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Existence of generalized solutions for ordinary
differential equations in Banach spaces*

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ABSTRACT:

1. Introduction

We will be considering the existence of solutions of ordinary differential equations in Banach spaces, taking these to have the nominal form

$$(1.1) \quad \dot{x} = \mathbf{A}x + f(x), \quad x(0) = \hat{\xi}_0$$

on some interval $[0, T]$. Here $x(\cdot)$ takes values in the Banach space \mathcal{X} and \mathbf{A} is the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$ of linear operators on \mathcal{X} . $f : \mathcal{X} \rightarrow \mathcal{X}$. We are indebted to [8] for an excellent survey of the extensive research on this problem; see also references there, especially [6].

For uniformly Lipschitzian $f : \mathcal{X} \rightarrow \mathcal{X}$, a standard formulation of ‘solution’ for (1.1) is in terms of the integral equation

$$(1.2) \quad x(t) = \mathbf{S}(t)\hat{\xi}_0 + \int_0^t \mathbf{S}(t-s)f(x(s))ds \quad (0 \leq t \leq T),$$

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where, through Banach's Contraction Mapping Principle (CMP), the uniform Lipschitz condition ensures existence. When f is merely continuous, however, Peano [7] showed local existence for $\mathcal{X} = \mathbb{R}^n$, but Godunov [5] (following an example by Dieudonné [3]) showed that this fails for all infinite dimensional Banach spaces. Further, for discontinuous f a more general solution notion is appropriate: we modify (1.2) to define a 'solution' of (1.1) as a pair of \mathcal{X} -valued functions $[x, y]$ on $[0, T]$ such that

$$(1.3) \quad x(t) = \mathbf{S}(t)\hat{\xi}_0 + \int_0^t \mathbf{S}(t-s)y(s) ds \quad (0 \leq t \leq T)$$

where, rather than merely having $y(s) = f(x(s))$ we extend the nonlinearity, following Fillipov [4], and require that

$$(1.4) \quad y(s) \in F_0(x(s)) \quad (0 \leq s \leq T) \text{ with } F_0(\hat{\xi}) = \bigcap_{\varepsilon > 0} F_\varepsilon(\hat{\xi})$$

or, more precisely, that

$$(1.5) \quad y(\cdot) \in \mathfrak{Y}_0(x(\cdot)) = \bigcap_{\varepsilon > 0} \mathfrak{Y}_\varepsilon(x) \quad \text{with} \\ \mathfrak{Y}_\varepsilon(x) = \{y \in L^2([0, T] \rightarrow \mathcal{X}) : y(s) \in F_\varepsilon(x(s)) \text{ ae } s \in [0, T]\}.$$

Noting that only the set-valued function $F_0(\cdot)$ is now relevant, we also permit $f(\cdot)$ to be set-valued — specifically permitting the possibility that $f(\xi) = \emptyset$ — and define

$$(1.6) \quad F_\varepsilon(\hat{\xi}) = F_\varepsilon(\hat{\xi}; f) = \overline{\text{hull}} \left\{ \zeta + \eta : \|\eta\| \leq \varepsilon, \zeta \in f(\xi), \|\xi - \hat{\xi}\| \leq \varepsilon \right\}.$$

Remark 1.1. Fillipov, working in a finite dimensional context, actually included exclusion of nullsets in the definition (1.6). The point of this was to restrict consideration to (locally) 'typical' values of $f(\xi)$ and we can accomplish the same purpose, instead, by 'trimming' $f(\cdot)$ — omitting atypical values even if this would give an empty 'value', which our present formulation admits. Although we do not consider this here, a similar trimming might be relevant to treating invariant sets, considering the existence of solutions which remain in some specified subset of \mathcal{X} . ■

Especially in view of the known nonexistence examples for continuous $f(\cdot)$, our interest here lies in providing a suitable condition on f to ensure existence of a solution in the sense described above. To this end, we consider a set Φ of functions $\varphi : \mathcal{X} \rightarrow \mathcal{X}$ such that

$$(1.7) \quad \begin{aligned} & i.] \quad \text{topologized by uniform convergence on compact sets,} \\ & \quad \Phi \text{ is compact,} \\ & ii.] \quad \Phi \text{ is convex as a subset of } C(\mathcal{X} \rightarrow \mathcal{X}), \\ & iii.] \quad \text{each } \varphi \in \Phi \text{ is Lipschitzian with a fixed constant } L. \end{aligned}$$

For $\xi \in \mathcal{X}$ we write $\Phi(\xi)$ for $\{\varphi(\xi) : \varphi \in \Phi\}$ and note that (1.7) ensures that each such $\Phi(\xi)$ is a compact convex subset of \mathcal{X} . Our main result, then, is the following:

Theorem 1.2. *Let \mathbf{A} be the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$ on the Banach space \mathcal{X} and let f be a set-valued function on \mathcal{X} . Suppose there is a set Φ , satisfying (1.7) as above, such that*

$$(1.8) \quad \mathcal{D} = \mathcal{D}(f, \Phi) = \left\{ \xi : \hat{f}(\xi) = f(\xi) \cap \Phi(\xi) \neq \emptyset \right\}$$

is dense in \mathcal{X} . Then the ordinary differential equation (1.1) has a global solution $x(\cdot)$ in the sense of (1.3), (1.5), (1.6).

Remark 1.3. The requirement that \mathcal{D} be dense will ensure that $F_0(\xi)$ is non-empty for each ξ , although it is worth noting at this point that $\xi \mapsto [\mathbf{A}\xi + f(\xi)]$ may actually be undefined for every ξ — the dense domain \mathcal{D} of (1.8) might be entirely disjoint from the dense domain of the infinitesimal generator \mathbf{A} . Thus, (1.1) cannot at all be interpreted pointwise, but only through the integrated form (1.3).

It is easy to see that in the finite dimensional case ($\mathcal{X} = \mathbb{R}^n$) the introduction of Φ is not at all a local constraint, but only a growth condition on the nonlinearity, irrelevant for local existence. It is in the infinite dimensional case that our hypotheses will provide a necessary compactness, e.g., excluding the examples of [3] and [5]. ■

We emphasize that our present concern is only with the *existence* of solutions, not with uniqueness. Standard examples in \mathbb{R} and \mathbb{R}^n show that additional hypotheses are generally needed to ensure uniqueness and this is,

of course, also a major concern of semigroup theory, both linear and nonlinear. For more general concerns with possible nonuniqueness we refer, e.g., to [4] and also note [10], [1], [2]. Nevertheless, we do have a modicum of well-posedness, in that the solution sets turn out to depend upper semicontinuously on the data.

2. Strategy

Our strategy for the proof of Theorem 1.2 will be the usual strategy for proving Peano's Theorem:

- Use the 'forward Euler' approach to construct a sequence of approximations $\{[x_n, y_n]\}$.
- Use compactness to show suitable convergence of some subsequence of these approximations to a limit $[\bar{x}, \bar{y}]$.
- Verify that this limit $[\bar{x}, \bar{y}]$ satisfies (1.3), (1.5), (1.6).

It is precisely the lack of local compactness in infinite dimensional spaces which leads to the counterexamples of [3] and [5]. The first and third steps of this argument are quite straightforward and the point of this paper is that that our hypothesis (1.7), (1.8) can provide the compactness needed for the second step. Our principal tools for this will be a representation for y and a general topological result from [9]:

Theorem 2.4. *Let (\mathcal{S}, d) be a complete metric space and \mathcal{M} an arbitrary index set; for $m \in \mathcal{M}$, let $G(\cdot, m) : \mathcal{S} \rightarrow \mathcal{S}$ satisfy a Lipschitz condition:*

$$(2.1) \quad d(G(\sigma, m), G(\sigma', m)) \leq \vartheta d(\sigma, \sigma') \quad (\sigma, \sigma' \in \mathcal{S})$$

(for some $\vartheta < 1$, independent of m) so there is a fixpoint $\sigma_m = G(\sigma_m, m)$ for each $m \in \mathcal{M}$. Now suppose that the set

$$G(\mathcal{K}, \mathcal{M}) = \{G(\sigma, m) : \sigma \in \mathcal{K}, m \in \mathcal{M}\}$$

has compact closure in \mathcal{S} for every compact subset $\mathcal{K} \subset \mathcal{S}$. Then every sequence $(\sigma_{m(k)})$ of fixpoints contains a subsequence converging in \mathcal{S} .

When we come to apply Theorem 2.4, the index set \mathcal{M} will have the form

$$(2.2) \quad \begin{aligned} \mathcal{M} &= L^2([0, T] \rightarrow \underline{M}) & \text{where} \\ \underline{M} &= \{\text{Borel measures on } \Phi\} \subset [C(\Phi)]^*; \end{aligned}$$

note that the measures in $[C(\Phi)]^*$ are signed measures while the ‘Borel measures’ $\mu(\cdot)$ in \underline{M} above are positive with $\mu(\Phi) = 1$. We will then work with a representation of the form

$$(2.3) \quad y(s) = \int_{\Phi} \varphi(x(s)) m(s, d\varphi) \quad \text{with } [s \rightarrow m(s, \cdot)] \in \mathcal{M}.$$

Lemma 2.5. *If we have any weak- * convergent sequence $m_j \xrightarrow{*} \bar{m}$ in \mathcal{M} and correspondingly define y_j, \bar{y} by (2.3) with $x = x_j$ where $x_j \rightarrow \bar{x}$ uniformly on $[0, T]$, then we have weak convergence $y_j \rightarrow \bar{y}$ in $\mathfrak{Y} = L^2([0, T] \rightarrow \mathcal{X})$.*

PROOF: For any $\eta \in \mathfrak{Y}^* = L^2([0, T] \rightarrow \mathcal{X}^*)$ and y as in (2.3), we consider

$$\begin{aligned} \langle \eta, y \rangle_{\mathfrak{Y}} &= \int_0^T \langle \eta(s), y(s) \rangle_{\mathcal{X}} ds \\ &= \int_0^T \left\langle \eta(s), \int_{\Phi} \varphi(x(s)) m(s, d\varphi) \right\rangle_{\mathcal{X}} ds = \langle \hat{\eta}, m \rangle_{\mathcal{M}} \\ &\quad \text{with } \hat{\eta}(s, \varphi) = \langle \eta(s), \varphi(x(s)) \rangle_{\mathcal{X}} \quad \text{for ae } s \in [0, T], \varphi \in \Phi. \end{aligned}$$

Applying the above to $\langle \eta, y_j \rangle_{\mathfrak{Y}}$ and letting $m_j \xrightarrow{*} \bar{m}$, noting that the Lipschitz condition ensures that $\varphi(x_j(\cdot)) \rightarrow \varphi(\bar{x}(\cdot))$ uniformly for $\varphi \in \Phi$, we then get $\langle \eta, y_j \rangle_{\mathfrak{Y}} \rightarrow \langle \eta, \bar{y} \rangle_{\mathfrak{Y}}$. ■

3. Proof of Theorem 1.2

In this section we follow the strategy described in Section 2 to prove Theorem 1.2. For global existence it is sufficient to prove existence on $[0, T]$ with $T > 0$ arbitrary. We now proceed sequentially with the three steps.

We first construct approximate solutions. Fix N , let $t_n = nT/N$ for $n = 0, 1, \dots, N$, and then recursively define the approximate solution pair $[x_N, y_N]$ on the intervals $[t_n, t_{n+1}]$ as follows:

Take $\hat{\xi}_0$ as given in the initial condition and then, given $\hat{\xi}_n$ for $n < N$, arbitrarily choose some $\xi_n \in \mathcal{D}$ with $\|\xi_n - \hat{\xi}_n\| \leq T/N$ — possible as we have assumed $\mathcal{D} = \mathcal{D}(f, \Phi)$ is dense in \mathcal{X} . We can then arbitrarily choose $\zeta_n \in \hat{f}(\xi_n) \neq \emptyset$ and define

$$(3.1) \quad y_N(s) \equiv \zeta_n \quad x_N(s) = \mathbf{S}(s - t_n)\hat{\xi}_n + \int_{t_n}^s \mathbf{S}(s - r)\zeta_n dr$$

for $s \in (t_n, t_{n+1}]$. Finally, we set $\hat{\xi}_{n+1} = x_N(t_{n+1})$. Note that this construction gives the integral relation (1.3) for $[x_N, y_N]$.

The next step is the compactness argument. This will be the longest part of the proof and will rely entirely on (1.3) and the fact that for $nT/N < s \leq (n+1)T/N$ we have

$$(3.2) \quad y_N(s) = \zeta_n \in \Phi(\xi_n) \quad \text{with } \|\xi_n - x_N(nT/N)\| \leq T/N.$$

Note first that $\zeta_n \in \Phi(\xi_n)$ just means that $\zeta_n = \varphi_n(\xi_n)$ for some $\varphi_n \in \Phi$ so, defining a Borel measure $\mu_n \in \underline{M}$ by

$$\mu_n(A) = \begin{cases} 1 & \text{if } \varphi_n \in A \\ 0 & \text{else} \end{cases} \quad \text{for Borel sets } A \subset \Phi,$$

we have $\zeta_n = \int_{\Phi} \varphi(\xi_n) \mu_n(d\varphi)$. Collecting these, we define a piecewise constant $m_N : [0, T] \rightarrow \underline{M}$ by

$$(3.3) \quad m_N(s, \cdot) = \mu_n(\cdot) \quad \text{for } nT/N < s \leq (n+1)T/N \quad (0 \leq n < N).$$

This gives us

$$(3.4) \quad \begin{aligned} y_N(s) &= \zeta_n = \int_{\Phi} \varphi(\xi_n) m_N(s, d\varphi) \\ &= \int_{\Phi} \varphi(x_N(s)) m_N(s, d\varphi) + \eta_N(s) \quad \text{where} \\ \eta_N(s) &= \int_{\Phi} [\varphi(\xi_n) - \varphi(x_N(s))] m_N(s, d\varphi). \end{aligned}$$

We now set $\mathfrak{X} = C([0, T] \rightarrow \mathcal{X})$ and define $G(\cdot, m) : \mathfrak{X} \rightarrow \mathfrak{X}$ by

$$(3.5) \quad [G(x(\cdot), m)](t) = \mathbf{S}(t)\hat{\xi}_0 + \int_0^t \int_{\Phi} \mathbf{S}(t-s) \varphi(x(s)) m(s, d\varphi) ds,$$

parametrized by $m \in \mathcal{M}$. Note that (3.1) then gives

$$(3.6) \quad x_N(t) = [G(x_N, m_N)](t) + \int_0^t \eta_N(s) ds,$$

i.e., x_N is a fixpoint of $[G(\cdot, m_N) + e_N] : \mathfrak{X} \rightarrow \mathfrak{X}$ where $e_N = \int_0^t \eta_N$.

At this point we use a standard trick: selecting for \mathfrak{X} the exponentially weighted norm

$$\|x\|_{\mathfrak{X}} = \max \{e^{-2\gamma Lt} \|x(t)\|_{\mathcal{X}} : t \in [0, T]\}$$

where γ is a bound for $\|\mathbf{S}(t)\|$ on $[0, T]$. We then have, for $0 \leq t \leq T$,

$$\begin{aligned}
& e^{-2\gamma Lt} \| [G(x_1, m) - G(x_2, m)](t) \|_{\mathcal{X}} \\
&= e^{-2\gamma Lt} \left\| \int_0^t \mathbf{S}(t-s) \int_{\Phi} [\varphi(x_1(s)) - \varphi(x_2(s))] m(s, d\varphi) ds \right\|_{\mathcal{X}} \\
&\leq e^{-2\gamma Lt} \int_0^t \|\mathbf{S}(t-s)\| \int_{\Phi} \|\varphi(x_1(s)) - \varphi(x_2(s))\|_{\mathcal{X}} m(s, d\varphi) ds \\
&\leq e^{-2\gamma Lt} \int_0^t \gamma \int_{\Phi} L \|x_1(s) - x_2(s)\|_{\mathcal{X}} m(s, d\varphi) ds \\
&= \gamma L \int_0^t e^{-2\gamma L(t-s)} [e^{-2\gamma Ls} \|x_1(s) - x_2(s)\|_{\mathcal{X}}] ds \\
&\leq \gamma L \int_0^t e^{-2\gamma L(t-s)} ds \|x_1 - x_2\|_{\mathcal{X}} \leq \frac{1}{2} \|x_1 - x_2\|_{\mathcal{X}}
\end{aligned}$$

Thus, using this norm, each $G(\cdot, m)$ is contractive on \mathcal{X} with the uniform contraction constant $\vartheta = 1/2$. As in Theorem 2.4, we will denote the unique fixpoint of $G(\cdot, m)$ by σ_m for each $m \in \mathcal{M}$.

We now complete verification of the hypotheses of Theorem 2.4, i.e., we proceed to show that $\overline{G(\mathcal{K}, \mathcal{M})}$ is compact in \mathcal{X} for any compact set $\mathcal{K} \subset \mathcal{X}$. To this end, note that continuity of the evaluation map: $[x, t] \mapsto x(t)$ shows that the image K_1 of $\mathcal{K} \times [0, T]$ is compact. Similarly, the image K_2 of $\Phi \times K_1$ under $[\varphi, \xi] \mapsto \varphi(\xi)$ is also compact in \mathcal{X} and $K_3 = \{\mathbf{S}(\tau)\zeta : 0 \leq \tau \leq T, \zeta \in K_2\}$ is compact. Also, $K_0 = \{\mathbf{S}(t)\hat{\xi}_0 : t \in [0, T]\}$ is compact. The definition of $G(\cdot, m)$ then ensures that each $z \in G(\mathcal{K}, \mathcal{M})$ takes its values in the compact set

$$K = K(\mathcal{K}) = K_0 + [0, T] \overline{\text{hull}} \{0, K_3\} \subset \mathcal{X}.$$

Now let

$$\begin{aligned}
\beta(\mathcal{K}) &= \max \|\zeta\|_{\mathcal{X}} : \zeta \in K(K) \\
\omega(h, \mathcal{K}) &= \max \{\|\mathbf{S}(\tau)\xi - \xi\| : \xi \in K(\mathcal{K}), \tau \in [0, h]\}.
\end{aligned}$$

By the compactness of $K = K(\mathcal{K})$, we have $\beta(\mathcal{K}) < \infty$ and $\omega(h, \mathcal{K}) \rightarrow 0$ as $h \rightarrow 0$ since \mathbf{S} is a C_0 semigroup. Next, we note that, for any $z(\cdot) \in G(\mathcal{K}, \mathcal{M})$ and any $0 \leq t < t' \leq T$, one has from (1.3)

$$z(t') = \mathbf{S}(t' - t)z(t) + \int_t^{t'} \mathbf{S}(t' - s)\zeta(s) ds$$

with $\zeta(s) \in K$. It follows that for $|t' - t| \leq h$ one has

$$\|z(t+h) - z(t)\|_{\mathcal{X}} \leq \omega(h, K) + h\gamma\beta(K)$$

so $G(\mathcal{K}, \mathcal{M})$ is uniformly equicontinuous. The desired compactness of $\overline{G(\mathcal{K}, \mathcal{M})}$ now follows from the (generalized) Arzelà-Ascoli Theorem.

We can now apply Theorem 2.4 to show that the ‘fixpoint set’ $\mathcal{K}_* = \overline{\{\sigma_m : m \in \mathcal{M}\}}$ is compact in \mathfrak{X} . In particular, for the sequence (σ_{m_N}) , there is a convergent subsequence, with $N = N(j) \rightarrow \infty$. Abusing notation slightly, we write σ_j for $\sigma_{m_{N(j)}}$ so we have $\sigma_j \rightarrow \bar{x}$ uniformly on $[0, T]$.

4. Proof of Theorem 1.2 (continued)

In the previous section we constructed a sequence of approximate solutions (x_N) which were fixpoints of maps $[G(\cdot, m_N) + e_N] : \mathfrak{X} \rightarrow \mathfrak{X}$. Applying Theorem 2.4, we showed subsequential convergence $\sigma_j \rightarrow \bar{x}$ for the fixpoints $\sigma_j = G(\sigma_j, m_{N(j)})$.

Our next task is to show that also $x_j \rightarrow \bar{x}$ (again abusing notation slightly in writing x_j for $x_{N(j)}$). Returning to (3.1) and (3.2) and taking $N = N(j)$, we now write

$$\begin{aligned} x_j(t) &= \mathbf{S}(t)\hat{\xi}_0 + \int_0^t \int_{\Phi} \mathbf{S}(t-s)\varphi(\hat{x}_j(s)) m_N(s, d\varphi) ds \\ (4.1) \quad \text{where} \quad \hat{x}_j(s) &= \zeta_n \quad \text{for } s \in [nT/N, (n+1)T/N] \\ \text{with} \quad &\|\zeta_n - x_j(nT/N)\|_{\mathcal{X}} \leq T/N, \\ \sigma_j(t) &= \mathbf{S}(t)\hat{\xi}_0 + \int_0^t \int_{\Phi} \mathbf{S}(t-s)\varphi(\sigma_j(s)) m_N(s, d\varphi) ds. \end{aligned}$$

For $s \in [nT/N, (n+1)T/N]$ we have

$$\begin{aligned} \|\hat{x}_j(s) - \sigma_j(s)\|_{\mathcal{X}} &\leq T/N + \|x_j(nT/N) - \sigma_j(nT/N)\|_{\mathcal{X}} + \|\sigma_j(nT/N) - \sigma_j(s)\|_{\mathcal{X}} \\ &\leq T/N + \rho_j(nT/N) + \hat{\omega}(T/N) \leq [T/N + \hat{\omega}(T/N)] + \rho_j(s) \end{aligned}$$

where

$$(4.2) \quad \rho_j(s) = \max \{\|x_j(t) - \sigma_j(t)\|_{\mathcal{X}} : 0 \leq t \leq s\}$$

(making $\rho_j(\cdot)$ nondecreasing, so $\rho_j(nT/N) \leq \rho_j(s)$) and

$$(4.3) \quad \hat{\omega}(h) = \max \{\|\sigma(t') - \sigma(t)\|_{\mathcal{X}} : |t' - t| \leq h, \sigma \in \mathcal{K}_*\}.$$

Note that the compactness of \mathcal{K}_* in \mathfrak{X} ensures that $\hat{\omega}(h) = \hat{\omega}(h, \mathcal{K})$ is well-defined with $\hat{\omega}(h) \rightarrow 0$ as $h \rightarrow 0$. We now proceed to estimate ρ_j : For any $t \in [0, T]$ we have

$$\begin{aligned} & \|x_j(t) - \sigma_j(t)\|_{\mathcal{X}} \\ &= \left\| \int_0^t \int_{\Phi} \mathbf{S}(t-s) [\varphi(\hat{x}_j(s)) - \varphi(\sigma_j(s))] m_N(s, d\varphi) ds \right\|_{\mathcal{X}} \\ &\leq \gamma L \int_0^t \|\hat{x}_j(s) - \sigma_j(s)\|_{\mathcal{X}} ds \end{aligned}$$

so, again noting that $\rho_j(\cdot)$ is nondecreasing, we have

$$\rho_j(t) \leq \gamma L t [h + \hat{\omega}(h)] + \gamma L \int_0^t \rho_j(s) ds$$

with $h = h_j = T/N(j) \rightarrow 0$. Applying the Gronwall Inequality to this, we have $\|x_j - \sigma_j\|_{\mathfrak{X}} \leq C[h + \hat{\omega}(h)]$ with a constant C depending only on $\mathcal{K}_*, T, \gamma, L$. Since $\sigma_j \rightarrow \bar{x}$, this shows that we also have $x_j \rightarrow \bar{x}$ uniformly on $[0, T]$ for some $\bar{x} \in \mathfrak{X}$.

We turn now to the sequence (y_j) which, corresponding to (4.1), is given by

$$(4.4) \quad y_j(s) = \int_{\Phi} \varphi(\hat{x}_j(s)) m_{N(j)}(s, d\varphi),$$

i.e., y_j is given by (2.3) with $m = m_{N(j)}$ and $x = \hat{x}_j$; note that we have shown uniform convergence $\hat{x}_j \rightarrow \bar{x}$ on $[0, T]$. Clearly \mathcal{M} is closed and, since $\mu \in \underline{M}$ gives $0 \leq \mu(\cdot) \leq 1$, is bounded as a subset of the dual space

$$L^2([0, T] \rightarrow [C(\Phi)]^*) = [L^2([0, T] \rightarrow C(\Phi))]^*.$$

Hence, by Alaoglu's Theorem, \mathcal{M} is weak-* compact and (again extracting a further subsequence, if necessary) we have $m_{N(j)} \xrightarrow{*} \bar{m}$ for some $\bar{m} \in \mathcal{M}$. We may then apply Lemma 2.5 to have weak convergence $y_j \rightharpoonup \bar{y}$ in $\mathfrak{Y} = L^2([0, T] \rightarrow \mathcal{X})$ with $[\bar{x}, \bar{y}]$ also related by (2.3) using \bar{m} . Given any $\xi \in \mathcal{X}^*$ and $t \in [0, T]$ we set $\eta(s) = [\mathbf{S}(t-s)]^* \xi$ on $[0, t]$ and $\eta(s) = 0$ for $s > t$. Then $\eta \in \mathfrak{Y}^*$ and

$$\langle \xi, x_j(t) \rangle_{\mathcal{X}} = \langle \xi, \mathbf{S}(t) \hat{\xi}_0 \rangle_{\mathcal{X}} + \langle \eta, y_j \rangle_{\mathfrak{Y}}$$

since our construction gave (1.3) for $[x_j, y_j]$, as noted following (3.1). Since we have $x_j(t) \rightarrow \bar{x}(t)$ in \mathcal{X} and $y_j \rightarrow \bar{y}$ in \mathfrak{Y} as $j \rightarrow \infty$, we also have (1.3) for $[\bar{x}, \bar{y}]$ in the limit.

To complete the proof of Theorem 1.2, we now verify (1.5) for \bar{y} . Note that each \mathfrak{Y}_ε (for $\varepsilon > 0$) is closed and convex, hence weakly closed in \mathfrak{Y} , with $\mathfrak{Y}_\varepsilon \subset \mathfrak{Y}_{\varepsilon'}$ for $0 < \varepsilon < \varepsilon'$. Our construction selected

$$y_j(s) \in \hat{f}(\hat{x}_j(s)) \subset f(\hat{x}_j(s))$$

so $y_j(s) \in F_\varepsilon(\bar{x}(s))$ whenever $\|\hat{x}_j(s) - \bar{x}(s)\|_{\mathcal{X}} \leq \varepsilon$. Since we have already noted that $\hat{x}_j \rightarrow \bar{x}$ uniformly on $[0, T]$, we have (for every $\varepsilon > 0$) $y_j \in \mathfrak{Y}_\varepsilon$ for large enough j , depending on ε . The weak convergence $y_j \rightharpoonup \bar{y}$ then also ensures that $\bar{y} \in \mathfrak{Y}_\varepsilon$ in the limit. This, for each $\varepsilon > 0$, gives (1.5).

This shows that $[\bar{x}, \bar{y}]$ is a solution of (1.1) in the sense we have determined and so completes the proof of Theorem 1.2. ■

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