppose $\{x_k\}$ is a sequence of functions in $\mathcal{A}$ and $\{c_N\}$ is a sequence of nonnegative functions in $\mathcal{A}$ such that

   $$\begin{array}{cc} 0 \le c_N(s) \le c_{N-1}(s) & \text{for a.e. } s \in \mathbb{R} \text{ and } N = 2, \ldots \\ \|c_N\| \to 0 \text{ as } N \to \infty & (L^2\text{-norm}) \\ |x_j(s) - x_k(s)| \le c_N(s) & \text{for a.e. } s \in \mathbb{R} \text{ and } j,k \le N = 1,2,\ldots \end{array}$$

   It follows that $\|x_j - x_k\| \le \|c_N\| \to 0$ so $\{x_k\}$ is a Cauchy sequence and so convergent in $L^2(\mathbb{R})$ to some L^2 function $\bar{x}(\cdot)$. Since (for each k and a.e. $s \in \mathbb{R}$) one has $|x_k(s)| \le |x_1(s)| + c_1(s) \le$ *bound*, it follows in the limit that $\bar{x}(\cdot)$ is bounded so the L^2-convergence $x_k \to \bar{x}$ is actually convergence in $\mathcal{A}$ and we have $\mathcal{P}$-completeness as desired.

   On the other hand, $\mathcal{A}_0$ is not complete in the usual sense with respect to its own metric topology, so it is not a Banach algebra. Further, one easily sees that there can be no renorming which makes this a Banach algebra, since one can find convergent sequences for which the product sequences do not converge.

2. This example also clarifies the distinction between norm boundedness and $\mathcal{P}$-boundedness.

   If we set $x_\lambda(s) = \{1 \text{ if } |s - \lambda| \le 1;\ 0 \text{ else }\}$, then $\|x_\lambda\| = \sqrt{2}$ so $A = \{x_\lambda : \lambda \in \mathbb{R}\}$ is norm bounded — but to have every $x_\lambda \in [-u, u]$ would require $u(s) \ge 1$ for all s: impossible for $u \in \mathcal{A} \subset L^2(\mathbb{R})$.

Example 6.9. *Let $\mathcal{A}$ be $\ell^1(\mathcal{A}_0)$, i.e., the set of sequences $\mathbf{x} = (x_0, x_1 \ldots)$ with each entry x_j taken from the specified Banach algebra $\mathcal{A}_0$ and with $\|\mathbf{x}\| = \sum_j \|x_j\| < \infty$. If we take convolution as multiplication so $\mathbf{xy}$ is given by $(xy)_j = \sum_{k=0}^{j} x_k y_{j-k}$, then $\mathcal{A}$ itself becomes a Banach algebra. We then take the usual componentwise partial order, assuming $\mathcal{A}_0$ is already partially ordered by a positive cone $\mathcal{P}_0$, and $\mathcal{A}$ is then a partially ordered Banach algebra, satisfying* [H1], [H2]. *One easily sees that if the norm of $\mathcal{A}_0$ is semimonotone with respect to $\mathcal{P}_0$ (e.g., for $\mathcal{A}_0 = \mathbb{R}$), then the $\mathcal{A}$-norm is also semimonotone so $\mathcal{A}$ is $\mathcal{P}$-complete by Theorem 3.1.*

Example 6.10. *Consider the algebra $\mathcal{F} = \mathcal{F}_R$ of all functions f given by a power series $f(\zeta) = \sum_k c_k \zeta^k$ with real coefficients and a radius of convergence $R_f > R$ and say f is nonnegative if each coefficient c_k is nonnegative. We* *define convergence $f_k \to f$ in $\mathcal{F}_R$ as coefficient-wise convergence (subject to the requirement that there is some $r > R$ and some $m > 0$ for which one has, uniformly, $|c_{k,N}| \leq mr^{-k}$).*

Although we have presented the topology somewhat differently here, this example is quite closely related to the function algebra used in Theorem 5.1 and essentially the same spectral calculus $F_x : f \mapsto f(x) = \sum_n c_n x^n$ applies here. We now note that the algebra homomorphism F_x is order preserving if $x \geq 0$. We emphasize, on the other hand, that the order relation here is quite different from that of Example 6.4 and note that *this* partially ordered algebra is clearly $\mathcal{P}$-complete.

7 Comments and remarks

We note, borrowing somewhat from [6], that a well-known matrix result by Varga [7, Theorem 3.32] also holds in this setting. For this we define a *regular splitting* of an element $z \in \mathcal{A}$ as a pair $[x, y]$ such that:

> $z = x - y$ with x invertible, both x^{-1} and y are nonnegative, and both x and $p = yx^{-1}$ are tame.

We refer to p as the *iteration element* for the splitting.

Theorem 7.1. *Let $\mathcal{A}$ be a $\mathcal{P}$-complete partially ordered algebra. Then, subject to a bound, the mapping: $x \mapsto \rho(p)$ is monotone increasing for regular splittings of a fixed element z. More precisely, given regular splittings $[x, y]$, $[u, v]$ of the same z ($y - x = z = v - u$), one has*

$$u \leq x \quad \textit{implies} \quad \rho(vu^{-1}) \leq \rho(yx^{-1}) \tag{7.1}$$

provided $\rho(yx^{-1}) \leq 1$ or, alternatively, if z is invertible with $xz^{-1} \geq 0$.

PROOF: Set $p := yx^{-1}$ and $q = vu^{-1}$.

Our first observation is that the alternative hypothesis $z^{-1} \geq 0$ already implies $\rho(p) < 1$ since $(1 - p)^{-1} = xz^{-1}$ so positivity gives $1 \in \mathcal{S}(p)$ by Lemma 5.6, with strict inequality as in the proof of Theorem 5.7 above.

We can now choose α arbitrarily close to $\bar{\alpha} = \sup \mathcal{S}(p) = 1/\rho(p) > 1$ such that $1 < \alpha < \bar{\alpha}$, giving $\alpha \in \mathcal{S}(p)$ so we have $\{[\alpha p]^k\}$ bounded uniformly in k. If we can show that this choice of α also gives $\mathcal{P}$-boundedness of the set $\left\{[\alpha q]^k\right\}$ so $\alpha \in \mathcal{S}(q)$, then we will have $\sup\{\mathcal{S}(q)\} \geq \bar{\alpha}$, giving (7.1).

To this end it is now convenient to introduce

$$\begin{aligned} a &:= 1 - ux^{-1} = (x-u)x^{-1} \geq 0, \qquad & b := p - a = vx^{-1} \geq 0 \\ z_\alpha &:= (1-\alpha p)^{-1} \geq 0 \qquad & q_\alpha = b(1-\alpha a)^{-1} \geq 0 \end{aligned} \tag{7.2}$$

where we note first that $0 \leq a \leq p$ so z_α is well-defined and nonnegative for $\alpha < \bar{\alpha}$ by Theorem 5.2 and, similarly, $(1-\alpha a)^{-1} \geq 0$ in view of Lemma 4.4-*2*. Note also that $(1-a)^{-1} = xu^{-1}$ so $q_1 = q$ and comparison of the (convergent) Neumann series shows that $(1-\alpha a)^{-1}$ is monotone increasing in α here so $q_1 \leq q_\alpha$ for $\alpha \geq 1$.

One can immediately compute the identities

$$z_\alpha = 1 + \alpha a z_\alpha + \alpha b z_\alpha \qquad b z_\alpha = q_\alpha\left(1 + \alpha b z_\alpha\right)$$

Multiply $1 = [1 - \alpha p + \alpha a + \alpha b]$ by z_α and, after noting that $b = q_\alpha(1-\alpha a)$, multiply $1 - \alpha a = [1 - \alpha p + \alpha b]$ on the left by q_α and on the right by z_α.

The first of these identities is the case $N = 0$ of the induction

$$\begin{aligned} z_\alpha &= \sum_0^N (\alpha q_\alpha)^k + \alpha a z_\alpha + \alpha\left(\alpha q_\alpha\right)^N b z_\alpha \\ &= \sum_0^N (\alpha q_\alpha)^k + \alpha a z_\alpha + \alpha\left(\alpha q_\alpha\right)^N q_\alpha(1 + \alpha b z_\alpha) \\ &= \sum_0^{N+1} (\alpha q_\alpha)^k + \alpha a z_\alpha + \alpha\left(\alpha q_\alpha\right)^{N+1} b z_\alpha. \end{aligned} \tag{7.3}$$

Since each term in (7.3) is nonnegative, this shows that $(\alpha q_\alpha)^k \leq z_\alpha$ for each k so $\alpha \in \mathcal{S}(q_\alpha)$ and we have shown that $\sup \mathcal{S}(q_\alpha) \geq \bar{\alpha}$ so $\rho(q_\alpha) \leq \rho(p)$. Since we took $\alpha \geq 1$, giving $0 \leq q \leq q_\alpha$, the desired result (7.1) follows by Lemma 4.4-*2*. ■

Finally, we note here some open questions:

[Q1] If x is any tame element of a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$, is it necessarily strongly tame? We are asking whether u_α in (2.6) can always be chosen in the maximal commutative subalgebra generated by x so restriction to that subalgebra would preserve tameness. [We do know this for $x \in \mathcal{P}$ by Lemma 5.6-*1*..]

[Q2] It is not really clear how much of [H2] is needed. Certainly our approach needs some notion of convergence to work with the Neumann series at all and will need that [if $u_n \to \bar{u}$ with $u_n \in \mathcal{P}$, then $\bar{u} \in \mathcal{P}$], but it is not clear that any more is needed (or that we need a full topology, other than consideration of some special sequences). Thus we ask whether it might be possible to find a purely algebraic conditions (or quasi-topological conditions as in [1]) which could still give the results of Section 5.

[Q3] If we have a weakened $\mathcal{P}$-completeness condition

$$\varepsilon_n \searrow 0, \quad \pm[x_m - x_n] \leq \varepsilon_n u (m > n) \Rightarrow \exists \bar{x} \ni x_n \to \bar{x}, \tag{7.4}$$

does this already imply the stronger condition

$$u_n \searrow 0, \quad \pm[x_m - x_n] \leq u_n (m > n) \Rightarrow \exists \bar{x} \ni x_n \to \bar{x} \tag{7.5}$$

which we have been using? — perhaps if it is also known that each x_n is tame and/or that the $\{x_n\}$ commute?

[Q4] In Example 3.5, if X satisfies (3.3), does it follow that for each $x \in \mathcal{A} = \mathcal{L}(X)$ there is some $u \in \mathcal{P}$ with $\pm x \leq u$? [This would imply that $\mathcal{P} + [-\mathcal{P}] = \mathcal{A}$.] Must we have (3.3) for $\mathcal{A}$? It would be even nicer if we could get this with $ux = xu$ and $\|u\| \leq C\|x\|$.

[Q5] We note that any algebra $\mathcal{A}$ with identity (more generally: such that $xy = 0$ for all y implies $x = 0$) is always (equivalent to) an algebra of linear operators on a vector space, since $\mathcal{A}$ is, of course, itself a vector space and we may identify each element x with the operator $T_x : y \mapsto xy$ so $\mathcal{A}$ can be identified with a subalgebra of the algebra of all bounded operators on $\mathcal{A}$ Note that if we use the partial order of $\mathcal{A}$ for the vector space, then there is an induced partial order for the operator algebra — and this coincides with the original partial order of $\mathcal{A}$. Can this be used — with some equivalent of compactness — to get some version of the 'positive eigenvector'component of the usual Perron-Frobenius Theory?

[Q6] If x is a tame element of a $\mathcal{P}$-complete partially ordered algebra $\mathcal{A}$, do we necessarily have $\rho(x) = \rho_*(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$? [We have the inequality $\rho \geq \rho_*$ in general by Theorem 5.2-*2.* and have equality for $x \in \mathcal{P}$ by Theorem 5.7.] It seems plausible to have equality in general and we conjecture that an argument for this might be based on a spectral calculus using formally the Cauchy Integral Formula

$$f(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z)\,(z-x)^{-1}\,dz$$

(reformulated to stay in $\mathbb{R}$ and necessitating the development of a corresponding version of the Cauchy Integral Theorem). We have not attempted this here.

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