Identification of $q(x)$ in $u_t = \mathbf{\Delta} u - qu$ from boundary observations [1]

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ABSTRACT: We consider the problem of recovering the coefficient $q(x)$ in the equation $u_t = \mathbf{\Delta} u - qu$ from boundary observations. Uniqueness of q based on knowledge of the 'Neumann $\mapsto$ Dirichlet response operator' is shown as an implication of (known) corresponding results concerning the inverse problem for the corresponding hyperbolic equation $w_{tt} = \mathbf{\Delta} w - qw$. This is then reduced to use of the response to a single input with some consideration of computational approximation.

KEY WORDS: *identification, parabolic, partial differential equation, uniqueness, approximation.*

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1.   Introduction

We consider the problem of identifying the (unknown) coefficient $q = q(x)$ in the parabolic partial differential equation

$$(1.1) \qquad u_t = \mathbf{\Delta} u - qu \qquad \text{on } \mathcal{Q} := (0, T) \times \Omega,$$

assuming input/output access only at the boundary $\Sigma = \Sigma_T := (0, T) \times \partial\Omega$. More precisely, we assume that we can specify the Neumann data for (1.1) with trivial initial data

$$(1.2) \qquad \frac{\partial u}{\partial \nu} = f \qquad \text{on } \Sigma_T \qquad u\big|_{t=0} = 0 \qquad \text{on } \Omega$$

and then observe the corresponding Dirichlet data:

$$(1.3) \qquad g := u\big|_\Sigma.$$

Formally, then, we have a linear input/output map ('Neumann $\mapsto$ Dirichlet response operator')

$$(1.4) \qquad \mathbf{R}_1 = \mathbf{R}_1(T; q) : f \mapsto g,$$

defined through (1.1), (1.2) and then the observation (1.3). Our principal result is that $\mathbf{R}_1$ (for any $T > 0$) uniquely determines the coefficient function $q(\cdot)$ appearing in (1.1), i.e., that

$$(1.5) \qquad q \mapsto \mathbf{R}_1(T, q) \ \text{ is injective on } \boldsymbol{\mathcal{A}}$$

when considered for q in some suitable set $\boldsymbol{\mathcal{A}}$ of 'admissible' functions.

We note that results like (1.5) are already available for the inverse problem for the corresponding hyperbolic equation

$$(1.6) \qquad w_{tt} = \mathbf{\Delta} w - qw \qquad \text{on } (0, \bar{T}) \times \Omega.$$

Our approach — exploiting the deep connection between (1.1) and (1.6) via transforms with respect to t — is stimulated by D. Russell's argument ([14], see also [15]), showing how to deduce exact nullcontrollability of the heat equation for a bounded region $\Omega \subset \mathbb{R}^N$ from a corresponding wave equation result. We may restate our description above to say that our primary result is the *implication*, under fairly general hypotheses, of (1.5) from

$$(1.7) \qquad q \mapsto \mathbf{R}_2(\bar{T}, q) \ \text{ is injective on } \boldsymbol{\mathcal{A}}$$

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where $\mathbf{R}_2$ is the corresponding 'Neumann $\mapsto$ Dirichlet response operator' for (1.6). This argument will be the content of Section 2.

Parenthetically, we note that a quite different argument could alternatively obtain parabolic identifiability from corresponding results, to the extent that these would be available, for the elliptic, rather than the hyperbolic case, i.e., deriving (1.5) from (cf., e.g., [11])

$$(1.8) \qquad q \mapsto \mathbf{R}_0(q) \text{ is injective on } \mathcal{A}$$

where $\mathbf{R}_0(q) : f \mapsto g$ is the 'Neumann $\mapsto$ Dirichlet operator' for the elliptic equation

$$(1.9) \qquad -\mathbf{\Delta}v + qv = 0 \quad \text{on } \Omega, \quad \frac{\partial v}{\partial \nu} = f(x) \qquad \left[g := v \Big|_{\partial\Omega} \right].$$

To see this, one applies (1.1), (1.2) to f constant in t which gives u analytic in t and, assuming $q > 0$, convergent to the steady state solution v of (1.9) as $t \to \infty$. This analyticity implies that $\mathbf{R}_1(T,q)f$ uniquely determines $g(\cdot) := u \Big|_{\partial\Omega}$ not only on $[0,T]$ but for all $t > 0$; compare the approach of [16]. The limit as $t \to \infty$ is then also uniquely determined so, for any such $f = f(x)$ and any $T > 0$, one sees that $\mathbf{R}_1(T,q)f = \mathbf{R}_1(T,\hat{q})$ implies $\mathbf{R}_0(q)f = \mathbf{R}_0(\hat{q})$; compare [10].

Whereas it seems that the entire response operator $\mathbf{R}_2$ may be needed for identifiability for (1.6), we will show in Section 3 that a single 'experiment', using a suitably chosen input f_* and observing the associated output

$$(1.10) \qquad g_* = \mathbf{\Gamma}(q) := \mathbf{R}_1(T;q)f_*,$$

suffices to identify q in (1.1), i.e., that f_* can be chosen so that $\mathbf{\Gamma}$ is injective on $\mathcal{A}$. Section 3 will also include some additional remarks on possible computational implementation.

2.   Principal results

We assume throughout that Ω is a bounded region in $\mathbb{R}^n$ with 'sufficiently smooth' boundary $\partial\Omega$ for the relevant trace theory to apply for the operators $\mathbf{B} : u \mapsto u \Big|_{\partial\Omega}$, $\mathbf{C} : u \mapsto \frac{\partial u}{\partial \nu}$ and for the consideration of Neumann conditions. We also assume that the unknown coefficient q is in $L^\infty(\Omega)$; there is then no further loss of generality in assuming, as we shall do, that $q > 0$ since a substitution $v := e^{-\alpha t}u$ replaces q by $q + \alpha$ — also just replacing f, g by

$e^{-\alpha t}f$, $e^{-\alpha t}g$ so $q \mapsto R_1(T,q)$ will be injective if and only if $q \mapsto R_1(T, q+\alpha)$ is injective.

Let $\mathbf{A} = \mathbf{A}_q$ be the elliptic operator $\mathbf{A} = -\mathbf{\Delta} + q$ on $\mathcal{H} := L^2(\Omega)$ with domain $\mathcal{D} = \mathcal{D}(\mathbf{A}) := \{u \in H^2(\Omega) : \mathbf{C}u = 0\}$. We note at this point the existence of an orthonormal (with respect to $\mathcal{H}$) basis of eigenfunctions

$$(2.1) \qquad \qquad \mathbf{A}e_k = \lambda_k e_k$$

with $0 < \lambda_1 \leq \lambda_2 \cdots \to \infty$ since we have taken $q > 0$.

We introduce the Green's operator $\mathbf{G}$ defined by $\mathbf{G} : \varphi \mapsto u$ with

$$(2.2) \qquad -\mathbf{\Delta}u + qu = 0 \text{ on } \Omega, \qquad \mathbf{C}u = \varphi \in \mathcal{X} := L^2(\partial\Omega).$$

We certainly have $u \in H^1(\Omega)$ for arbitrary $\varphi \in \mathcal{X} = L^2(\partial\Omega)$ so, noting [7], [8] the equivalence of $H^s(\Omega)$ and $\mathcal{D}(\mathbf{A}^\vartheta)$ for $\vartheta = 2s$, we have

$$(2.3) \qquad \qquad \mathbf{A}^{1/2}\mathbf{G} : \mathcal{X} \xrightarrow{\text{cont.}} \mathcal{H}$$

(with $\mathbf{A}^\vartheta\mathbf{G} : \mathcal{X} \to \mathcal{H}$ for any $\vartheta < 3/4$ if $\partial\Omega$ is, e.g., in C^1). Then the solution u of

$$(2.4) \qquad \dot{u} + \mathbf{A}u = 0, \quad \mathbf{C}u = f(t) \quad \text{with } u\big|_{t=0} = 0$$

has the representation [3]

$$(2.5) \qquad \qquad u(t) = \int_0^t [\mathbf{A}^{1/2}\mathbf{S}(t-s)][\mathbf{A}^{1/2}\mathbf{G}]f(s)\,ds$$

where $\mathbf{S}(\cdot)$ is the (analytic) semigroup on $\mathcal{H}$ generated by $-\mathbf{A}$ so

$$(2.6) \qquad \qquad \|\mathbf{A}^\nu\mathbf{S}(t)\| \leq Mt^{-\nu}.$$

From (1.3) and the form of (2.5), we then see that $\mathbf{R}_1$ is a convolution operator:

$$(2.7) \qquad [\mathbf{R}_1 f](t) = g(t) := \mathbf{B}u(t) = \int_0^\infty \mathbf{K}_1(t-s)f(s)\,ds$$

with the kernel $\mathbf{K}_1(\cdot) = \mathbf{K}_1(\cdot; q)$ given by

$$(2.8) \qquad \qquad \mathbf{K}_1(t) := \begin{cases} 0 & \text{for } t \leq 0 \\ \mathbf{BAS}(t)\mathbf{G} & \text{for } t > 0 \end{cases}$$

where, noting (2.3), (2.6), and

$$(2.9) \qquad \qquad \mathbf{BA}^{-\gamma} : \mathcal{H} \xrightarrow{\text{cont.}} \mathcal{X} \qquad (\text{any } \gamma > 1/4).$$

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we may write

$$\mathbf{BAS}(t)\mathbf{G} = \left[\mathbf{BA}^{-\gamma}\right]\left[\mathbf{A}^{1/2+\gamma}\mathbf{S}(t)\right]\left[\mathbf{A}^{1/2}\mathbf{G}\right]$$

with $1/4 < \gamma < 1/2$ to see that $\|\mathbf{K}_1(\cdot)\|$ is integrable whence $\mathbf{R}_1$ is, e.g., a continuous operator from $\mathcal{F}_T := L^2((0,T) \times \partial\Omega)$ to itself

At this point it is convenient to shift to the Fourier representation for the semigroup. Using (2.1) in (2.8) gives the series representation

(2.10) $$\mathbf{K}_1(t)\xi = \sum_k \lambda_k e^{-\lambda_k t} \langle e_k, \mathbf{G}\xi\rangle \mathbf{B}e_k$$

for $t > 0$. What we will actually need is the Laplace transform of this:

(2.11)
$$
\begin{aligned}
\hat{\mathbf{K}}_1(s)\xi &:= \int_0^\infty e^{-st}\mathbf{K}_1(t)\xi\, dt \\
&= \sum_k \lambda_k \int_0^\infty e^{-st}e^{-\lambda_k t}dt\langle e_k, \mathbf{G}\xi\rangle \mathbf{B}e_k \\
&= \sum_k \frac{\lambda_k}{s + \lambda_k}\langle e_k, \mathbf{G}\xi\rangle \mathbf{B}e_k \\
&= \mathbf{BA}(s + \mathbf{A})^{-1}\mathbf{G}\xi \qquad \text{for } s > 0.
\end{aligned}
$$

Note that the final form of this easily gives boundedness on $\mathcal{X}$ of $\hat{\mathbf{K}}_1(s)$ for $s \geq 0$ so we have no difficulties justifying convergence for the series and our manipulations. More precisely, we observe that everything certainly works well for the 'core' of the operator (specification for ξ in a suitable dense set of 'nice' functions) and then we can extend by continuity, using the final form.

With boundary conditions and initial conditions, the wave equation (1.6) now becomes

(2.12) $$\ddot{w} + \mathbf{A}w = 0, \quad \mathbf{C}w = f \quad \text{with } w = 0 = \dot{w} \text{ at } t = 0$$

and the response operator is the map

$$\mathbf{R}_2 = \mathbf{R}_2(\bar{T}, q) : f \mapsto \mathbf{B}w$$

with w defined by (2.12) for the time interval $(0, \bar{T})$. It is well-known that this $\mathbf{R}_2$ is a bounded operator from, e.g., $\mathcal{F}_{\bar{T}} := L^2((0, \bar{T}) \times \partial\Omega)$ to itself.

We proceed directly to the 'separation of variables' solution, again expanding with respect to the orthonormal basis $\{e_k\}$,

$$w = \sum_k y_k(t)e_k, \qquad \mathbf{G}f = \sum_k \varphi_k(t)e_k,$$

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one easily verifies from (2.12) that each $y_k(\cdot)$ is the solution of the ordinary differential equation

$$\ddot{y} + \lambda_k y = \lambda_k \varphi_k(t) \quad \text{with } y(0) = 0 = \dot{y}(0)$$

whence, noting that the assumed positivity $q > 0$ gives $\lambda_k > 0$, one has

$$y_k(t) = \mu_k \int_0^t [\sin \mu_k(t-s)] \varphi_k(s)\, ds \quad \left(\mu_k := \sqrt{\lambda_k} \right).$$

Substituting, this gives the series representation

$$(2.13) \quad [\mathbf{R}_2 f](t) = \mathbf{B} w(t) = \int_0^t \sum_k \mu_k [\sin \mu_k(t-s)] \langle e_k, \mathbf{G} f(s) \rangle \mathbf{B} e_k\, ds$$

so we see that $\mathbf{R}_2$ is a convolution operator: $f \mapsto \mathbf{K}_2 * f$ with the kernel $\mathbf{K}_2(\cdot)$ given, corresponding to (2.10), by the series

$$(2.14) \qquad\qquad \mathbf{K}_2(t) \xi = \sum_k \mu_k [\sin \mu_k t] \langle e_k, \mathbf{G} \xi \rangle \mathbf{B} e_k.$$

Again, we need the Laplace transform of this:

$$
(2.15) \quad
\begin{aligned}
\hat{\mathbf{K}}_2(s) \xi \;\; &:= \;\; \int_0^\infty e^{-st} \mathbf{K}_2(t) \xi\, dt \\
&= \;\; \int_0^\infty e^{-st} \sum_k \mu_k [\sin \mu_k t] \langle e_k, \mathbf{G} \xi \rangle \mathbf{B} e_k\, dt \\
&= \;\; \sum_k \mu_k \int_0^\infty e^{-st} [\sin \mu_k t]\, dt\, \langle e_k, \mathbf{G} \xi \rangle \mathbf{B} e_k \\
&= \;\; \sum_k \frac{\lambda_k}{s^2 + \lambda_k} \langle e_k, \mathbf{G} \xi \rangle \mathbf{B} e_k \\
&= \;\; \mathbf{B} \mathbf{A} (s^2 + \mathbf{A})^{-1} \mathbf{G} \xi \qquad \text{for } s > 0.
\end{aligned}
$$

Again, we think of these manipulations as performed for 'nice' ξ with the result then extended by continuity, using the final form. Comparing (2.15) with (2.11) gives our key identity:

$$(2.16) \qquad\qquad \hat{\mathbf{K}}_2(s; q) \equiv \hat{\mathbf{K}}_1(s^2; q) \quad \text{for } s > 0.$$

Returning to (2.8), we observe that, since $\mathbf{S}(\cdot)$ is an analytic semigroup, the operator function: $t \mapsto \mathbf{K}_1(t; q)$ is itself analytic in t (for complex t with positive real part). It follows that specification of $\mathbf{R}_1(T; q)$ implies specification of the kernel $\mathbf{K}_1(t; q)$ for $0 < t < T$ and so, by analyticity, uniqueness of

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the determination of $\mathbf{K}_1(\cdot\,; q)$ on $(0, \infty)$. This means that the Laplace transform $\hat{\mathbf{K}}_1(\cdot\,; q)$ is uniquely determined — as is the Laplace transform $\hat{\mathbf{K}}_2(\cdot\,; q)$ by the identity (2.16). By the standard uniqueness results for Laplace transforms, this means that $\mathbf{K}_2(t; q)$ is determined for $t > 0$ so the convolution operator $\mathbf{R}_2(\bar{T}; q)$ is uniquely determined for arbitrary $\bar{T} > 0$. We have thus proved[2] the asserted implication:

THEOREM 1: *Suppose it is known, for some bounded Ω in $\mathbb{R}^n$ and a set $\boldsymbol{\mathcal{A}}$ of bounded functions on Ω, that (1.7) holds for some $\bar{T}$. Then (1.5) holds for arbitrary $T > 0$.*
[Equivalently, if $q, \hat{q} \in \boldsymbol{\mathcal{A}}$ with $q \not\equiv \hat{q}$, then $\mathbf{R}_1(T, q) \not\equiv \mathbf{R}_1(T, \hat{q})$ for all $T > 0$.]

From [12] we have, re-stated in our notation, the following[3] result:

THEOREM (R-S): *Let Ω be a bounded region in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Then (1.7) holds for $\boldsymbol{\mathcal{A}} = L^\infty(\Omega)$.*

Combining this with Theorem 1, we immediately obtain the desired indentifiability result for (1.1).

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[2]The argument also provides a partial converse to the implication: if one could independently show uniqueness of the correspondence $\mathbf{R}_1(T; q) \leftrightarrow q$ (for some T and some class of q), then one would necessarily have uniqueness for $\mathbf{R}_2(\cdot\,; q) \leftrightarrow q$ — with observation now needed on all of $\mathbb{R}_+$ since analyticity in t is unavailable for $\mathbf{R}_2$ to get uniqueness from an interval without further information.

[3]We are indebted, for the reference to [12], to a referee for a previous version of this paper which referred, instead, to a sequence of recent papers [4], [5], [1], [2], [6] by M. Belishev and others which provide a reconstruction algorithm for q, justified under a control-theoretic hypothesis that the pair $[\Omega, q]$ is *normal* — for $0 < t < T_*$ the set of approximately reachable states by boundary control on $(0, t)$ is all of $\mathcal{H}_t := \{v \in \mathcal{H} : v(x) = 0 \text{ if } |x - \partial\Omega| > t\}$. Using duality, a sufficient condition for this normality is that $\partial\Omega$ and q be analytic for applicability of the classical Holmgren–John Uniqueness Theorem — although we note that this has quite recently been extended to the non-analytic case by Tataru [17] (see, also, related results by Robbiano [13] and by Hörmander [9]). In comparison with [12], we observe that considerable regularity may be needed for normality in the reconstruction, but not for the (nonconstructive) injectivity of: $q \mapsto R_2(\bar{T}, q)$. On the other hand, using the results in [17] one can get uniqueness results applying to observation on a part of the boundary while, also, the results in, e.g., [6] consider more general wave equations

$$(2.17) \qquad\qquad \rho(x)w_{tt} = \boldsymbol{\nabla} \cdot (\mu(x)\boldsymbol{\nabla}w) - qw$$

where any two of the three coefficient functions ρ, μ, q are assumed known with the third coefficient to be recovered. The argument in Section 2 for our key identity (2.16) is valid also for these settings so one would obtain corresponding identifiability results for the parabolic case. We view these as directions for future extensions of our present results.

COROLLARY: *Let Ω be a bounded region in $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$. Assume it is known that the coefficient q in (1.1) is in $\boldsymbol{\mathcal{A}} = L^\infty(\Omega)$. Then q is uniqely determined by $\mathbf{R}_1(T, q)$.*

The observation that verification of our manipulations on a dense set is sufficient could become more significant if we wished to consider variations on the operator. In particular, if we wished to use Dirichlet data as input instead and then observe the corresponding Neumann data (reversing the roles of $\mathbf{B}$, $\mathbf{C}$), then the regularity results would not be as cooperative and it is useful to observe that equality on a core suffices. Alternatively, one could obtain continuity using other boundary operators $\mathbf{B}, \mathbf{C}$ by suitable adjustment of the spaces, perhaps admitting different $\mathcal{X}_{\text{in}}$ for f and $\mathcal{X}_{\text{out}}$ for g so that one can then proceed exactly as we have done. For such possible generalization, we also note that we do not need the full strength of the present self-adjointness of $\mathbf{A}$, giving orthonormality of the eigenfunctions in (2.1) but only, e.g., that $\Sigma\alpha_k e_k \mapsto [\Sigma|\alpha_k|^2]^{1/2}$ is an equivalent norm.

3.   Identification with a single input

Theorem 1 and its Corollary require complete knowledge of $\mathbf{R}_1$ in order to determine q. Interpreted directly, this would mean that one would need knowledge of 'all possible' input/output pairs $[f, g]$ corresponding to (1.1), (1.2), (1.3) — requiring an infinite number of input/output 'experiments'. Using the form of $\mathbf{R}_1$ given in (2.7), (2.8) together with the regularity associated with (1.1), we now wish to show that a single experiment, observing the output g_* for a single properly chosen input f_*, will suffice to determine $\mathbf{K}(\cdot)$ and so q.

Taking any total set (e.g., an orthonormal basis) $\{\xi_k\}$ for $\mathcal{X} = L^2(\partial\Omega)$ and a sequence of times $0 = t_1 < t_2 < \ldots \to T$, we may set

$$(3.1) \qquad\qquad f_*(t) := \sum_{t_k < t} c_k \xi_k \qquad (0 < t < T)$$

with, e.g., $c_k := 2^{-k}$ ensuring convergence in $\mathcal{F}_T$. We set $\mathcal{I}_k := (t_k, t_{k+1})$, $\hat{\mathcal{I}}_k := (0, t_{k+1} - t_k)$ for $k = 1, 2, \ldots$ so $\mathcal{I} := \bigcup_k \mathcal{I}_k = [0, T) \setminus \{t_1, t_2, \ldots\}$. Clearly, $g_* := \mathbf{R}_1 f_*$ will be analytic in t on $\mathcal{I}$ with

$$(3.2) \qquad\qquad \dot{g}_*(t) = \sum_{t_k < t} c_k \mathbf{K}(t - t_k)\xi_k \qquad (t \in \mathcal{I}).$$

Although g_* certainly depends on q, we note that no *a priori* information about q is needed for this construction of f_*.

From (3.2) one first notes that knowledge of g_* on $\mathcal{I}_1$ just gives $\mathbf{K}(\cdot)\xi_1$ on $\hat{\mathcal{I}}_1$ by differentiation and therefore determines $\mathbf{K}(t)\xi_1$ for all $t > 0$ by analyticity. Next, knowing g_* on $\mathcal{I}_2$ we may subtract the now-known $\frac{1}{2}\mathbf{K}(t)\xi_1$ from $\dot{g}_*$ to obtain $\mathbf{K}(\cdot)\xi_2$ on $\hat{\mathcal{I}}_2$ whence, again by analyticity, $\mathbf{K}(t)\xi_2$ would be known for all $t > 0$. Recursively, we similarly obtain each $\mathbf{K}(\cdot)\xi_k$ on $\hat{\mathcal{I}}_k$ and so on $\mathbb{R}_+$ for $k = 3, 4, \dots$. Thus, a single pair $[f_*, g_*]$ constructed in this fashion will uniquely determine $\mathbf{K}(t)\xi_k$ for each k and all $t > 0$, hence will determine q.

The input function f_* is here piecewise constant in t but we note that replacing f_* as input by its time integral just produces the time integral of g_* as output and so also determines the original $\dot{g}_*$ of (3.2). Iterating this idea, we can use an input which is C^m in t for arbitrary m. We can get any desired spatial regularity by a suitable choice of $\{\xi_k(\cdot)\}$ as smooth functions on $\partial\Omega$.

THEOREM 2:   *Given Ω and any $T > 0$ one can select a suitable (smooth) function $f_* \in \mathcal{F}_T$ such that the corresponding map $\boldsymbol{\Gamma}$ of (1.10) is injective when considered on $\boldsymbol{\mathcal{A}} \subset L^\infty(\Omega)$.*

Fixing Ω, T, f_* as above, the injectivity of $\boldsymbol{\Gamma}$ in Theorem 2 means that (exact) observation of the output $g_* = \boldsymbol{\Gamma}(q)$ uniquely determines q. The obvious next question is whether this determination can be realized computationally: we would like an implementable procedure to recover q to any desired degree of accuracy provided we are able to compute to arbitrary accuracy and to produce the input and measure the output with arbitrary accuracy. This is far from obvious in view of the ill-posedness of the problem for any reasonable topologies.

The argument for justification of any computational schema for the problem sets this in the context of a sequence of increasingly accurate approximating problems and then asserts the convergence of the computed approximants q_j to the true coefficient q. We begin by writing our *a priori* information about q in the form:

(3.3) $$q \in \mathcal{K} \subset \boldsymbol{\mathcal{A}} \subset L^\infty(\Omega).$$

Our principal assumptions, here, are that $\mathcal{K}$ is a closed subset of $L^\infty(\Omega)$ and that $\boldsymbol{\Gamma} : \mathcal{K} \xrightarrow{\text{cont.}} \mathcal{G}$ for some suitable $\mathcal{G}$ topology with respect to which we can assume an increasingly accurate sequence of measurements $g_j \to g_*$.

Standard techniques of numerical analysis enable us to provide computational solutions for the defining equations, giving a sequence of approximations $\mathbf{\Gamma}_j \to \mathbf{\Gamma}$. We assume here that this is uniform convergence on $\mathcal{K}$ but note that the convergence need only be at q if, instead, we would have uniform equicontinuity on $\mathcal{K}$ of the $\mathbf{\Gamma}_j$. The various approaches to ill-posed problems now each provide some selection procedure: given $g_j, \mathbf{\Gamma}_j$ (with some accuracy estimate), there is a way to select $q_j \in \mathcal{K}$ so that $\mathbf{\Gamma}_j(q_j) \approx g_j$ and we may assume this is done in such a way as to have

(3.4) $$[\mathbf{\Gamma}_j(q_j) - g_j] \to 0 \quad \text{as } k \to \infty.$$

If $\mathcal{K}$ is compact, the generic argument is to obtain (for a subsequence) convergence $q_j \to \bar{q}$ for some $\bar{q}$. We then have

$$\mathbf{\Gamma}(\bar{q}) - g_* = [\mathbf{\Gamma}(\bar{q}) - \mathbf{\Gamma}(q_j)] + [\mathbf{\Gamma}(q_j) - \mathbf{\Gamma}_j(q_j)] + [\mathbf{\Gamma}_j(q_j) - g_j] + [g_j - g_*]$$

and, since each term on the right goes to 0, we conclude that $\mathbf{\Gamma}(\bar{q}) = g_*$. By our uniqueness theorem, we must then have $\bar{q} = q$. Finally, uniqueness of the limit makes the subsequence extraction irrelevant so, as desired, one has convergence of the sequence of computed approximants to the true solution $(q_j \to q)$ in the sense of the $\mathcal{K}$ topology.

As a variant of this, suppose one were to know *a priori* only that $q \in L^\infty(\Omega)$ but did not know any specific bound. We then propose the selection procedure: Choose

(3.5) $$\|q_j\|_{L^2(\Omega)} + \|q_j\|_{L^\infty(\Omega)} \le \min + \varepsilon_j$$

subject to a constraint on the residual error

(3.6) $$\|\mathbf{\Gamma}_j(q_j) - g_j\| \le \varepsilon_j'.$$

We make the assumptions that $\varepsilon_j \to 0$ and also that $\varepsilon_j' \to 0$, giving (3.4) but with ε_j' large enough (in comparison to the error estimates for the computational map $\mathbf{\Gamma}_j$ and for the observation g_j) that q, itself, is permitted to compete in the minimization, i.e., that (3.6) is satisfied with $q_j = q$.

THEOREM 3: *The computational procedure determined by (3.5), (3.6) provides a sequence (q_j) which converges strongly to the true q in $L^p(\Omega)$ for all finite p.*

PROOF: If we set

$$\alpha_j := \|q_j\|_{L^2(\Omega)}, \ \alpha := \|q\|_{L^2(\Omega)}, \ \beta_j := \|q_j\|_{L^\infty(\Omega)}, \ \beta := \|q\|_{L^\infty(\Omega)},$$

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then (3.5), with the admissibility of q in (3.6), gives

$$(3.7) \qquad \limsup[\alpha_j + \beta_j] \le [\alpha + \beta].$$

Since this means $\{\alpha_j\}$ is bounded, we must have, for a subsequence, weak convergence in $L^2(\Omega)$, i.e., $q_j \rightharpoonup \hat{q}$. Further, convexity gives

$$(3.8) \qquad \|\hat{q}\|_{L^2(\Omega)} =: \hat{\alpha} \le \liminf \alpha_j, \quad \|\hat{q}\|_{L^\infty(\Omega)} =: \hat{\beta} \le \liminf \beta_j.$$

The hypotheses, together with (3.6), ensure that

$$\lim \mathbf{\Gamma}(q_j) = \lim \mathbf{\Gamma}_j(q_j) = \lim g_j = g_* := \mathbf{\Gamma}(q).$$

Now let u_j be the solution of

$$(3.9) \qquad u_t = \mathbf{\Delta}u - q_j u, \quad u_\nu = f_*, \quad u\big|_{t=0} = 0$$

and observe that the uniform L^∞ bound on q_j gives the standard (uniform) bound on u_j in $L^2([0,T] \to H^1(\Omega))$ and so also a uniform bound on $\dot{u}_j$ in $L^2([0,T] \to H^{-1}(\Omega))$. Using the Aubin Compactness Theorem, we may extract a further subsequence to have $u_j \to \hat{u}$ in, say, $L^2([0,T] \to H^s(\Omega))$ for any $s < 1$. From the weak formulation of the problem, one easily sees that for $q_j \rightharpoonup \hat{q}$ one has $\hat{u}$ satisfying the limit equation. Since the boundary trace is closed when applied to solutions of (3.9) and we already know that $\mathbf{C}u_j = g_j \to g_*$, it follows that $\mathbf{\Gamma}(\hat{q}) = g_*$, i.e., $\mathbf{\Gamma}(\hat{q}) = \mathbf{\Gamma}(q)$. Since (3.8) gives $\hat{q} \in L^\infty(\Omega)$, Theorem 2 now gives $\hat{q} = q$ and uniqueness of this limit means that we may ignore the previous extractions of subsequences. Since this gives $\hat{\alpha} = \alpha$ and $\hat{\beta} = \beta$, it follows from (3.7), (3.8) that $\alpha_j \to \alpha$. This, together with the weak convergence $q_j \rightharpoonup q$, gives strong convergence $q_j \to q$ in the Hilbert space $L^2(\Omega)$. As Ω is bounded, this immediately gives $L^p(\Omega)$. convergence for $p \le 2$ and the presence of an $L^\infty(\Omega)$ bound also gives $L^p(\Omega)$ convergence for all $p < \infty$. ∎

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