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Nonconvergence Results for the Application of Least-Squares Estimation to Ill-Posed Problems[1]

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Abstract. One standard approach to solving $f(x)=b$ is the minimization of $\|f(x)-b\|^2$ over x in $\tilde{\mathfrak{X}}$, where $\tilde{\mathfrak{X}}$ corresponds to a parametric representation providing sufficiently good approximation to the true solution x^*. Call the minimizer $x = \mathscr{A}(\tilde{\mathfrak{X}})$. Take $\tilde{\mathfrak{X}} = \mathfrak{X}_N$ for a sequence $\{\mathfrak{X}_N\}$ of subspaces becoming dense, and so determine an approximating sequence $\{x_N := \mathscr{A}(\mathfrak{X}_N)\}$. It is shown, with f linear and one-to-one, that one need not have $x_N \to x^*$ if f^{-1} is not continuous.

Key Words. Ill-posed problems, least-squares estimation, approximation, parametric representation, convergence, nonconvergence.

1. Introduction

Many problems of practical importance involve the solution of equations

$$f(x)=b, \tag{1}$$

in which the unknown is an element of some infinite-dimensional space $\mathfrak{X}$ (e.g., a function or set of functions). For example, one might seek to determine a coefficient function in a partial differential equation from observational data taken from a solution; here, b denotes the observation and x would denote the unknown coefficient function, together with such other data as must be adjoined to what is known already to make a

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well-defined map: $x \mapsto b$. For more, related applications, see e.g. Refs. 1, 3, 6.

If b is to be given (in practice, only approximately) as an element of a certain space $\mathfrak{Y}$, then it may happen that f has no continuous inverse. In that case, we say that the problem (1) is *ill posed.* We assume that only the continuity of f^{-1} is at issue, i.e., that f is one-to-one and that the given b is in the range of f.

It is clear that one may attempt to solve (1) by minimizing the square of the residual error, replacing (1) by the *optimization problem*:

$$\text{minimize } \mathcal{J}(x) := \tfrac{1}{2}\|f(x) - b\|_{\mathfrak{Y}}^2 \qquad \text{over } x \text{ in } \mathfrak{X}. \tag{2}$$

Since any feasible computation must be finitary, a standard approach is to assume a parametric representation for x, treat $\mathcal{J}$ as a function of the finite number of parameters, and use (2) to estimate the parameter values and so determine x. Typical representations might be spline approximations or truncated power series or Fourier series expansions.

For simplicity of analysis, we shall assume that the problem is linear,

$$f(x) := Ax,$$

and that the spaces $\mathfrak{X}$, $\mathfrak{Y}$ are Hilbert spaces. The approach described above now consists of determining x by solving the finite-dimensional quadratic optimization problem:

$$\text{minimize } \mathcal{J}(x) := \tfrac{1}{2}\|Ax - \tilde{b}\|_{\mathfrak{Y}}^2 \qquad \text{over } x \text{ in } \tilde{\mathfrak{X}}, \tag{3}$$

where $\tilde{\mathfrak{X}}$ is a finite-dimensional subspace of $\mathfrak{X}$ and $\tilde{b}$ is the actual observation,

$$\tilde{b} \approx b^* := Ax^*.$$

Here, x^* denotes the *true* solution, and b^* the corresponding *true* observational data. Note that, even if b^* were somehow available exactly, computational imprecision and the exigencies of finitary representation would introduce such potential perturbation. The approximant determined by (3) is denoted by $\mathcal{A}(\tilde{\mathfrak{X}})$.

As with most computational schemes, one applies (3) but analyzes the procedure asymptotically, embedding $\tilde{\mathfrak{X}}$ as one of an increasing family of subspaces $\{\mathfrak{X}_N\}$ and $\tilde{b}$ as one of an approximating sequence $\{b_N\}$. Such a scheme would be termed *convergent* if $b_N \to b^*$ implies that $x_N \to x^*$, where each $x_N = \mathcal{A}(\mathfrak{X}_N)$ is the solution of the following problem:

$$\text{minimize } \mathcal{J}_N(x) := \tfrac{1}{2}\|Ax - b_N\|_{\mathfrak{Y}}^2 \qquad \text{over } x \text{ in } \mathfrak{X}_N. \tag{4}$$

If A is one-to-one, then (4) always has a unique solution, and it is easy to see that this solution is just A^{-1} acting on the orthogonal projection of b_N on the finite-dimensional subspace

$$\mathfrak{Y}_N := A\mathfrak{X}_N \subset \mathfrak{Y}.$$

If A has a continuous inverse, it is clear that the scheme (4) is convergent, since the sequence of projections onto $\{\mathfrak{Y}_N\}$ converges strongly to the identity. On the other hand, our aim in this paper is to show that this is *always false* if the problem is ill posed, i.e., if A^{-1} is unbounded. There are two principal results in that case.

(i) For any $\{\mathfrak{X}_N\}$ and any $b^* = Ax^*$, there exist sequences $b_N \to b^*$ such that $\|x_N\| \to \infty$ and also such that $\{x_N = \mathscr{A}(\mathfrak{X}_N)\}$ is bounded, but $x_N \not\to x^*$. If $\{x_N\}$ is bounded, one always has *weak* convergence: $x_N \to x^*$.

(ii) Even if b^* would be given *exactly* ($b_N = b^* = Ax^*$ for each N), for almost any such b^* there exist $\{\mathfrak{X}_N\}$ for which $\{x_N = \mathscr{A}(\mathfrak{X}_N)\}$ is unbounded.

The first of these results is unsurprising; after all, the instability of the solution under perturbation of the data is characteristic of ill-posed problems. Only its simplicity in use and its usefulness in the well-posed case can account for the otherwise inexplicable persistence of this approach in engineering practice, despite its obvious shortcomings (even after reduction to the parametrized formulation; see the discussion and references in Ref. 1). It is the second result which is somewhat astonishing; even with exact data (and arbitrarily rapid convergence of the eigenfunction expansion of the right-hand side), one cannot be confident of convergence or even of boundedness of the sequence of computed *approximants.*

These results indicate the necessity for extreme caution in dealing with ill-posed problems. For a discussion of some *convergent* computational approaches to ill-posed problems, see Ref. 6. Note that, for particular classes of applications, such projection–estimation schemes, with $\{\mathfrak{X}_N\}$ of specified form, can be justified (see Refs. 4, 5). It is the necessity of doing this which is implied by the present results.

2. Nonconvergence with Perturbation

As above, let $\mathfrak{X}$ and $\mathfrak{Y}$ be infinite-dimensional Hilbert spaces; and let $A_0: \mathfrak{X} \to \mathfrak{Y}$ be compact, injective, and with dense range (this last hypothesis is not significant; otherwise, replace $\mathfrak{Y}$ by the closure of the range). Let the eigenvalues of $A_0^*A_0: \mathfrak{X} \to \mathfrak{X}$, taken with multiplicities, be $\{\alpha_1^2, \alpha_2^2, \ldots\}$; and let $\{a_1, a_2, \ldots\}$ be the corresponding eigenvectors. The eigenvalues are positive (we also take $\alpha > 0$) and converge to zero; assume that they are ordered decreasingly so $\alpha_1 \geq \alpha_2 \ldots$. The eigenvectors may be taken to form an orthonormal basis of $\mathfrak{X}$.

Now, define $U: \mathfrak{X} \to \mathfrak{Y}$ such that

$$Ua_n = \alpha_n^{-1} A_0 a_n.$$

It is easily verified that $\{Ua_n\}$ is orthonormal, so U is unitary and, as A_0 has dense range, is surjective. Let

$$A := U^* A_0 : \mathfrak{X} \to \mathfrak{X},$$

and observe that A is self-adjoint, with eigenvalues $\{\alpha_n\}$ and eigenvectors $\{a_n\}$, compact, injective, and has dense range. Note that, for any $x \in \mathfrak{X}$, $y_0 \in \mathfrak{Y}$, one has

$$\|A_0 x - y_0\|_{\mathfrak{Y}} = \|Ax - y\|_{\mathfrak{X}},$$

where

$$y := U^* y_0.$$

We have thus shown that, with no loss of generality, it is always possible to reduce considerations to *positive self-adjoint operators*, such as $A: \mathfrak{X} \to \mathfrak{X}$. Henceforth, we assume that A_0 is already in the form of A, and we omit the subscript on the norm as only one space is involved.

Consider an arbitrary increasing sequence of subspaces $\{\mathfrak{X}_N\}$, with $\bigcup_N \mathfrak{X}_N$ dense in $\mathfrak{X}$. Assume, for simplicity, that

$$\dim \mathfrak{X}_N = N,$$

so there is an orthonormal basis $\{e_1, e_2, \ldots\}$ of $\mathfrak{X}$ such that

$$\mathfrak{X}_N = \mathrm{sp}\{e_1, \ldots, e_N\}.$$

Let

$$\mathfrak{Y}_N := A\mathfrak{X}_N \subset \mathfrak{X},$$

and let

$$A_N : \mathfrak{X}_N \to \mathfrak{Y}_N$$

be the restriction of A. Clearly, A_N is invertible, and we set

$$\nu_N := \|A_N^{-1}\|.$$

It can be shown (see e.g. Lemma 4.6 of Ref. 2) that

$$\nu_N \geq 1/\alpha_N,$$

so $\nu_N \to \infty$.

Theorem 2.1. Let $A, \{\mathfrak{X}_N\}$, etc., be as above. Let $x^* \in \mathfrak{X}$ and $b^* := Ax^*$. Then, there exists a sequence $b_N \to b^*$ for which the corresponding sequence $\{x_N = \mathscr{A}(\mathfrak{X}_N)\}$, given by (4), has $\|x_N - x^*\| \to \infty$.

Note that, although we have written just $\mathscr{A}(\mathfrak{X}_N)$, x_N depends on b_N as well as on the choice of $\mathfrak{X}_N$.

Proof. There exists v_N in $\mathfrak{Y}_N$, with

$$\|v_N\| = \nu_N^{-1/2},$$

such that

$$v_N = Au_N,$$

i.e.,

$$u_N := A_N^{-1} v_N \in \mathfrak{X}_N,$$

with

$$\|u_N\| = \nu_N^{1/2}.$$

Let $\tilde{b}_N$ be the orthogonal projection of b^* on $\mathfrak{Y}_N$ and

$$\tilde{x}_N := A_N^{-1} \tilde{b}_N;$$

$\tilde{x}_N$ is what one would obtain from (4) using $b_N = b^*$. One of $\|\tilde{x}_N \pm u_N\|$ is greater than $\|u_N\|$; and, with that choice of sign, take

$$b_N := b^* \pm v_N.$$

Then,

$$x_N = A_N^{-1}(\tilde{b}_N \pm v_N) = \tilde{x}_N \pm u_N,$$

so

$$\|x_N\| \geq \|u_N\| \to \infty,$$

whence

$$\|x_N - x^*\| \to \infty,$$

while

$$\|b_N - b^*\| \leq \|b_N - \tilde{b}_N\| + \|\tilde{b}_N - b^*\| \to 0,$$

since

$$\|\tilde{b}_N - b_N\| = \|v_N\| \to 0,$$

and the projections on $\mathfrak{Y}_N$ go strongly to the identity as A has dense range and $\bigcup_N \mathfrak{X}_N$ is dense. □

To construct $b_N \to b^*$, with $\{x_N\}$ bounded but not converging to x^*, let $\hat{x}_N$ be the orthogonal projection of x^* on $\mathfrak{X}_N$, and then set

$$b_N := A\hat{x}_N + \hat{v}_N,$$

where

$$\hat{v}_N = A\hat{u}_N \in \mathfrak{Y}_N$$

has

$$\|\hat{u}_N\| = 1, \qquad \|\hat{v}_N\| = 1/\nu_N,$$

i.e.,

$$\hat{v}_N = \nu_N^{-1/2} v_N.$$

Since $\hat{x}_N \in \mathfrak{X}_N$, we have

$$x_N = \hat{x}_N + \hat{u}_N.$$

Since $\hat{x}_N \to x^*$, we have

$$\|x_N - x^*\| \to 1 = \|u_N\|,$$

so $x_n \not\to x^*$. Also,

$$A\hat{x}_N \to Ax^* = b^* \quad \text{and} \quad \hat{v}_N \to 0,$$

so $b_N \to b^*$. □

For any b_N one has

$$Ax_N = \tilde{b}_N,$$

the projection of b_N on $\mathfrak{Y}_N$, so $b_N \to b^*$ implies

$$\tilde{b}_N = Ax_N \to b^*.$$

If any subsequence of $\{x_N\}$ were bounded, then any further subsequence would contain a weakly convergent subsequence, $x_{N(k)} \rightharpoonup x_*$. Since $Ax_{N(k)} \to b^*$ and the graph of A is closed in the weak topology of $\mathfrak{X} \times \mathfrak{X}$, one has

$$Ax_* = b^*,$$

so $x_* = x^*$. Thus, the original subsequence converges to x^* weakly, although as shown, not necessarily strongly. □

3. Nonconvergence with Exact Data

We first show by an example that, even using the unperturbed data b^*, one may encounter the behavior noted above. Let $\{\mathfrak{X}_N = \text{sp}\{e_1, \ldots, e_N\}\}$ be any increasing sequence of subspaces as above, $\{e_n\}$ an orthonormal basis of the Hilbert space $\mathfrak{X}$. We wish to construct

$$A_0 : \mathfrak{X} \to \mathfrak{X} \quad \text{and} \quad b^* = A_0 x^*$$

(recalling the beginning of Section 2, we write A_0, since it is convenient here *not* to assume a reduction to the self-adjoint case) for which $\{x_N = \mathscr{A}(\mathfrak{X}_N)\}$, as computed by (4) using the exact right-hand side $b_N = b^*$ for each N, has $\|x_N - x^*\| \to \infty$.

Example 3.1. We take A_0 to be given in the form

$$A_0 : \sum_{n=1}^{\infty} \xi_n e_n \mapsto \sum_{n=1}^{\infty} (\alpha_n \xi_n + \beta_n \xi_1) e_n, \tag{5}$$

with

$$\alpha_1 = 1, \qquad \beta_1 = 0, \qquad \sum \beta_n^2 < \infty, \qquad 0 \neq \alpha_n \to 0.$$

This defines a compact, injective linear operator $A_0 : \mathfrak{X} \to \mathfrak{X}$ with dense range. Writing

$$x^* = \sum_1^{\infty} \xi_n^* e_n, \qquad x = \sum_1^{\infty} \xi_n e_n,$$

we have

$$\mathscr{J}(x) = \tfrac{1}{2}\|A_0(x - x^*)\|^2 = \tfrac{1}{2} \sum_{n=1}^{\infty} [\alpha_n(\xi_n - \xi_n^*) + \beta_n(\xi_1 - \xi_1^*)]^2$$

in (4).

Taking $x = \mathscr{A}(\mathfrak{X}_N)$, one has

$$0 = \partial\mathscr{J}/\partial\xi_n = \alpha_n[\alpha_n(\xi_n - \xi_n^*) + \beta_n(\xi_1 - \xi_1^*)], \qquad \text{for } n = 2, \ldots, N; \tag{6}$$

and, using (6), and noting that $\xi_m = 0$ for $n > N$,

$$0 = \partial\mathscr{J}/\partial\xi_1 = (\xi_1 - \xi_1^*) + \sum_{n=N+1}^{\infty} \beta_n[\alpha_n(-\xi_n^*) + \beta_n(\xi_1 - \xi_1^*)],$$

so

$$\xi_1 - \xi_1^* = \left[\sum_{N+1}^{\infty} \alpha_n \beta_n \xi_n^*\right] \Big/ \left[1 + \sum_{N+1}^{\infty} \beta_n^2\right]. \tag{7}$$

Thus, using (6) and (7),

$$\begin{aligned} \|x_N - x^*\|^2 &= \sum_1^N (\xi_n - \xi_n^*)^2 + \sum_{N+1}^{\infty} \xi_n^{*2} \\ &= \left[1 + \sum_2^N (\beta_n/\alpha_n)^2\right]\left[\sum_{N+1}^{\infty} \alpha_n \beta_n \xi_n^*\right] \Big/ \left[1 + \sum_{N+1}^{\infty} \beta_n^2\right] + \sum_{N+1}^{\infty} \xi_n^{*2}. \end{aligned} \tag{8}$$

Suppose, now, that $\{\beta_n\}_2^\infty$ and $\{\xi_n^*\}_1^\infty$ have been given arbitrarily, but with

$$\sum_n \beta_n^2 < \infty, \qquad \sum_n \xi_n^{*2} < \infty,$$

and all terms nonzero. We wish to construct $\{\alpha_n\}_2^\infty$, so

$$0 \neq \alpha_n, \qquad \alpha_n \to 0, \qquad \|x_N - x^*\| \to \infty.$$

For example, take $\alpha_n = 1/n$ for n even; and, for n odd, take $\alpha_n \leq 1/n$ and so small that

$$(\beta_n/\alpha_n)(\alpha_{n+3}\beta_{n+3}\xi_{n+3}^*) \geq (n+1)^2\left[1 + \sum_2^\infty \beta_n^2\right]. \tag{9}$$

Clearly, the use of (9) in (8) for $N = n, n+1$ gives

$$\|x_N - x^*\| \geq N,$$

so

$$\|x_N - x^*\| \to \infty,$$

as desired for this operator A_0. It is somewhat more delicate to arrange that

$$\|x_N - x^*\| \to c \neq 0, \infty,$$

but this can also be done. □

We now return to the notation of the self-adjoint case. Thus, A is now assumed to have an orthonormal basis $\{a_n\}_1^\infty$ of eigenvectors with positive eigenvalues $\{\alpha_n\}$.

Theorem 3.1. Let A be a compact, injective linear operator with dense range. Let $b^* := Ax^*$ be given, with x^* not a finite linear combination of eigenvectors. Then, there is an orthonormal basis $\{e_n\}$ of $\mathfrak{X}$ such that the sequence $x_N = \mathscr{A}(\mathfrak{X}_N)$, determined using (4) with

$$\mathfrak{X}_N := \mathrm{sp}\{e_1, \ldots, e_N\} \quad \text{and} \quad b_N = b^*$$

for each N, is unbounded.

Proof. The construction presented here gives

$$\|x_{2J-1}\| \to \infty,$$

but $x_{2J} \to x^*$, under the simplifying assumption that

$$x^* = \sum_1^\infty \beta_n^* a_n,$$

with $\beta_n \neq 0$ for every n. It is not too difficult to modify the construction to admit having infinitely many $\beta_n = 0$, provided also that infinitely many $\beta_n \neq 0$.

The eigenvalues and eigenvectors are given with *some* ordering, and we start by reordering them recursively to suit our purposes. At each step, let S_m be the set of indices as yet not chosen when one comes to select the mth in the new ordering; we proceed to choose in the order: $m = 2, 1, 4, 3, 6, 5, \ldots, 2j, 2j-1, \ldots$. For $m = 2j$, let $n(2j)$ be simply the least n in S_{2j}. For $m = 2j-1$, let $n(2j-1)$ be the least n in $S_{2j-1} := S_{2j} \setminus \{n(2j)\}$ for which α_n is small enough that

$$\rho_j := [\alpha_n/\alpha_{n(2j)}] < \min\{|\beta^*_{n(2j)}|/j, 1/\sqrt{2}\}. \qquad (10\text{-}1)$$

For simplicity of notation, we now simply write $\hat{a}_m, \hat{\alpha}_m, \hat{\beta}^*_m$ for $a_{n(m)}$, $\alpha_{n(m)}, \beta^*_{n(m)}$, assuming the new ordering. Thus,

$$0 < \rho_j := \hat{\alpha}_{2j-1}/\hat{\alpha}_{2j} < |\hat{\beta}^*_{2j}|/j, \qquad 1 - \rho_j^2 \geq 1/2, \qquad (10\text{-}2)$$

for $j = 1, 2, \ldots$. Now, set

$$s_j = \pm\rho_j, \qquad c_j = \sqrt{(1-\rho_j^2)} \quad \text{with} \quad s_j \hat{\beta}^*_{2j-1} \hat{\beta}^*_{2j} > 0, \qquad (11)$$

and then define a new basis $\{e_n\}$ in terms of $\{\hat{a}_m\}$ by

$$e_{2j-1} := c_j \hat{a}_{2j-1} - s_j \hat{a}_{2j}, \qquad e_{2j} := s_j \hat{a}_{2j-1} + c_j \hat{a}_j. \qquad (12)$$

Note that, since

$$c_j^2 + s_j^2 = 1 \qquad \text{for each } j,$$

$\{e_n\}_1^\infty$ is again an orthonormal basis. We will take

$$\mathfrak{X}_N := \mathrm{sp}\{e_1, \ldots, e_N\} \quad \text{and} \quad x_N = \mathscr{A}(\mathfrak{X}_N).$$

Expansions with respect to the bases $\{\hat{a}_m\}, \{e_n\}$ are given by

$$x = \sum_m \beta_m \hat{a}_m = \sum_n \gamma_n e_n,$$

with

$$\begin{aligned} \gamma_{2j-1} &= c_j \beta_{2j-1} - s_j \beta_{2j}, & \gamma_{2j} &= s_j \beta_{2j-1} + c_j \beta_{2j}, \\ \beta_{2j-1} &= c_j \gamma_{2j-1} + s_j \gamma_{2j}, & \beta_{2j} &= -s_j \gamma_{2j-1} + c_j \gamma_{2j}. \end{aligned} \qquad (13)$$

Since we take

$$b^* = Ax^*,$$

so

$$Ax - b^* = A(x - x^*),$$

we have for consideration in (4):

$$\begin{aligned}\mathcal{J}(x) &:= \tfrac{1}{2}\|A(x-x^*)\|^2 = \tfrac{1}{2}\Big\|\sum_k (\beta_k - \hat{\beta}_k^*)\hat{\alpha}_k\hat{a}_k\Big\|^2 \\ &= \tfrac{1}{2}\sum_{j=1}^{\infty} \|(\gamma_{2j-1}-\gamma_{2j-1}^*)Ae_{2j-1} + (\gamma_{2j}-\gamma_{2j}^*)Ae_{2j}\|^2 \\ &= \tfrac{1}{2}\sum_1^{\infty}[(c_j[\gamma_{2j-1}-\gamma_{2j-1}^*] + s_j[\gamma_{2j}-\gamma_{2j}^*])^2\hat{\alpha}_{2j-1}^2 \\ &\qquad + (-s_j[\gamma_{2j-1}-\gamma_{2j-1}^*] + c_j[\gamma_{2j}-\gamma_{2j}^*])^2\hat{\alpha}_{2j}^2].\end{aligned}$$

For minimizing $\mathcal{J}$, we consider $\partial\mathcal{J}/\partial\gamma_n$. With some manipulation,

$$\begin{aligned}\partial\mathcal{J}/\partial\gamma_{2j-1} &= [c_j^2\hat{\alpha}_{2j-1}^2 + s_j^2\hat{\alpha}_{2j}^2](\gamma_{2j-1}-\gamma_{2j-1}^*) + [\hat{\alpha}_{2j-1}^2 - \hat{\alpha}_{2j}^2]s_jc_j(\gamma_{2j}-\gamma_{2j}^*), \\ \partial\mathcal{J}/\partial\gamma_{2j} &= [\alpha_{2j-1}^2 - \hat{\alpha}_{2j}^2]s_jc_j(\gamma_{2j-1}-\gamma_{2j-1}^*) + [s_j^2\hat{\alpha}_{2j-1}^2 + c_j^2\hat{\alpha}_{2j}^2](\gamma_{2j}-\gamma_{2j}^*).\end{aligned} \tag{14}$$

If both of these expressions vanish [e.g., for $x = x_N = \mathcal{A}(\mathfrak{X}_N)$ with $N \geq 2j$], then

$$\gamma_{2j-1} = \gamma_{2j-1}^* \qquad \text{and} \qquad \gamma_{2j} = \gamma_{2j}^*,$$

since the determinant of the system is

$$\hat{\alpha}_{2j-1}^2\hat{\alpha}_{2j}^2 \neq 0.$$

For $N = 2J$, this makes x_{2J} the projection of x^* on $\mathfrak{X}_{2J}$, so the subsequence $\{x_{2J}\}$ will converge to x^*.

Suppose, now, that

$$N = 2J - 1,$$

on the other hand. One has

$$\gamma_n = \gamma_n^* \qquad \text{for } n < 2J-1 = N;$$

but, for

$$n = 2J - 1,$$

one has

$$\partial\mathcal{J}/\partial\gamma_{2J-1} = 0$$

in (14), with $j = J$, but must have $\gamma_{2J} = 0$ as $\gamma_n = 0$ for $n > N$. Thus,

$$\begin{aligned}\gamma_{2J-1} - \gamma_{2J-1}^* &= \frac{[\hat{\alpha}_{2J-1}^2 - \hat{\alpha}_{2J}^2]s_Jc_J}{c_J^2\hat{\alpha}_{2J-1}^2 + s_J^2\hat{\alpha}_{2J}^2}\gamma_{2J}^* \\ &= \frac{[\rho_J^2 - 1]s_Jc_J}{[\rho_J^2-1]c_J^2 + 1}[s_J\hat{\beta}_{2J-1}^* + c_J\hat{\beta}_{2J}^*],\end{aligned} \tag{15}$$

and so

$$(\gamma_{2J-1}-\gamma_{2J-1}^*)^2=\frac{[\rho_J^2-1]^2\rho_J^2(1-\rho_J^2)}{([\rho_J^2-1](1-\rho_J^2)+1)^2}[\pm\rho_J\hat{\beta}_{2J-1}^*+\surd(1-\rho_J^2)\hat{\beta}_{2J}^*]^2$$

$$>\hat{\beta}_{2J}^{*2}/64\rho_J^2>J^2/64,$$

using (10-2), (11), (13). Thus,

$$\|x_{2J-1}-x^*\|\geq J/8,$$

and the sequence $\{x_N\}$ is unbounded ($\|x_{2J-1}\|\to\infty$). □

It is easy to see from (15) that replacing (11) by a different choice of (s_J, c_J), still subject to

$$s_J^2+c_J^2=1,$$

permits construction of a new basis $\{e_n\}$ giving to $|\gamma_{2J-1}-\gamma_{2J-1}^*|$ an arbitrary value in the interval $[0, J/8]$ without affecting the results of any other computations, i.e., results for x_N with N other than $2J-1$. Thus, one can also arrange that $\|x_{2J-1}-x^*\|\to c$ for any choice of c in $[0,\infty]$; so, in particular, one can have $\{x_N\}$ bounded with $x_N\not\to x^*$.

It is clear that the construction above requires $\beta_n^*\neq 0$ for infinitely many n for (10-2) to be possible. If $x^*\neq 0$, then it seems likely that a suitable orthonormal basis $\{e_n\}$, *not* of the form (12) or indeed block-related to $\{\alpha_k\}$ at all, could be found, say of the form

$$e_n=\sum_1^n c_{n,j}\hat{a}_j$$

with $c_{n,n}\neq 0$ and $\{a_n\}$ a suitable reordering of $\{a_k\}$, for which the expansion coefficients $\{\langle x^*, e_n\rangle\}$ decay slowly enough and (4) gives $\{\|x_N-x^*\|\}$ unbounded or even, more strongly, $\|x_N-x^*\|\to\infty$. We have preferred here to present Theorem 3.1, rather than to cope with the computational complexities of this more general form.

4. Discussion

The use of computations based on an assumed (approximate) parametric representation for an *unknown* function to be estimated is pervasive in engineering practice and system theory. Indeed, the very term *lumped parameter* indicates such an approach to system structure. The use of *minimization of the residual* is then a standard approach to estimation of the parameter values.

The results above show that, for ill-posed problems, this is an unacceptable procedure in the absence of detailed justificatory analysis of $\{A_N^{-1}\}$ and, for the particular b^* involved, of the convergence to b^* of the projections $\tilde{b}_N$ on $\mathfrak{Y}_N := A\mathfrak{X}_N$. The analysis presented was only for *linear* problems (whereas, e.g., system identification problems are nonlinear even in the case of linear dynamics). However assuming, as is typically done, that f in (1) is smoothly Fréchet differentiable near the desired solution x^*, convergence to x^* of the approximating sequence $\{x_N\}$ would suggest applicability of the linearized model, and so would suggest the relevance of the results above. At present, no rigorous realization of this argument is available, even for Theorem 2.1 under strong smoothness and uniformity hypotheses on the Fréchet derivative. It is equally true, of course, that the present results preclude the possibility of any justification via linearization of the convergence of the algorithm:

$$\text{minimize } \mathscr{J}_N(\lambda) := \tfrac{1}{2}\|f(x_n(\lambda)) - b_N\|^2 \qquad \text{over } \lambda \in \Lambda_N, \tag{16}$$

where $\{\Lambda_N\}$ is a sequence of parameter spaces and $\{\lambda \mapsto x_N(\lambda)\}$ are the corresponding parametrizations; now,

$$\mathfrak{X}_N := \{x_N(\lambda) : \lambda \in \Lambda_N\}$$

need no longer be a linear subspace but *is* locally diffeomorphic to Λ_N.

It should be emphasized that it is not the method of least squares *per se* which is causing the problem.[3] The real difficulty[4] lies with the use of approximating subspaces which may be poorly related to the operator A. The constructions of Example 3.1 and Theorem 3.1 are, of course, quite artificial and one might expect (and would hope) that *natural* choices of subspaces would (as e.g. in Ref. 5) lead to convergent approximation sequences. On the other hand, the existence of even such artificial constructions makes that expectation and hope less confident and emphasizes the need for careful examination of the procedure.

In particular, for ill-posed problems, it is inadequate to verify that particular computational procedures apply to effective treatment of (1) or

[3] Another popular approximation method (for A positive) selects x_N in $\mathfrak{X}_N$ to make the residual $Ax_N - b$ orthogonal to $\mathfrak{X}_N$ and an essentially identical construction can be used to demonstrate the possibility of nonconvergence for *that* method. On the other hand, if $\mathfrak{X}_N$ were in the range of A^*, one could select x_N in $\mathfrak{X}_N$ to make $Ax_N - b$ orthogonal to $\mathfrak{Y}_N$, where $\mathfrak{X}_N = A^*\mathfrak{Y}_N$ and the x_N so selected will be the best approximation in $\mathfrak{X}_N$ to the true solution.

[4] The instability under perturbation exhibited in Theorem 2.1 is inherent in the ill-posedness of the problem, but is not the real difficulty. If the difficulty exhibited in Example 3.1 and Theorem 3.1 were not to occur, then these perturbations could be controlled by requiring that $b_N \to b^*$ rapidly enough, $\|b_N - b^*\| = o(1/\nu_N)$, i.e., if the accuracy of measurement and calculation are suitably improved as the approximation is expected to improve.

(2) *after* the introduction of an approximating parametric form. In the language of statistical practice, such procedures are not *robust* enough. As is well known, the results are overwhelmingly sensitive to *noise* in the modes (discarded by the parametrization) associated with eigenvectors of AA^* [for this, take $A = f'(x^*)$ in the nonlinear case] corresponding to very small eigenvalues. Some desensitization of the computation and the possibility of making use of *a priori* knowledge of special properties of the solution x^* (e.g., extra smoothness beyond membership in $\mathfrak{X}$) can be obtained by the use of approximation procedures specifically addressed to the difficulties associated with ill-posed problems (see e.g. Refs. 6 and 7).

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