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A convergent approximation scheme for the inverse Sturm–Liouville problem†

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Abstract. For the Sturm–Liouville operator $L=L_p: y\mapsto -y''+py$ one seeks to reconstruct the coefficient p from knowledge of the sequence of eigen-frequencies (λ_j with $Ly_j=\lambda_j y_j$ for some $y_j\neq 0$). An implementable scheme is: for some N determine p_N so (approximately) p_N has minimum norm with eigen-frequencies $\{\bar\lambda_1,\ldots,\bar\lambda_N\}$ as given. This is the method of 'generalised interpolation' and is shown to give a convergent approximation scheme: $p_N\to\bar p$. The principal technical difficulties are the continuities of the functionals $p\mapsto\lambda_j$, which are shown for p topologised by weak convergence in $(H^1)'$, and the injectivity of $p\mapsto\{\lambda_j: j=1,2,\ldots\}$.

1. Introduction

We consider the ordinary differential operators

$$L=L_p: y\mapsto -y''+py \qquad (1.1)$$
$$: L^2(-1,1)\supset\mathscr{D}\to L^2(-1,1)$$

(where $'=\mathrm{d}/\mathrm{d}x$) with homogeneous boundary conditions of the form

$$-y'(-1)+hy(-1)=0=y'(1)+hy(1). \qquad (1.2)$$

Thus, the domain $\mathscr{D}$ of L has the form

$$\mathscr{D}=\mathscr{D}_p:=\{y\in L^2(-1,1): (1.2)\text{ holds and } Ly\in L^2(-1,1)\}. \qquad (1.3)$$

Essentially, one takes $\mathscr{D}$ to be the maximal domain for L given by (1.1) and (1.2), and this will make L a self-adjoint operator with compact resolvent. By standard Sturm–Liouville theory one has a sequence

$$\lambda_1<\lambda_2<\lambda_3<\ldots\to\infty \qquad (1.4)$$

of eigenvalues $\lambda_j=\lambda_j(p)$ with a corresponding orthonormal sequence of eigenfunctions $y_j=y_j(p)$ such that

$$Ly_j=\lambda_j y_j \qquad y_j\in\mathscr{D}_p \qquad (1.5)$$

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where $\| \ \|$ denotes the norm of $L^2(-1, 1)$ and $\|y_j\| = 1$. Under the hypothesis that p is *even* on $(-1, 1)$ so

$$p(x) = p(-x) \qquad \text{for } x \in (-1, 1) \tag{1.6}$$

it is known (cf, [1, 3, 8], etc) that the map

$$p \mapsto (\lambda_1, \lambda_2, \ldots) \tag{1.7}$$

defined by (1.4) and (1.5) is injective—*the sequence* $\{\lambda_j\}$ *of eigenfrequencies uniquely determines the coefficient p.*

Given the above, it makes sense to attempt to identify p from the observation/knowledge of $(\lambda_1, \lambda_2, \ldots)$, inverting the map (1.7). This is the inverse Sturm–Liouville problem for which we seek a constructive solution in the form of an (implementable) approximation algorithm.

By analogy with the classical interpolation problem in which an approximate reconstruction of an otherwise unknown function is obtained from a finite set of values of certain functionals, the approximation scheme considered [8, 9] may be viewed as a form of *generalised interpolation.*

Procedure (P_*). For $N = 1, 2, \ldots$, obtain p_N to satisfy

$$\|p_N\|_* = \min \tag{1.8}$$

subject to

$$\lambda_j(p_N) = \bar{\lambda}_j \qquad \text{for } j = 1, \ldots, N \tag{1.9}$$

where $\{\bar{\lambda}_j\}$ is the given set of eigenvalues (so $\bar{\lambda}_j := \lambda_j(\bar{p})$ for the 'true' potential $\bar{p}$) and $\| \ \|_*$ is a suitably chosen norm corresponding to a Banach space $\mathbf{X}_*$ of admissible p—for which (1.1) makes sense and (1.6) holds. The functionals considered are then

$$p \mapsto \lambda_j = \lambda_j(p) \colon \mathbf{X}_* \to \mathbb{R}, \tag{1.10}$$

defined by (1.4) and (1.5) for $j = 1, 2, \ldots$ and, as with the evaluation functionals $f \mapsto f(t_j)$ occurring in classical interpolation, we will use the fact that specification of the complete set of values $(j = 1, 2, \ldots)$ uniquely determines the unknown function. Our object is to prove, under suitable hypotheses, that

$$p_N \to \bar{p} \quad \text{as} \quad N \to \infty \tag{1.11}$$

using (P_*). More precisely, we will introduce an approximate version of (P_*)—call it (P_a)—in which (1.8) and (1.9) are realised only approximately in determining $\tilde{p}_N$ and show that $\tilde{p}_N \to \bar{p}$.

2. The operator

The principal technical difficulty arising in the consideration of the scheme (P_*) introduced above is the continuity of the functionals $p \mapsto \lambda_j(p)$. As this is of some interest in its own right, we will present a result somewhat stronger than will be used in § 4 for showing the convergence of the approximation scheme. Thus, we will consider $p \in [H^1(-1, 1)]'$ and will show that the functional $p \mapsto \lambda_j(p)$ is continuous from $(H^1)'$ with its weak topology for each $j = 1, 2, \ldots$.

When p is smooth—or, indeed, when $p \in \mathbf{Y} := L^2(-1, 1)$—one has

$$\mathscr{D} = \mathscr{D}_0 := \{y \in H^2(-1, 1) \colon (1.2)\}$$

since this gives $y'' \in \mathbf{Y}$ and, as y is then continuous, one also has $py \in \mathbf{Y}$. On the other hand, for $p \notin \mathbf{Y}$ one cannot take the domain of L_p to be contained in $\mathscr{D}_0$ since smooth y then gives $py \notin \mathbf{Y}$. Thus, before proceeding, it is necessary to discuss briefly the interpretation of $\mathscr{D}_p, L_p$ for general $p \in (H^1)'$.

First, note an elementary functional analysis inequality: given Banach spaces $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and linear maps E: $\mathbf{A} \to \mathbf{B}$ and F: $\mathbf{B} \to \mathbf{C}$ with E compact and F injective, one has

$$\|Ex\|_{\mathbf{B}} \leqslant \varepsilon \|x\|_{\mathbf{A}} + C_\varepsilon \|FEx\|_{\mathbf{C}}$$

for all $x \in \mathbf{A}$, for all $\varepsilon > 0$ (some C_ε). Since the embedding $H^1(-1, 1) \hookrightarrow \mathscr{C}[-1, 1]$ is compact, it follows that

$$\|y\|_{\mathscr{C}} \leqslant \varepsilon \|y\|_1 + C_\varepsilon \|y\| \tag{2.1}$$

(all $\varepsilon > 0$, some C_ε) where, now and hereafter, we use $\| \ \|_{\mathscr{C}}$, $\| \ \|_1$ and $\| \ \|$ to denote, respectively, the norms in $\mathscr{C}[-1, 1]$, in $H^1(-1, 1)$ and in $\mathbf{Y} := L^2(-1, 1)$. Note, also, that H^1 is an algebra with

$$\|uv\|_1 \leqslant \|u\|_{\mathscr{C}} \|v\|_1 + \|u\|_1 \|v\|_{\mathscr{C}}$$

so that, using (2.1), one has

$$\begin{aligned} \|y^2\|_{\mathscr{C}} &\leqslant 2\varepsilon^2 \|y\|_1^2 + 2C_\varepsilon^2 \|y\|^2 \\ \|y^2\|_1 &\leqslant 4\varepsilon \|y\|_1^2 + (C_\varepsilon^2/2\varepsilon)\|y\|^2. \end{aligned} \tag{2.2}$$

Now consider y and z such that $y \in H^1$, $z \in \mathbf{Y}$ and

$$-y'' + py = z + \lambda y \qquad -y'(-1) + hy(-1) = 0 = y'(1) + hy(1) \tag{2.3}$$

so, for any $v \in H^1$ one has

$$\langle z, v \rangle + \lambda \langle y, v \rangle = \langle y', v' \rangle + h[yv|_1 + yv|_{-1}] + \langle p, yv \rangle. \tag{2.4}$$

One can view (2.4) as a weak definition of L_p as a map: $H^1 \to (H^1)'$ with the domain $\mathscr{D}_p$ taken as the pre-image of $\mathbf{Y}$ so that it remains necessary to show L_p: $\mathbf{Y} \supset \mathscr{D}_p \to \mathbf{Y}$ is closed and densely defined. (As usual, for smooth p and y one can recover the boundary condition (1.2) as well as the definition (1.1) from the weak formulation (2.4).)

Assuming $\|p\|_* \leqslant M$ (where, now, $\| \ \|_*$ denotes the norm of $(H^1)'$ dual to that of H^1, one takes $v = y$ to obtain

$$\begin{aligned} \|y\|_1^2 = \|y'\|^2 + \|y\|^2 &= \langle z, y \rangle + (\lambda + 1)\|y\|^2 - h[y^2(1) + y^2(-1)] - \langle p, y^2 \rangle \\ &\leqslant \langle z, y \rangle + (\lambda + 1)\|y\|^2 + 2|h| \, \|y^2\|_{\mathscr{C}} + \|p\|_* \|y^2\|_1 \\ &\leqslant \langle z, y \rangle + (\lambda + 1)\|y\|^2 + 4|h|[\varepsilon^2 \|y\|_1^2 + C_\varepsilon^2 \|y\|^2] + M[4\varepsilon \|y\|_1^2 + (C_\varepsilon^2/2\varepsilon)\|y\|^2]. \end{aligned}$$

Setting

$$K_M := 1 + C_\varepsilon^2/2\varepsilon^2 \quad \text{with} \quad M\varepsilon + |h|\varepsilon^2 \leqslant \tfrac{1}{8} \tag{2.5}$$

this gives the fundamental estimate

$$\|y\|_1^2 \leqslant 2\langle z, y \rangle + 2(\lambda + K_M)\|y\|^2 \tag{2.6}$$

given (2.3) with $\|p\|_* \leqslant M$ and (2.5).

Now choose $\lambda = \lambda_*$ with $\lambda_* \leqslant -(K_M + 1)$ and (2.6) gives $\|y\|_1^2 + 2\|y\|^2 \leqslant 2\|z\| \, \|y\|$ so that

$$\|y\| \leqslant \|z\| \qquad \|y\|_1 \leqslant \sqrt{2}\|z\| \tag{2.7}$$

and the map $z \mapsto y$ determined by (2.3) is then continuous (from $\mathbf{Y}$ to $H^1 \hookrightarrow \mathbf{Y}$) where defined.

For smooth p the map $z \mapsto y$ is defined on all of $\mathbf{Y}$. Given general $\bar{p} \in (H^1)'$ let (p_k) be a sequence of smooth functions with $p_k \to \bar{p}$ in $(H^1)'$ and, with $z \in \mathbf{Y}$ fixed, let y_k be defined by (2.3) using p_k for p and λ_* for λ. Then, uniformly, $\|y_k\|_1 \leqslant \sqrt{2}\|z\|$ so we may assume weak convergence in H^1, i.e. $y_k \rightharpoonup \bar{y}$ for some $\bar{y} \in H^1$. For any $v \in H^1$ one then has as in (2.4)

$$\langle z, v\rangle + \lambda_* \langle y_k, v\rangle = \langle y_k', v'\rangle + h[y_k v|_1 + y_k v|_{-1}] + \langle p_k, y_k v\rangle.$$

Since $y_k \rightharpoonup \bar{y}$ weakly in H^1, one has $y_k \to \bar{y}$ in $\mathscr{C}[-1, 1]$ by the compactness of the embedding so, going to the limit, one has

$$\langle z, v\rangle + \lambda_* \langle \bar{y}, v\rangle = \langle \bar{y}', v'\rangle + h[\bar{y}v|_1 + \bar{y}v|_{-1}] + \langle \bar{p}, \bar{y}v\rangle$$

which was the interpretation of (2.3) for $\bar{p}$. Hence, the map $z \mapsto y$ defined by (2.3) using $\bar{p}$ and λ_* is defined for all $z \in \mathbf{Y}$ and continuous to H^1 (well defined as the bound $\|\bar{y}\|_1 \leqslant \sqrt{2}\|z\|$ gives uniqueness as well as continuity so $\bar{y}$ is independent of the choice of the approximating sequence (p_k)).

For any $p \in (H^1)'$, then we may take $\mathscr{D}_p$ to be the range of the map $z \mapsto y$. (Note that

$$-y'' + py = z + \lambda_* y = \hat{z}' + \hat{\lambda}'_* y \qquad (\hat{z}' := z + (\lambda_* - \hat{\lambda}'_*)y)$$

so this range is independent of the choice of λ_*.) We must finally show that $\mathscr{D}_p$, so obtained, is dense. If not, one would have the existence of $v \in \mathbf{Y}$ orthogonal to all y satisfying (2.3) for some z; let u, corresponding to y, be obtained from (2.3) with v for z—i.e., $\boldsymbol{L}_p u = v + \lambda_* u$ with $u \in \mathscr{D}_p$. Then, for any y and z, as in (2.3),

$$\begin{aligned} 0 &= \langle v, y\rangle \\ &= \langle L_p u, y\rangle - \lambda_* \langle u, y\rangle \\ &= \langle u', y'\rangle + h[uy|_1 + uy|_{-1}] + \langle p, uy\rangle - \lambda_* \langle u, y\rangle \\ &= \langle u, z\rangle. \end{aligned} \tag{2.8}$$

Since $z \in \mathbf{Y}$ is arbitrary, this gives $u = 0$, so $v = 0$. Hence $\mathscr{D}_p$ is dense in $\mathbf{Y}$.

Remark. For smooth p and for $y \in H^2$ one can interpret (2.3) almost classically: taking y'' as a distributional derivative which is then a function in Y and actually evaluating y and y' at ± 1 to verify the boundary conditions (1.2). For more general p we have used the weak interpretation (2.4)—actually viewing $\boldsymbol{L}_p$ as a map from H^1 to $(H^1)'$ (with y'' and the boundary conditions considered weakly and, as $y, v \in H^1$ gives $yv \in H^1$, interpreting $py \in (H^1)'$ by $\langle py, v\rangle := \langle p, yv\rangle$) and then restricting to $\mathscr{D}_p \subset \mathrm{H}^1$ such that the result is actually in $\mathbf{Y}$. This interpretation embeds the boundary conditions in the definition of the operator by (2.4) together with the construction of $\mathscr{D}_p$. We do note, however, that if one were to have $p \in L^2_{\mathrm{loc}}$ for neighbourhoods of ± 1 (i.e., if there were α and β with $-1 < \alpha < \beta < 1$ and $\hat{p} \in \mathbf{Y}$ such that $\langle p - \hat{p}, f\rangle = 0$ for f with support in $[\alpha, \beta]$) then one would have correspondingly $y \in H^2_{\mathrm{loc}}$ near ± 1 for y determined by (2.4) with, say, $\lambda = \lambda_*$ (i.e., for all $y \in \mathscr{D}_p$) and so could then interpret (1.2) classically.

We have constructed $\boldsymbol{L}_p$ as a densely defined operator—basically by defining $\boldsymbol{T}_* = (\boldsymbol{L}_p - \lambda_*)^{-1}$ and taking $\boldsymbol{L}_p$ to be $[\boldsymbol{T}_*^{-1} + \lambda_*]$ with $\mathscr{D}_p := \mathscr{R}(\boldsymbol{T}_*)$ as domain. Clearly $\boldsymbol{L}_p$, so constructed, is closed and, as in (2.8), is self-adjoint. Since $\boldsymbol{T}_* : \mathbf{Y} \to H^1$, the operator $\boldsymbol{L}_p$ has compact resolvent.

We finally show that $\boldsymbol{L}_p$ is semi-bounded, i.e. that $\lambda_1 := \inf\{\langle y, \boldsymbol{L}_p y\rangle : y \in \mathscr{D}_p, \|y\| = 1\}$ is finite. It is sufficient to consider y for which $\langle y, \boldsymbol{L}_p y\rangle \leqslant 0$ (else $\lambda_1 \geqslant 0$). For such y one applies (2.6) with $\lambda = 0$ to obtain

$$\langle y, \boldsymbol{L}_p y\rangle \geqslant \|y\|_1^2 - 2K_M\|y\|^2 \geqslant -2K_M$$

whence $\lambda_1 \geqslant -2K_M$.

The standard spectral theory for self-adjoint operators (on a Hilbert space) having compact resolvent gives $\lambda_1 \in \sigma(\boldsymbol{L}_p)$—$\lambda_1$ is the smallest eigenvalue. More generally we know that the spectrum $\sigma(\boldsymbol{L}_p)$ is a sequence

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots \tag{2.9}$$

(multiplicities are permitted but each is, at most, finite) with a corresponding set of eigenfunctions (y_k) satisfying

$$\boldsymbol{L}_p y_k = \lambda_k y_k \qquad y_k \in \mathscr{D}_p \tag{2.10}$$

with $\|y_k\| = 1$. If no non-trivial multiplicities occur, the sequence (y_k) is automatically orthonormal but can be taken orthonormal in any case. Since $\boldsymbol{T}_*$ is injective and $\sigma(\boldsymbol{L}_p) = 1/\sigma(\boldsymbol{T}_*) - \lambda_*$, one has $\lambda_k \to \infty$ and (y_k) is an orthonormal basis of y. One easily obtains from this that

$$\begin{aligned} &y = \sum \eta_k y_k & &\boldsymbol{L}_p y = \sum \lambda_k \eta_k y_k \\ &\langle y, \boldsymbol{L}_p y\rangle = \sum \lambda_k \eta_k & &\|y\|^2 = \sum \eta_k^2 \end{aligned} \tag{2.11}$$

and hence

$$\inf\{\langle y, \boldsymbol{L}_p y\rangle : y \in H^1, \|y\| = 1, y \perp y_k \text{ for } k < j\} = \lambda_j. \tag{2.12}$$

Note that in the inf in (2.12) one need not require $y \in \mathscr{D}_p$ but only $y \in H^1 \subset \mathscr{D}_p$. Note also that if the inf in (2.12) is attained at any $\hat{y} \in H^1$ with $\|\hat{y}\| = 1$ and $\hat{y} \perp y_k$ for $k < j$, then one necessarily has $\boldsymbol{L}_p \hat{y} = \lambda_j \hat{y}$ so one may take $\hat{y}$ to be y_j.

Proof. To see that the inf in (2.12) is actually attained, note that for a minimising sequence (y^ν) in $\mathscr{S}_j := \{y \in H^1 : \|y\| = 1, y \perp y_k \text{ for } k < j\}$ one has from (2.6) with $y = y^\nu$, $\lambda = 0$ and $z = z^\nu := \boldsymbol{L}_p y^\nu$ that $\{\|y^\nu\|_1\}$ is bounded, so $y^\nu \rightharpoonup \hat{y}^\nu$ for a sub-sequence. Since, then, $\|\hat{y}\|_1 \leqslant \liminf \|y^\nu\|_1$ and since (2.4) gives

$$\langle y^\nu, z^\nu\rangle = \|y^\nu\|_1^2 - 1 - \langle p, y^\nu\rangle - h([y^\nu(1)]^2 + [y^\nu(-1)]^2),$$

the minimum is attained at $\hat{y}$ (and $\|y^\nu\|_1 \to \|\hat{y}\|_1$ so $y^\nu \to \hat{y}$ in H^1). Clearly $\hat{y} \in \mathscr{S}_j$. Now, for $w \in \mathscr{S}_j$ with $w \perp \hat{y}$, one sets $\psi(t) = \psi(t; w) := \langle \hat{y} + tw, L_p(\hat{y} + tw)\rangle$ and has $\psi'(0) = 0$, so $\boldsymbol{L}_p \hat{y} =: \hat{z} \perp w$ and hence $\hat{z} \in \sup\{y_1, \dots, y_{j-1}, \hat{y}\}$. But $\langle y_k, \hat{z}\rangle = \langle \boldsymbol{L}_p y_k, \hat{y}\rangle = \lambda_k \langle y_k, \hat{y}\rangle = 0$. Thus $\hat{z} \in \sup\{\hat{y}\}$ and $\boldsymbol{L}_p \hat{y} = \lambda \hat{y}$; clearly $\lambda = \lambda_j$. Since $\hat{z} \in H^1 \subset \mathbf{Y}$, one necessarily has $\hat{y} \in \mathscr{D}_p$. Thus, $\hat{y} = y_j$. $\square$

3. Continuity of the spectrum

The construction of the preceding section—(2.6) and (2.11) in particular—can now be used to show the desired continuity result for the functionals $p \mapsto \lambda_j(p)$. Throughout we will define $\boldsymbol{L}_p$, $\mathscr{D}_p$, etc, for $p \in (H^1)'$ as in § 2 and let $\{[\lambda_k(p), y_k(p)] : k = 1, 2, \dots\}$ denote the corresponding sequence of eigenpairs. These are taken so $\lambda_1(p) \leqslant \lambda_2(p) \leqslant \dots$ with $\{y_k(p)\}$ orthonormal. We topologise $p \in (H^1)'$ by (sequential) weak convergence.

Theorem 1. Suppose $\bar{p} \in (H^1)'$ and $p = p_\nu \rightharpoonup \bar{p}$ (weak convergence in $(H^1)'$). Set $\bar{\lambda}_k := \lambda_k(\bar{p})$ and $\bar{y}_k := y_k(\bar{p})$ for each k and $\lambda_k = \lambda_{k,\nu} := \lambda_k(p_\nu)$ and $y_k = y_{k,\nu} := y_k(p_\nu)$ for each k and ν. Then $\lambda_{k,\nu} \to \bar{\lambda}_k$ and (with some modification allowing for the non-uniqueness in the determination of $\{y_{k,\nu}\}$ and $\bar{y}_k$) $y_{k,\nu} \to \bar{y}_k$ (strong convergence in H^1) for each $k = 1, 2, \dots$.

Proof. The proof is by induction, using the characterisation (2.12) to show firstly, that $\limsup \lambda_{k,\nu} \leqslant \bar{\lambda}_k$, secondly, that $\liminf \lambda_{k,\nu} \geqslant \bar{\lambda}_k$ (so $\lambda_{k,\nu} \to \bar{\lambda}_k$) and finally, that $y_{k,\nu} \to \bar{y}_k$ in H^1 (for a sub-sequence and with possible respecification of $\bar{y}_k$).

Given j, suppose that the result is known for all $k < j$. Define y by applying the Gram–Schmidt procedure to $\{y_{1,\nu}, \dots, y_{j-1,\nu}, \bar{y}_j\}$ so

$$\begin{aligned} \tilde{y}_\nu &= N_\nu \left[\bar{y}_j - \sum_{k<j} \langle y_{k,\nu}, \bar{y}_j \rangle y_{k,\nu} \right] \\ &= N_\nu \bar{y}_j - \sum_{k<j} c_{k,\nu} y_{k,\nu} \end{aligned} \tag{3.1}$$

where N_ν is a normalising constant and $c_{k,\nu} = N_\nu \langle y_{k,\nu}, \bar{y}_j \rangle$. Since $y_{k,\nu} \to \bar{y}_k$ in $\mathbf{Y}$, one has $\langle y_{k,\nu}, \bar{y}_j \rangle \to \langle \bar{y}_k, \bar{y}_j \rangle = 0$ for $k < j$, and so

$$N_\nu \to 1 \qquad c_{k,\nu} \to 0 \tag{3.2}$$

(j fixed, $k < j$). From (2.12) and (2.4) with $p = p_\nu$, $\lambda = 0$ and $v = \tilde{y}_\nu$, we have

$$\begin{aligned} \lambda_j = \lambda_{j,\nu} &= \min\{\langle y, \boldsymbol{L}_p y \rangle : y \in H^1, \|y\| = 1, y \perp y_k \text{ for } k < j\} \\ &\leqslant \langle \tilde{y}_\nu, \boldsymbol{L}_p \tilde{y}_\nu \rangle \\ &= \|\tilde{y}'_\nu\|^2 + h[\tilde{y}^2_\nu(1) + \tilde{y}^2_\nu(-1)] + \langle p, \tilde{y}^2_\nu \rangle. \end{aligned} \tag{3.3}$$

Since (3.1) and (3.2) give $\tilde{y}_\nu \to \bar{y}_j$ in H^1 (as $y_{k,\nu} \to \bar{y}_k$ in H^1 so $\{y_{k,\nu}\}$ is bounded in H^1) one has $\|\tilde{y}'_\nu\| \to \|\bar{y}'_j\|$ and $\tilde{y}^2_\nu \to \bar{y}^2_j$ in $\mathscr{C}[-1, 1]$ and in H^1. It follows, noting that $p_\nu \rightharpoonup \bar{p}$, that the right-hand side of (3.3) converges to

$$\|\bar{y}'_j\|^2 + h[\bar{y}^2_j(1) + \bar{y}^2_j(-1)] + \langle \bar{p}, \bar{y}^2_j \rangle = \langle \bar{y}_j, \boldsymbol{L}_{\bar{p}} \bar{y}_j \rangle = \bar{\lambda}_j.$$

Hence,

$$\limsup \lambda_{j,\nu} \leqslant \bar{\lambda}_j. \tag{3.4}$$

Now, using the characterisation (2.10) of $y_{j,\nu}$, we have from (2.6) with $\lambda = \lambda_{j,\nu}$ and $z = 0$ that

$$\|y_{j,\nu}\|_1^2 \leqslant 2(\lambda_{j,\nu} + K_M)$$

which is uniformly bounded. Thus, there is a sub-sequence for which $y_{j,\nu} \rightharpoonup \hat{y}$ (weak convergence in H^1 and so strong convergence in $\mathscr{C}[-1, 1]$ and in $\mathbf{Y}$). Thus, $\|\hat{y}\| = 1$ and, since

$$\begin{aligned} \langle y_{j,\nu}, \bar{y}_k \rangle &= \langle y_{j,\nu}, y_{k,\nu} \rangle + \langle y_{j,\nu}, \bar{y}_k - y_{k,\nu} \rangle \\ &= \langle y_{j,\nu}, \bar{y}_k - y_{k,\nu} \rangle \to 0 \qquad \text{for } k < j, \end{aligned}$$

one has $\hat{y}\perp\bar{y}_k$ for $k<j$. Also,

$$\bar{\lambda}_{j,\nu}=\langle y_{j,\nu}, \boldsymbol{L}_p y_{j,\nu}\rangle \qquad (p=p_\nu)$$
$$=\|y_{j,\nu}\|_1^2-1+h[y_{j,\nu}^2(1)+y_{j,\nu}^2(-1)]+\langle p_\nu, y_{j,\nu}^2\rangle$$

and

$$\liminf \lambda_{j,\nu}\geqslant\|y\|_1^2-1+h[\hat{y}^2(1)+\hat{y}^2(-1)]+\langle\bar{p},\hat{y}^2\rangle$$
$$=\langle\hat{y},\boldsymbol{L}_{\bar{p}}\hat{y}\rangle$$
$$\geqslant\min\{\langle y,\boldsymbol{L}_{\bar{p}}y\rangle: y\in H^1, \|y\|=1, y\perp y_k \text{ for } k<j\}=\bar{\lambda}_j. \tag{3.5}$$

Comparing with (3.4) gives

$$\lambda_{j,\nu}\to\bar{\lambda}_j \qquad \text{as } \nu\to\infty. \tag{3.6}$$

This was shown only along sub-sequences ($\nu=\nu_i$) for which $y_{j,\nu}$ converges weakly in H^1 but a standard argument, noting the uniqueness of the limit, shows that (3.6) holds along the full sequence ($\lambda_{j,\nu}$: $\nu=1,2,\ldots$).

Using (3.6) in (3.5) shows that $\langle\hat{y},\boldsymbol{L}_{\bar{p}}\hat{y}\rangle=\bar{\lambda}_j$ and, as noted following (2.12), this means that $\hat{y}$ may be taken as $\bar{y}_j$ since we have already seen that $\|\hat{y}\|=1$ and $\hat{y}\perp\bar{y}_k$ for $k<j$. If (2.10) uniquely characterises $\bar{y}_k$ apart from the inevitable arbitrary choice of orientation, then one can consistently select orientations for $y_{j,\nu}$ so $y_{j,\nu}\to\bar{y}_j$ without extracting a sub-sequence. (If $p\in\mathbf{Y}$, classical Sturm–Liouville theory ensures that this is the case since non-trivial multiplicities cannot occur; it is not immediately clear whether this remains valid for general $p\in(H^1)'$. Otherwise, if multiple eigenvalues could occur for $\boldsymbol{L}_{\bar{p}}$, one could not be certain that $y_{j,\nu}\to\bar{y}_j$ without extracting sub-sequences, although one would still have suitable convergence for the projections on sp$\{y_{j,\nu},\ldots,y_{j+m,\nu}\}$ where $\lambda_j=\ldots=\lambda_{j+m}$.)

Observe that the argument giving (3.5) now shows, with (3.6) and $\boldsymbol{L}_{\bar{p}}\hat{y}=\bar{\lambda}_j\hat{y}$, that

$$\|y_{j,\nu}\|_1^2\to\|\hat{y}\|_1^2.$$

This, with weak convergence in H^1, implies the desired strong convergence

$$y_{j,\nu}\to\hat{y}=\bar{y}_j \tag{3.7}$$

in H^1 along the sub-sequence—or for the full sequence as noted above.

Since the inductive hypothesis is vacuous for $j=1$, this shows (3.6) and (3.7) for each $j=1,2,\ldots$. Note that, if necessary, we proceed from j to $j+1$ after extracting a sub-sequence but, since the limit in (3.6) is the same for any such (repeated) extraction of sub-sequences, one nevertheless has (3.6) for the full sequence. (It is only (3.7) which may require modification in view of non-uniqueness in the specification of $\bar{y}_j$.) $\square$

4. The approximation scheme

The arguments of this section, given theorem 1 and uniqueness, are quite simple, following [7]. We show convergence for an abstract approximation procedure (P_a) and then remark on implementation. The basic assumptions are

(i) $\mathscr{P}$ is a closed convex subset of a dual Banach space $\mathbf{X}_*$ embedded in $[H^1(-1,1)]'$; we now let $\|\ \|_*$ denote the norm of $\mathbf{X}_*$.

(ii) For $p\in\mathbf{X}_*$ one defines $\boldsymbol{L}_p$, etc, as in § 2 (with h fixed).

(iii) $\bar{p}\in\mathscr{P}$ is such that the sequence $\{\bar{\lambda}_j:=\lambda_j(\bar{p})\colon j=1,2,\ldots\}$ uniquely determines $\bar{p}$. (This is known to hold for moderately smooth and symmetric $\bar{p}$: $p(x)=p(-x)$ on $(-1,1)$.)

The approximation procedure under consideration is a relaxed version of 'generalised interpolation'.

Procedure (P_a). For each $N=1, 2, \ldots$ consider $p \in \mathscr{P}$ satisfying

$$|\lambda_k(p)-\bar{\lambda}_k| \leqslant \varepsilon_{k,N} \qquad \text{for } k=1, \ldots, N \tag{4.1}$$

and suppose p_N satisfies (4.1) and that

$$\|p_N\|_* \leqslant \inf\{\|p\|_* : p \in \mathscr{P}, (4.1)\} + \delta_N. \tag{4.2}$$

Theorem 2. Let $\bar{p} \in \mathscr{P}$ satisfy conditions (i), (ii) and (iii) and suppose $(p_N: N=1, 2, \ldots)$ is obtained as in procedure (P_a) above with $0<\varepsilon_{k,N} \to 0$ (as $N \to \infty$ for each k; $N \geqslant k$) and $0<\delta_N \to 0$. Then $p_N \xrightarrow{*} \bar{p}$ (weak $*$ convergence in $\mathbf{X}_*$). Further, $\|p_N\|_* \to \|\bar{p}\|_*$ so, if in addition $\mathbf{X}_*$ has the Efimov–Stečkin property (e.g., for any uniformly convex space such as L^p with $1<p<\infty$), one has strong convergence: $p_N \to \bar{p}$ in $\mathbf{X}_*$. If $\mathscr{P}$ is compact in $\mathbf{X}_*$ (in this case one may omit the hypothesised convexity of $\mathscr{P}$), then there is a convergence rate.

Proof. Clearly $\bar{p}$ itself satisfies (4.1) so (4.2) gives

$$\|p_N\|_* \leqslant \|\bar{p}\|_* + \delta_N \tag{4.3}$$

which is uniformly bounded. By Alaoglu's theorem there is then a sub-sequence (p_ν) such that $p_\nu \xrightarrow{*} \hat{p}$ (weak$*$ convergence in $\mathbf{X}_*$ to some $\hat{p} \in \mathscr{P}$). Since $\mathbf{X}_*$ embeds in $(H^1)'$, this gives $p_\nu \rightharpoonup \hat{p}$ (weak convergence in $(H^1)'$) so theorem 1 gives

$$\lambda_{j,\nu} := \lambda_j(p_\nu) \to \hat{\lambda} := \lambda_j(\hat{p}) \qquad \text{for } j=1, 2, \ldots. \tag{4.4}$$

On the other hand, (4.1) gives (for each j) $\lambda_{j,\nu} \to \bar{\lambda}_j$ since $\varepsilon_{k,N} \to 0$, so one has $\lambda_j(\hat{p})=\lambda_j(\bar{p})$ for $j=1, 2, \ldots$. By the uniqueness assumption (iii) this implies $\hat{p}=\bar{p}$. Since the limit is independent of the extracted sub-sequence, one has $p_N \to \bar{p}$ as desired.

The weak$*$ lower semicontinuity of the norm then gives $\|\bar{p}\|_* \leqslant \liminf \|p_N\|_*$ and combining this with (4.3) shows $\|\bar{p}\|_* = \lim \|p_N\|_*$ since $\delta_N \to 0$. Hence one has strong convergence $p_N \to \bar{p}$ in $\mathbf{X}_*$ if, for example, $\mathbf{X}_*$ is uniformly convex. Note that if $\mathscr{P}$ is compact in $\mathbf{X}_*$ then no such assumption on $\mathbf{X}_*$ is needed to ensure strong convergence.

Finally, if $\mathscr{P}$ is compact and (iii) holds for every $\bar{p} \in \mathscr{P}$, then a standard topological theorem (a continuous bijection from a compact metric space is a homeomorphism) shows that the map

$$\mathbf{\Lambda}: p \mapsto (\lambda_j(p): j=1, 2, \ldots): \mathscr{P} \to \mathbb{R}^\infty \tag{4.5}$$

has a uniformly continuous inverse. Here, the sequence space $\mathbb{R}^\infty$ is topologised say, by the metric

$$\rho[(\lambda), (\bar{\lambda})] := \sum_{j=1}^{\infty} 2^{-j}|\lambda_j-\bar{\lambda}_j|/[1+|\lambda_j-\bar{\lambda}_j|] \tag{4.6}$$

which makes $(\lambda) \to (\bar{\lambda})$ iff $\lambda_j \to \bar{\lambda}_j$ for each j. Continuity of $\mathbf{\Lambda}$ is then given by theorem 1 and uniform continuity of $\mathbf{\Lambda}^{-1}$ (from its range) means that the inverse problem is no longer ill-posed in this context (i.e., subject to the *a priori* constraint $\bar{p} \in \mathscr{P}$) and there is a convergence rate for (P_a); for a discussion of a related situation see [5]. □

Remark. The uniqueness condition (iii) may obviously be weakened to require only that

(iii′) there is a unique $\bar{p} \in \mathscr{P}$ *of minimum norm* such that $\bar{\lambda}_j = \lambda_j(\bar{p})$ for $j = 1, 2, \ldots$

without modifying the result and the procedure will give convergence to this $\bar{p}$. If no uniqueness condition is imposed at all, then the result still holds (after possibly extracting a sub-sequence) with convergence to some $\bar{p}$ of minimum norm in $\mathscr{P} \subset \mathbf{X}_*$. Note, also, that there is essentially no change in the argument for theorem 2 if in (4.2) one were to minimise (approximately) the distance to some more convenient estimate p_*, not necessarily 0, so

$$\|p_N - p_*\|_* \leqslant \inf\{\|p - p_*\|_* : p \in \mathscr{P}, (4.1)\} + \delta_N \tag{4.2′}$$

would replace (4.2).

If the given sequence $\{\bar{\lambda}_j\}$ does not actually correspond to $\{\lambda_j(\bar{p})\}$ for some $\bar{p} \in \mathscr{P}$, then either (4.1) becomes impossible (for $N > N_0$ the set of $p \in \mathscr{P}$ satisfying (4.1) is empty) or, provided $\mathscr{P}$ is unbounded in $\mathbf{X}_*$, one could have a sequence $\{p_N\}$ with $\|p_N\|_* \to \infty$. □

Remark. Especially since there exist other computational approaches to the inverse Sturm–Liouville problem, such an abstract convergence theorem is of practical interest only to the extent that one can propose a computationally feasible implementation. The present paper, although inspired primarily by [9] and conversation with T Suzuki, may be considered a simplification and generalisation of the considerations of [4] (see also the references therein for other computational studies). We observe that the method of generalised interpolation—procedure (P_a)—requires precisely what is made available by standard approaches to the direct problem.

Consider, for example, the Galerkin approximation as an approach to the eigenvalue problem. Given a subspace $\mathscr{V}$ of H^1 and $p \in (H^1)'$, recursively define

$$\begin{aligned}
&\mathscr{S}_j(p; \mathscr{V}) := \{y \in \mathscr{V} : \|y\| = 1, y \perp y_k(p; \mathscr{V}) \text{ for } k < j\} \\
&\lambda_j(p; \mathscr{V}) := \min\{\langle y, L_p y\rangle : y \in \mathscr{S}_j(p; \mathscr{V})\} \\
&y_j(p; \mathscr{V}) := \arg\min\{\langle y, L_p y\rangle : y \in \mathscr{S}_j(p; \mathscr{V})\} \qquad \text{for } j = 1, \ldots, \dim \mathscr{V}.
\end{aligned} \tag{4.7}$$

If one considers a sequence $\{\mathscr{V}_m\}$ of such subspaces 'becoming dense' in H^1 (so, for any $\hat{y} \in H^1$, there is a sequence $\hat{y}_m \to \hat{y}$ in H^1 with $\hat{y}_m \in \mathscr{V}_m$) then the proof given for (2.12) shows

$$\lambda_j(p; \mathscr{V}_m) \to \lambda_j(p) \qquad \text{as } m \to \infty \text{ (for } j = 1, 2, \ldots). \tag{4.8}$$

At the same time we introduce (finite dimensional) subspaces $\{\mathscr{U}_n\}$ becoming dense in $\mathbf{X}_*$ (in particular, for any $\bar{p} \in \mathscr{P}$ there exists $p_n \in \mathscr{P}_n \subset \mathscr{U}_n$ such that $p_n \to \bar{p}$ in $\mathbf{X}_*$). If we assume that $\{\varepsilon_{j,N} : j \leqslant N, N = 1, 2, \ldots\}$, $\{\delta_N : N = 1, 2, \ldots\}$ are given with $\varepsilon_{j,N} \to 0$, $\delta_N \to 0$ as $N \to \infty$ and that suitable dependencies $m(N)$, $n(N)$ are specified, then the basic step of the Galerkin-implemented procedure is defined as follows.

Procedure (P_G): Given N, consider the coefficients p satisfying

$$\begin{aligned}
&p \in \mathscr{P}_n \subset \mathscr{U}_n && \text{with } n = n(n) \\
&|\lambda_k(p; \mathscr{V}_m) - \bar{\lambda}_k| \leqslant \varepsilon_{k,N} && \text{for } k = 1, \ldots, N \text{ with } m = m(N)
\end{aligned} \tag{4.9}$$

and then find p_N satisfying (4.9) and also

$$\|p_N\|_* \leqslant \min\{\|p\|_* : (4.9)\} + \delta_N. \tag{4.10}$$

The significance of a 'suitable' dependence of n and m on N is essentially that the set defined by (4.9) not be empty (at least for large enough N), more specifically, that one knows *a priori* the existence of a sequence $\hat{p}_N \to \bar{p}$ with each $\hat{p}_N$ satisfying (4.9) for that N. Exactly what this means as a specific growth rate for $n(\,)$, $m(\,)$ depends on how rapidly $\{\mathscr{U}_n\}$ and $\{\mathscr{V}_m\}$ become dense, the moduli of continuity for the functionals $p \mapsto \lambda_j(p)$ at $\bar{p}$ and how rapidly $\varepsilon_{k,N} \to 0$.

The assumptions already made do ensure, however, that it is always possible to have $n(\,)$ and $m(\,)$ grow rapidly enough for this to hold. Assume one has moduli of continuity

$$\|p-\bar{p}\|_* \leqslant \hat{\delta}_k(\varepsilon) \Rightarrow |\lambda_k(p)-\lambda_k(\bar{p})| \leqslant \varepsilon \tag{4.11}$$

for $k=1,2,\ldots$; let $\hat{\delta}_N := \min\{\delta_k(\varepsilon_{k,N}/2): k=1,\ldots,N\}$. Now take $n=n(N)$ large enough to ensure existence of $\hat{p}_N \in \mathscr{P}_n$ such that $\|p_N-\bar{p}\|_* \leqslant \delta_N$. Then take $m=m(N)$ large enough so that

$$|\lambda_k(\hat{p}_N; \mathscr{V}_m)-\lambda_k(\hat{p}_N)| \leqslant \varepsilon_{k,N}/2 \qquad \text{for } k=1,\ldots,N.$$

It follows that $\hat{p}_N$ so constructed satisfies (4.9) with $\hat{p}_N \to \bar{p}$. As in the proof of theorem 2, this ensures that $\{p_N\}$ satisfying (4.10) is bounded in $\mathbf{X}_*$ and converges (weak $*$ or strong, as appropriate) to $\bar{p}$. □

Remark. There is a sense in which it is unnecessary, in the construction above, to introduce $\{\mathscr{U}_n\}$. Observe that in the definition of $\lambda_k(p;\mathscr{V})$ one applies $\boldsymbol{L}_p$ only to $y \in \mathscr{V}$ and p appears only in the term $\langle p, y^2\rangle$ for $y \in \mathscr{V}$. If $\mathscr{V}=\mathscr{V}_{\mathrm{m}}$ is finite dimensional with a basis $\{\eta_i\}$, then $y=\Sigma\, c_i\eta_i$ and

$$\langle p, y^2\rangle = \sum_{i,j} c_i c_j \langle p, \eta_i\eta_j\rangle$$

so it is sufficient to know the effect of p acting as an element of the finite dimensional space $[\mathrm{sp}\{\eta_i\eta_j\}]^*$, indeed, in a certain subspace of this since the matrix $((\langle p, \eta_i\eta_j\rangle))$ is, for example, necessarily symmetric. This actually embeds $(H^1)'$ in the space of symmetric linear maps: $H^1 \to (H^1)'$ (i.e., $y \mapsto py$ defined by $py\colon v \mapsto \langle p, yv\rangle$) and not all such maps are associated with 'multiplication' operators as desired here. For the spline spaces $\mathscr{V}_m$ typically used for the Galerkin approach to the eigenvalue problem one has 'localised bases' (each η_i with 'small' support near $s_i \in [-1,1]$ so $\eta_i\eta_j \equiv 0$ unless s_i, s_j are close) and use of generalised $\hat{p} \in \{$symmetric matrices $\langle p, \eta_i\eta_j\rangle\}$ with an appropriate norm gives 'localisation in the limit' and—with suitable interpretation—convergence to $\bar{p}$. This would mean that N and $\mathscr{V}_{\mathrm{m}(N)}$ alone would determine a finite dimensional space in which to seek $\hat{p}$ satisfying (4.10) but this does not seem worth the interpretative complications. □

5. Discussion

After discussing the definition of the Sturm–Liouville operator formally given by (1.1) and (1.2) in the context of 'rough' p ($p \in H^{-1}(-1,1)$), it was shown that the spectrum $\sigma(L_p)$ depends continuously on p, topologised by weak sequential convergence in H^{-1} with a corresponding continuity result (strongly in H^1) for the associated eigenfunctions (subject to possible modification for multiple eigenvalues).

The choice of boundary conditions (1.2) was determined by the availability of the simplest uniqueness result (subject to symmetry) for the inverse problem but, clearly, the identical discussion would apply to the slightly more general boundary conditions

$$-y'(-1)+h_1 y(-1)=0=y'(1)+h_2 y(1)$$

(without taking $h_1=h_2$). With slight modification one can also treat the Dirichlet conditions in a similar fashion. At present we are not certain as to whether (with or without such extensions) the classical Sturm–Liouville theory generalises to ensure for $p\in H^{-1}$ that all eigenvalues of L_p are simple.

This continuity result, theorem 1, was used in demonstrating convergence for the method of generalised interpolation, theorem 2. A brief discussion was also given indicating an approach to computational implementation. Note that in the presence of compactness—for example, an *a priori* estimate of $\|p\|_E$ for a space E which embeds compactly in the space $\mathbf{X}_*$ corresponding to the norm used for convergence—there will be a convergence rate for the approximation scheme, including the considerations involved in the implementation (P_G). While we have indicated some elements to an approach to explicit determination of convergence rates (compare, also, the related considerations of [7] in the context of linear ill-posed problems), no such complete computation has been carried through here.

The method of generalised interpolation used for the approximation scheme here was initially proposed, in the special context of [2], by the late William Chewning. An indication of its applicability to the inverse Sturm–Liouville problem was noted in [8] in connection with system identification, a connection which has provided much of the motivation for work in this area [5, 6, 10, 11].

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