DRAFT

Global stability of the equilibrium for scalar delay differential equations

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Abstract

This version has a lot of inserted Comments to indicate what must be done and what will be in here when this is completed. [These Comments also note open questions, where possible, so for suggested results not noted as open one may assume the proofs are known, even if not yet included.]

This was originally intended as a revised version of what became [5] before publication, then became comments (initially addressed to Giang and Lenbury) toward the possibility of a successor to that paper since the earlier version of it had actually been submitted (and then accepted).

KEY WORDS: *delay differential equations, comparison theorem, convergence to equilibrium, Nicholson's blowfly model.*

1 Introduction

COMMENT: *Some additional explanatory material is needed in the Introduction as to our concerns (in particular, that we are* not *looking at all at questions regarding periodic solutions) and our strategy for the analysis.*

Our concerns have largely been motivated by consideration of population models, but the presentation of this paper in terms of interval maps has been strongly influenced by Proposition 1.1 of [6]. We will be considering, primarily, delay differential equations of the form

$$\dot{x} + x = f(x)\Big|_{t-\tau} \qquad \text{for } t > 0 \tag{1.1}$$

with $f(\cdot)$ continuous and fixed lag $\tau > 0$. Proposition 1.1 of [6] (see Lemma 3.1, below) asserts that, for any closed interval $\mathcal{I} \subset \mathbb{R}$, one has:

$$\begin{array}{l} \text{If } f(\mathcal{I}) \subset \mathcal{I}, \quad \text{then} \\ \;x(t) \in \mathcal{I} \text{ on } [s, s+\tau] \text{ implies } x(t) \in \mathcal{I} \text{ for all } t \geq s. \end{array} \tag{1.2}$$

for any solution x of (1.1), assuming $s + \tau \geq 0$. We will usually be assuming that the constitutive function satisfies the structural hypothesis

$$\begin{cases} f(r) > r & \text{if } r < 0, \\ f(r) < r & \text{if } r > 0, \end{cases} \tag{1.3}$$

or, equivalently, that

$$f(r)/r < 1 \quad \text{for } r \neq 0. \tag{1.4}$$

The hypothesis (1.3) implies that f has a (unique) fixed point $f(0) = 0$, so the constant $x \equiv 0$ is a steady state equilibrium solution. Our concern is the global asymptotic stability of this equilibrium solution, i.e., we seek conditions ensuring that, for every solution x of (1.1), the asymptotic range

$$\overline{\mathcal{I}} = [\overline{m}, \overline{M}] \quad \text{where} \begin{cases} \overline{m} = \liminf_{t\to\infty} x(t) \\ \overline{M} = \limsup_{t\to\infty} x(t) \end{cases} \tag{1.5}$$

will just be the single point $\{0\}$.

COMMENT: *It should be remarked that our analysis proceeds primarily by looking at the dependence of global asymptotic stability on the constitutive function $f(\cdot)$ for fixed delay $\tau > 0$. This must then be interpreted, for any fixed f, to see the dependence on τ.*

A variety of population models lead to scalar delay differential equations of the form

$$\frac{dy}{dt}(t) = [g(y)y](t-\tau) - \delta y(t) \tag{1.6}$$

where y is the population size, $g(\cdot)$ gives the population dependent reproduction rate, the lag τ is a maturation time or gestation period, and the constant δ is a death rate. For such a model, all of these are necessarily positive.

Not only can the population size never be meaningfully negative, but the model cannot represent a real population when $y \approx 0$. [For example, it would be unlikely to have births if $y(t) = 0$, even if $y(t-\tau)$ would seem to permit this in (1.6).] The derivation of (1.6) is by the deterministic evolution of the mean of a stochastic situation, justified by the Law of Large Numbers for large populations, but subject to stochastic fluctuation when y is too small for that.

We are then looking at autonomous situations in which there is a (unique) population size $\bar{r} > 0$ at which the birth and death rates balance: $g(\bar{r}) = \delta$. It is typical that deaths exceed births for greater populations (so this population size represents the steady state *carrying capacity* of the environment) while births exceed deaths when the population is below this capacity, i.e.,

$$\begin{cases} g(r) > \delta & \text{if } 0 < r < \bar{r}, \\ g(r) < \delta & \text{if } r > \bar{r}. \end{cases} \tag{1.7}$$

Thus, $\bar{r}$ provides a unique equilibrium: the constant function $y \equiv \bar{r}$ is a solution and there is no other nontrivial constant solution.

We will be specifically interested in Nicholson's blowfly model, for which g takes the form $g(r) = \mu e^{-\gamma r}$ with parameters $\mu, \gamma > 0$ — giving $\bar{r} = [\ln \mu/\delta]/\gamma > 0$ provided $\mu > \delta$. See Section 5.

COMMENT: *It is, of course, Section 5 which now is most in need of* *completion.*

It is immediate that this equilibrium solution is stable for the ordinary differential equation model, neglecting the delay by setting $\tau = 0$ in (1.6) — and, indeed, is a global attractor in this ordinary differential equation setting. Analyzing the ordinary differential equation, one obtains this stability for quite general continuous functions $g(\cdot)$, subject to (1.7), but it is well-known that this can fail in the presence of delay, especially when the delay is large.

In interpreting 'global' here, we first note that we are only considering $y \geq 0$ as meaningful and will neglect the trivial solution $y \equiv 0$. It turns out that our hypotheses then ensure strict positivity of nontrivial solutions of (1.6) for $t \geq \tau$ and the stability assertion we seek is the asymptotic convergence to equilibrium as $t \to \infty$ (in an appropriate topology) of all solutions with nontrivial non-negative initial data. We are also interested, when possible, in demonstrating *persistence* — that $\liminf_{t\to\infty} y(t) > 0$ for all nontrivial solutions — even when we cannot demonstrate global asymptotic stability of the equilibrium.

Similar equations arise in modeling ...

COMMENT: *This should be continued a bit, noting some situations other than population dynamics with similar equations ...; for example, compare the Introduction of [6].*

It will be convenient to transform the equation (1.6) somewhat. First choose the time unit to make $\delta = 1$ (assuming, of course that we start with $\delta > 0$), noting that there is a corresponding change in τ and $g(\cdot)$. We then could set $\hat{f}(r) = rg(r)$ but find it more convenient to work with the deviation from equilibrium. Thus, finally, we set

$$x(t) = y(t) - \bar{r}, \qquad f(r) = [r + \bar{r}]g(r + \bar{r}) - \bar{r} \tag{1.8}$$

and obtain for $x(\cdot)$ the delay differential equation (1.1). It is easy to verify that the modelling assumption (1.7) above for the population model (1.6) immediately gives our structural hypothesis (1.3) for this constitutive function $f(\cdot)$. It will be useful to note that the positivity of g gives also

$$f(r) \geq -\bar{r} \qquad \text{for } r \geq 0. \tag{1.9}$$

For constitutive functions f satisfying (1.3), we introduce the function $\psi = \psi(\cdot; f)$ defined by

$$\psi(\tilde{r}) = \begin{cases} \sup\{f(r) : \tilde{r} \le r \le 0\} & \text{for } \tilde{r} \le 0, \\ \inf\{f(r) : 0 \le r \le \tilde{r}\} & \text{for } \tilde{r} \ge 0. \end{cases} \tag{1.10}$$

Note that ψ is nonincreasing with $\psi(0) = 0$. In the presence of (1.3), one has $f(\mathcal{I}) = \psi(I)$ for any interval $\mathcal{I}$ containing 0. Our main sharpening of (1.2) will then be a delay-dependent estimate for the asymptotic range of a bounded solution:

$$\overline{m} \ge \left(1 - e^{-\tau}\right) \psi(\overline{M}), \qquad \overline{M} \le \left(1 - e^{-\tau}\right) \psi(\overline{m}) \tag{1.11}$$

COMMENT: *Complete this section by outlining the paper.*

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2 Some background

We begin this section by noting the equivalence of the delay differential equation (1.1) with the integral equation

$$x(t) = e^{-(t-s)} x(s) + \int_s^t e^{-(t-r)} f(x(r - \tau))\, dr \tag{2.12}$$

for $s \le t < \infty$. Of course, for solutions on $\mathbb{R}_+$ we take $s \ge 0$ here. For, e.g., $s = 0$, the requisite values of $x(0)$ and of $x(r - \tau)$ for $0 \le r \le \tau$ are *initial data* for the problem. It is immediately apparent from (2.12) that such solutions will be continuous for $t \ge 0$ and C^1 for $t > \tau$ even if the initial data is only measurable (with $f(x(\cdot))$ integrable).

More generally, if we denote by $z = z_{\nu,\tau}(\cdot)$ the fundamental solution of the linear delay differential equation

$$\dot{z} + z = \nu z(\cdot - \tau) \tag{2.13}$$

so $z(t) = 0$ for $t < 0$ with $z(0) = 1$, then we can easily verify the integral representation

$$x(t) = \left[z(t-s)x(s) + \nu \int_{-\tau}^{0} z([t-s]-[r+\tau])x(s+r)\,dr\right] + \int_{0}^{t-s} z(t-s-r)\,[f(x)-\nu x]\,(s+r-\tau)\,dr \tag{2.14}$$

for arbitrary ν. Note that

$$z_{\nu,\tau}(t) = e^{-t} \sum_{k=0}^{\lfloor t/\tau \rfloor} \frac{\left[\nu e^{\tau}\,(t-k\tau)_{+}\right]^{k}}{k!} \tag{2.15}$$

for $t \geq 0$, as again may easily be verified by the method of steps. Using (2.15), some manipulation shows that (2.14) reduces to (2.12) for arbitrary ν if $t \leq s + \tau$.

COMMENT: *Check the details of this.*

The equation (2.13) is the special case $f(r) = \nu r$ of (1.1). We will have exponential decay to 0 of all solutions of (2.13)— i.e., global asymptotic stability — if and only if all complex solutions λ of the characteristic equation

$$\lambda + 1 = \nu e^{-\tau\lambda} \tag{2.16}$$

have negative real parts: note, e.g., that the Laplace transform of $z_{\nu,\tau}$ is just $1/h$ where the characteristic function $h(\lambda) = \lambda + 1 - \nu e^{-\tau\lambda}$. This stability is impossible if $\nu \geq 1$, which we have, in any case, excluded by (1.3) and otherwise certainly holds for $\tau = 0$. By continuation as τ increases from 0, the critical case (Hopf bifurcation) occurs when one would have a pure imaginary solution $\lambda = i\omega$ of (2.16) so

$$1 = \nu \cos \tau\omega, \qquad \omega = \nu \sin \tau\omega,$$

which gives $1 + \omega^2 = \nu^2$ so $\omega = \pm\sqrt{\nu^2 - 1}$. Thus, we have global asymptotic stability for (2.13), exponential decay to 0,

- for $0 \leq \tau < \tau_0$ with

  $$\tau_0 = \tau_0(\nu) = \frac{\arccos(1/\nu)}{\sqrt{\nu^2 - 1}} \tag{2.17}$$

  (so $\tau(\nu) \sim \pi/2|\nu|$ as $\nu \to -\infty$) when $\nu < -1$.

- for arbitrary delay $\tau > 0$ when $-1 \leq \nu < 1$, so we then set $\tau_0(\nu) = \infty$.

For $0 \leq \tau < \tau_0(\nu)$ (with $\nu < 1$) we have exponential decay of the fundamental solution $z = z_{\nu,\tau}(\cdot)$ of (2.14) and (2.15) so, certainly, z will then be in $L^1(0, \infty)$; we set

$$Z_1 = Z_1(\nu, \tau) = \|z\|_{L^1(0,\infty)} \qquad \text{when } \tau < \tau_0(\nu). \tag{2.18}$$

COMMENT: *We would like an estimate for Z_1, giving the dependence on ν, τ. We see from (2.15) that z is positive on $\mathbb{R}_+$ when $0 \leq \nu$ (all $\tau \geq 0$) so then, using (2.13), we can calculate explicitly*

$$\begin{aligned} Z_1 \quad &= I := \int_0^\infty z(t)\,dt \\ &= \int_0^\infty [\nu z - \dot{z}]\,dt \\ &= \nu I \ - z\,\big|_0^\infty = \nu I + 1, \end{aligned}$$

giving $Z_1 = 1/(1-\nu)$. Since $|z_{\nu,\tau}| \leq z_{|\nu|,\tau}$, this also gives the rather crude estimate $Z_1 \leq 1/(1-|\nu|)$ for $-1 < \nu \leq 0$. Unfortunately, the range $-1 < \nu < 1$ is not really interesting for this and we have no *estimate (necessarily τ-dependent) when $\nu \leq -1$.*

We note that when $\tau = \tau_0(\nu) < \infty$ the linear delay differential equation (2.13) has periodic solutions $x(t) = c\cos\omega t$ with $\omega = \sqrt{\nu^2 - 1}$; the minimal period T is then $2\pi/\omega$ so

$$\frac{T}{\tau} = \frac{2\pi}{\arccos(1/\nu)} \ \longrightarrow 4- \quad \text{as } \nu \to -\infty.$$

[In general, the (minimal) period T would not be an integer multiple of the lag τ, although that is not impossible: for $\nu = \cos(2\pi/3) = \sqrt{3}/2$ this periodic solution of (2.13) has minimal period 3τ.]

COMMENT: *Check this.*

For the nonlinear problem (1.1), the linearization around equilibrium is just (1.1) with $\nu = f'(0)$ and we expect the analysis of *local* asymptotic stability to be as above. We note that it is known (cf., e.g., [4], [3], [1], using the Non-Ejective FixPoint Theorem of [2]), once one has bounded solutions, that there exist periodic solutions of (1.1) — so even local asymptotic stability is impossible — when $f'(0) = \nu < -1$ and the delay τ is past the onset of instability for the linearization, i.e., when τ is greater than $\tau_0(\nu)$ as given by (2.17). This represents a fundamental limitation on the attainable positive results for global asymptotic stability which we seek.

3 First results

We begin with (1.2), including here a slightly different proof from that in [6]; see also Remark 6.2.

Lemma 3.1 *Let $\mathcal{I} = [m, M]$ be a closed subinterval of $\mathbb{R}$ (possibly infinite) and let x be a solution of* (1.1):

$$\dot{x} + x = f(x(\cdot - \tau))$$

such that $\mathcal{I} \supset x([s, s+\tau])$ — i.e., $m \le x(t) \le M$ for $s \le t \le s + \tau$. Suppose also that $f(\mathcal{I}) \subset \mathcal{I}$ — i.e., that

$$m \le r \le M \text{ implies } m \le f(r) \le M. \tag{3.19}$$

Then $m \le x(t) \le M$ for all $t \ge s$.

Proof: We wish to show that $T = \sup\{t : m \le x(r) \le M \text{ for } s \le r \le t\}$ cannot be finite. If so, there would be some $t_* \in (T, T + \tau)$ with either $x(t_*) < m$ and $\dot{x}(t_*) \le 0$ or $x(t_*) > M$ and $\dot{x}(t_*) \ge 0$; suppose the former, necessarily with m finite. Then (1.1) gives

$$f(x(t_* - \tau)) = \dot{x}(t_*) + x(t_*) < m.$$

On the other hand, setting $r = x(t_* - \tau)$ and noting that $t_* - \tau < T$, so $r \in \mathcal{I}$ by the definition of T, we see that $f(r) \ge m$ by (3.19) — a contradiction. We get a similar contradiction in the alternative case, $x(t_*) > M$. ■

Theorem 3.2 *Suppose the constitutive function $f(\cdot)$ satisfies* (1.3) *and that there is a sequence $r_k \to \infty$ such that $\psi(\psi(r_k)) < r_k$ (alternatively, if there is a sequence $\tilde{r}_k \to -\infty$ such that $\psi(\psi(\tilde{r}_k)) > \tilde{r}_k$). Then every solution of* (1.1) *is bounded.*

PROOF: We can take r_k to be an increasing sequence so, as ψ is decreasing, $\tilde{r}_k = \psi(r_k)$ will be a decreasing sequence and we distinguish two cases: $\tilde{r}_k \to -\infty$ or $\tilde{r}_k \geq \tilde{r}_*$; in the latter case, $\tilde{r}_* = \psi(\infty) > -\infty$. Given any particular solution $x(\cdot)$ of (1.1), let $\mathcal{I}_0 = [m_0, M_0] \supset x([s, s+\tau])$ for some $s \geq 0$ so $m_0 \leq x(t) \leq M_0$ on $[s, s+\tau]$. If we are in the first case or if we are in the second case with $\tilde{r}_* < m_0$, we can choose k large enough that $r_k \geq M_0$ and $\tilde{r}_k \leq m_0$. We then can set $\mathcal{I} = [\tilde{r}_k, r_k] \supset \mathcal{I}_0$ and have $\psi(r_k) = \tilde{r}_k$ and $\psi(\tilde{r}_k) = \psi(\psi(r_k)) < r_k$ so $\psi(\mathcal{I}) \subset \mathcal{I}$. On the other hand, if we are in the second case with $m_0 \leq \tilde{r}_*$, we can choose k so $r_k \geq \max\{M_0, \psi(m_0)\}$ and set $\mathcal{I} = [m_0, r_k]$. We then have $\psi(r_k) \geq \tilde{r}_* \geq m_0$ and, of course, $r_k \geq \psi(m_0)$ so again $\psi(\mathcal{I}) \subset \mathcal{I}$. In either of these situations we may then apply Lemma 3.1 to see that $x(\cdot)$ remains in the bounded interval $\mathcal{I}$. ∎

We remark that if $\psi(-\infty)$ is finite as in (1.9), then $\psi(\psi(r)) < r$ for all $r > \psi(\psi(\infty))$ so Theorem 3.2 applies (and similarly if $\psi(\infty)$ is finite.

4 τ-dependent estimates

COMMENT: *As suggested by Giang, this Lemma is used for the proof of Theorem 4.2 – so we include a proof for completeness – but there is an alternative proof of that (which I sent you) using Ekeland's Approximate Variational Principle instead of Lemma 4.1. On the other hand, if this paper would be rewritten as a successor to the version now submitted to JMAA, it might be interesting to omit Lemma 4.1 entirely and give the alternative proof of Theorem 4.2 instead.*

Lemma 4.1 *Let $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ be the asymptotic range of a bounded solution x*

of (1.1)*. Then there are* C^1 *functions* u, v *on* $\mathbb{R}$ *such that*

$$(4.20)\qquad \begin{array}{rl} i. & u, v \textit{ satisfy } (1.1) \textit{ on all of } \mathbb{R} \\ ii. & \overline{m} \le u(t), v(t) \le \overline{M} \textit{for all } t \textit{ so} \\ & \min\{\overline{m}, \psi(\overline{M})\} \le f(u(t)), f(v(t)) \le \max\{\overline{M}, \psi(\overline{m})\} \\ iii. & u(0) = \overline{m},\ \dot{u}(0) = 0; \qquad v(0) = \overline{M},\ \dot{v}(0) = 0. \end{array}$$

PROOF: By the definition (1.5) of $\overline{m}$, there is a sequence $t_k \to \infty$ such that $x(t_k) \to \overline{m}$ and we set $u_k(t) = x(t_k + t)$ — e.g., for $t \ge -\frac{1}{2}t_k$ with $u_k(t) \equiv x(\frac{1}{2}t_k)$ for $t \le -\frac{1}{2}t_k$. The set $\{u_k(\cdot)\}$ is uniformly bounded (with $\overline{m} - \varepsilon \le u_k(t) \le \overline{M} + \varepsilon$ for large enough t_k) with uniformly bounded derivatives, so there is a function u such that $u_k \to u$ uniformly on compact sets in $\mathbb{R}$. Clearly $\overline{m} \le u(t) \le \overline{M}$ in the limit. Since each u_k satisfies (1.1) on $[-T, T]$ for $t_k > 2T$, the derivatives also converge uniformly on compacta we see that u satisfies (1.1) on each interval $[-T, T]$, giving (i). We already have $\overline{m} \le u(t) \le \overline{M}$. If $u(t) = r < 0$ for some t, then, by the definition of $\overline{m}$ and by (1.3), we have $\overline{m} \le r < f(r)$ with $\overline{m} < 0$ so the definition (1.10) gives $f(r) \le \psi(r)$. Similarly, if $u(t) = r > 0$, then $\psi(r) \le f(r) < r \le \overline{M}$. Thus, we always have either $\overline{m} < f(r)$ (if $r = u(t) < 0$) or $\psi(r) \le f(r)$ (if $r \ge 0$), etc., so, for either possible sign of $u(t)$, we have the inequality of (ii) for $f(u(t))$. Since $u_k(0) = x(t_k) \to \overline{m}$, we have $u(0) = \overline{m}$ and, as that is necessarily a minimum by (ii), we also have $\dot{u}(0) = 0$. The construction of $v(\cdot)$ is similar. ■

Theorem 4.2 *Suppose the constitutive function* $f(\cdot)$ *satisfies* (1.3) *and that* $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ *is the asymptotic range of a bounded solution* $x(\cdot)$ *of* (1.1)*. Then* $\overline{\mathcal{I}} \subset \psi_\tau(\overline{\mathcal{I}})$*, i.e.,*

$$(4.21)\qquad \begin{array}{cc} \overline{m} \ge \psi_\tau(\overline{M}), & \overline{M} \le \psi_\tau(\overline{m}) \\ \textit{where} & \psi_\tau(\tilde{r}) = (1 - e^{-\tau})\, \psi(\tilde{r}). \end{array}$$

PROOF: Let u, v be as in Lemma 4.1. Since u satisfies (1.1) and $\dot{u}(0) = 0$, we have $\overline{m} = u(0) = f(u(-\tau))$. Noting that $u(-\tau) = r \ge \overline{m} = f(r)$, the hypothesis (1.3) gives $u(-\tau) > 0$. Applying (2.12) to the solution u with

$t = 0$ and $s = -\tau$, we then obtain, using Lemma 4.1(ii),

$$\begin{aligned} \overline{m} = u(0) &= e^{-\tau} u(-\tau) + \int_{-\tau}^{0} e^{r} f(u(r-\tau))\,dr \\ &> \int_{-\tau}^{0} e^{r} \min\{\overline{m}, \psi(\overline{M})\}\,dr \\ &= \left(1 - e^{-\tau}\right) \min\{\overline{m}, \psi(\overline{M})\}. \end{aligned}$$

Since $0 < (1 - e^{-\tau}) < 1$, this gives $\overline{m} > (1 - e^{-\tau})\,\psi(\overline{M}) = \psi_\tau(\overline{M})$ unless $\overline{m} = \psi(\overline{M}) = 0$. Essentially the same argument, using v, shows that $\overline{M} < (1 - e^{-\tau})\,\psi(\overline{m}) = \psi_\tau(\overline{m})$ unless $\overline{M} = \psi(\overline{m}) = 0$. Combining these shows that $\overline{\mathcal{I}} \subset \psi_\tau(\mathcal{I})$. ■

Corollary 4.3 *Suppose the constitutive function* $f(\cdot)$ *satisfies* (1.3) *and that* $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ *is the asymptotic range of a bounded solution* $x(\cdot)$ *of* (1.1)*. Then either* $\overline{m} < 0 < \overline{M}$ *(so* $x(\cdot)$ *oscillates between approximately* $\overline{m}$ *and* $\overline{M}$ *with infinitely many intermediate zeroes) or* $\overline{m} = 0 = \overline{M}$ *so* x *converges to equilibrium.*

PROOF: Suppose we were to have $\overline{M} \geq \overline{m} > 0$. Then (1.10) gives $\psi(\overline{m}) \leq 0$, contradicting $0 < \overline{M} \leq \psi_\tau(\overline{m})$ as in (4.21). Similarly we cannot have $\overline{m} \leq \overline{M} < 0$. Further, if $\overline{m} = 0$ then (4.21) gives $0 = \overline{m} \leq \overline{M} \leq \psi_\tau(\overline{m}) = 0$; of course, $\overline{M} = 0$ similarly implies $\overline{m} = 0$. ■

Corollary 4.4 *Suppose the constitutive function* $f(\cdot)$ *satisfies* (1.3) *and that* $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ *is the asymptotic range of a bounded solution* $x(\cdot)$ *of* (1.1)*. For any bounded interval* $\mathcal{I} = [m, M] \supset \overline{\mathcal{I}}$*, one has* $\overline{\mathcal{I}} \subset \psi_\tau(\mathcal{I})$ *and*

$$\overline{\mathcal{I}} \subset \bigcap_k {\psi_\tau}^{[k]}(\mathcal{I}). \tag{4.22}$$

PROOF: Since $m \leq \overline{m} \leq \overline{M} \leq M$ with $\psi_\tau(\cdot)$ decreasing, we have $\psi_\tau(M) \leq \psi_\tau(\overline{M}) \leq \psi_\tau(\overline{m}) \leq \psi_\tau(m)$ — i.e., $\psi_\tau(\mathcal{I}) \supset \psi_\tau(\overline{\mathcal{I}}) \supset \overline{\mathcal{I}}$. By iteration, this gives (4.22). ■

Corollary 4.5 *Suppose the constitutive function $f(\cdot)$ satisfies* (1.3) *and that $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ is the asymptotic range of a bounded solution $x(\cdot)$ of* (1.1). *For any bounded interval $\mathcal{I} = [m, M] \supset \overline{\mathcal{I}}$, suppose f satisfies on $\mathcal{I}$ the one-sided linear bounds*

$$\begin{aligned} f(r) &\le \underline{a} + \underline{b}(-r) \quad \text{when } m \le r \le 0 \\ f(r) &\ge -\overline{a} - \overline{b}r \quad \text{when } 0 \le r \le M. \end{aligned} \tag{4.23}$$

Then $\overline{\mathcal{I}} \subset \mathcal{I}' = [m', M']$ where

$$m' = -\frac{(1-e^{-\tau})\left(\overline{a} + \overline{b}\underline{a}\right)}{1 - (1-e^{-\tau})^2\,\underline{b}\overline{b}}, \qquad M' = \frac{(1-e^{-\tau})\left(\underline{a} + \underline{b}\overline{a}\right)}{1 - (1-e^{-\tau})^2\,\underline{b}\overline{b}}$$

provided $(1-e^{-\tau}) < 1/\sqrt{\underline{b}\overline{b}}$. In particular, if each solution of (1.1) *eventually lies in an interval $\mathcal{I}$ for which* (4.23) *holds with $\underline{a} = \overline{a} = 0$, then the equilibrium is globally asymptotic stable.*

COMMENT: *Add: For example, if $f(r) \ge 0$ for all $r > 0$ (alternatively, if $f(r) \le 0$ for $r < 0$), then we would have global asymptotic stability.* PROOF: *If $f(r) \ge 0$ for all $r > 0$, then Theorem 3.2* **Complete..**

PROOF: From (4.23) one gets corresponding linear bounds for ψ on $\mathcal{I} \supset \overline{\mathcal{I}}$ so, using (4.21),

$$\begin{aligned} \overline{m} \ge \psi_\tau(\overline{M}) &\ge (1-e^{-\tau})\left(-\overline{a} - \overline{b}\overline{M})\right) \\ &\ge (1-e^{-\tau})\left[-\overline{a} - \overline{b}\psi_\tau(\overline{m})\right] \\ &\ge (1-e^{-\tau})\left[-\overline{a} - \overline{b}\,(1-e^{-\tau})\left(\underline{a} + \underline{b}(-\overline{m})\right))\right] \\ &= -\left[(1-e^{-\tau})\,\overline{a} + (1-e^{-\tau})^2\,\overline{b}\underline{a}\right] + (1-e^{-\tau})^2\,\overline{b}\underline{b}\overline{m} \end{aligned}$$

whence $\overline{m} \ge m'$ — provided $(1-e^{-\tau}) < 1/\sqrt{\underline{b}\overline{b}}$ so $[1 - (1-e^{-\tau})^2\,\underline{b}\overline{b}]$ would be positive. One gets $\overline{M} \le M'$ similarly. If $\underline{a} = \overline{a} = 0$, then this gives $\overline{m} = \overline{M} = 0$ so one would have asymptotic convergence to equilibrium and this, for each solution, is the asserted global asymptotic stability. ∎

For our next result we wish to exploit the representation (2.14) to obtain a bound for $\overline{\mathcal{I}}$ in terms of

$$D = D_\nu(r) = f(r) - \nu r, \qquad \overline{D}_\nu(\mathcal{I}) = \|D\|_{L^\infty(\mathcal{I})}. \tag{4.24}$$

The results provided will be a bit less satisfactory than those of Theorem 4.2, since the fundamental solution z of (2.13) does not have a fixed sign for $\nu < 0$. Note that ν cannot be entirely arbitrary here as $Z_1(\nu, \tau)$ of (2.18) will only be finite when $\tau < \tau_0$.

Theorem 4.6 *Suppose the constitutive function $f(\cdot)$ satisfies* (1.3). *Now suppose $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ is the asymptotic range of a bounded solution $x(\cdot)$ of* (1.1). *Then, for any ν such that $\tau < \tau_0(\nu)$ we have*

$$\overline{\mathcal{I}} \subset [-\alpha, \alpha] \qquad \text{with } \alpha = Z_1(\nu, \tau)\overline{D}_\nu(\overline{\mathcal{I}}), \tag{4.25}$$

i.e., $-\overline{m}, M \leq \alpha$.

Proof: Since f is continuous, we can find $a = a_\varepsilon$ large enough that $\overline{m} - \delta \leq x(t) \leq \overline{M} + \delta$ for $t \geq a_\varepsilon - \tau$ giving

$$|D(x(t))| \leq \max\{|D(r)| : r \in \overline{\mathcal{I}}\} + \varepsilon = \overline{D}(\overline{\mathcal{I}}) + \varepsilon. \tag{4.26}$$

Then, for large t, (2.14) with $s = 0$ gives

$$\begin{aligned} x(T) = \quad & \Big[z(T)x(0) + \nu \int_{-\tau}^{0} z(T - [r + \tau])x(r)\, dr \\ & \qquad + \int_0^a z(T - r)[D(x)](r - \tau)\, dr \Big] \\ & + \int_a^T z(T - r)[D(x)](r - \tau)\, dr. \end{aligned}$$

By the choice of $a = a_\varepsilon$ we can apply (4.26) in the last term here so that is bounded by $Z_1[\overline{D}(\overline{\mathcal{I}}) + \varepsilon]$ for arbitrary $T > a$. We can choose $T = T_k \to \infty$ so, e.g., $x(T_k) \to \overline{m}$ and note that, with a fixed, the bracketed terms $[\cdots] \to 0$ as $T_k \to \infty$ by the exponential decay of $z(\cdot)$. In the limit $T_k \to \infty$ this gives $|\overline{m}| \leq Z_1[\overline{D}(\overline{\mathcal{I}}) + \varepsilon]$; since $\varepsilon > 0$ was arbitrary, we obtain: $|\overline{m}| \leq \alpha$. Similarly, choosing $T_k \to \infty$ so $x(T_k) \to \overline{M}$ leads to $|\overline{M}| \leq \alpha$ and combing gives (4.25) as desired. ■

Corollary 4.7 *Suppose $\overline{\mathcal{I}} = [\overline{m}, \overline{M}]$ is the asymptotic range of a bounded solution $x(\cdot)$ of* (1.1). *Suppose the constitutive function f satisfies (1.3) and* *is 'sufficiently flat' on some symmetric bounded interval* $\mathcal{I} = [-M, M] \supset \overline{\mathcal{I}}$ *that, for some choice of* ν,

$$\begin{aligned} &|f(r) - \nu r| \le \overline{a} + \vartheta |r| \qquad \textit{with} \\ &0 \le \vartheta < \frac{1}{Z_1(\nu,\tau)}, \ \textit{and} \ \ 0 \le \overline{a} \le \frac{(1 - Z_1\vartheta)M}{Z_1}. \end{aligned} \tag{4.27}$$

Then

$$\overline{\mathcal{I}} \subset \mathcal{I}' = [-M', M'] \quad \textit{with } M' = \frac{Z_1 \overline{a}}{1 - Z_1\vartheta}.$$

[If $\overline{a} = 0$ *in* (4.27)*, one has asymptotic stability. If each solution of* (1.1) *eventually lies in a symmetric interval* $\mathcal{I}$ *on which* (4.27) *holds with* $\overline{a} = 0$*, then one has global asymptotic stability of the equilibrium.]*

PROOF:

COMMENT: *Finish this proof...*

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5 Examples

COMMENT: *We should work out the specific implications of the above for persistence and convergence to equilibrium in the context of the Nicholson blowfly model. [Should any other model(s) be worked out explicitly?]*

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6 A comparison theorem

Let $\mathbf{A}$ be the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$ of positive linear operators on the partially ordered Banach space $\mathcal{X}$ (i.e., $0 \le x \in \mathcal{X}$ implies $\mathbf{S}(t)x \ge 0$ for all $t \ge 0$). Fixing an interval $\mathcal{I} = [-\tau, -\delta]$ with $0 < \delta < \tau$, we set $\boldsymbol{\mathcal{X}} = C(\mathcal{I} \to \mathcal{X})$, partially ordered by taking $\xi_1 \le \xi_2$ in $\boldsymbol{\mathcal{X}}$ whenever $\xi_1(\vartheta) \le \xi_2(\vartheta)$ in $\mathcal{X}$ for each ϑ in $\mathcal{I}$. For a continuous $\mathcal{X}$-valued function $x(\cdot)$, we now denote by $x^t \in \boldsymbol{\mathcal{X}}$ the restriction of $x(t + \cdot)$ to $\mathcal{I}$. For any continuous function $f : \mathbb{R}_+ \times \boldsymbol{\mathcal{X}} \to \mathcal{X}$, we can then consider the vector delay differential equation

$$\dot{x} - \mathbf{A}x = f\left(t, x^t\right) \tag{6.28}$$

for which, much as in (2.12), we have

$$x(t) = \mathbf{S}(t-s)x(s) + \int_s^t \mathbf{S}(t-r)f\left(r, x^r\right)\, dr \tag{6.29}$$

for $t > s$. [Given continuous data on $[-\tau, 0]$, the method of steps (taking steps of size δ) then ensures global existence of C^1 solutions.]

Theorem 6.1 *Suppose f_j $(j = 1, 2)$ are continuousfunctions: $\mathbb{R}_+ \times \boldsymbol{\mathcal{X}} \to \mathcal{X}$ and let x_j be solutions of $(6.28)_j$ — $f = f_j$ for $j = 1, 2$ — such that $x_1 \le x_2$ on $[s - \tau, s]$. Suppose f_1, f_2 satisfy*

$$\xi_1 \le \xi_2 \textit{in } \boldsymbol{\mathcal{X}} \textit{ implies } f_1(t, \xi_1) \le f_2(t, \xi_2) \textit{in } \mathcal{X}. \tag{6.30}$$

Then $x_1(t) \le x_2(t)$ for all $t \ge s$.

Remark 6.1 It is easy to see that if either f_1 or f_2 is isotone ($x_1 \le x_2$ implies $f_j(t, x_1) \le f_j(t, x_2)$), then the simple comparison $f_1 \le f_2$ implies (6.30). Further, it should be clear from the proof below that (6.30) need only hold 'where relevant' so, for example, if $x_2(\cdot)$ is known, we need to verify (6.30) only for ξ_1 such that $\xi_1 \le \xi_2 = [x_2]^t$ for $t \in [s, T + \delta]$ to ensure that $x_1(t) \le x_2(t)$ for $s \le t \le T$. Finally, we note that essentially the same proof supports consideration of a sandwiching comparison to get propagation of the inequality $x_1 \le x_2 \le x_3$ under a condition on f_j $(j = 1, 2, 3)$ generalizing (6.30). ■

Proof: Let $T_* = \max\{T : x_1(t) \le x_2(t) \text{for} s - \tau \le t \le T\}$; we have $T_* \ge s$ by assumption and, in contradiction to the desired result, assume $T_* < \infty$. By the definition of T_* there would then exist $t \in (T_*, T_* + \delta)$ for which $x_1(t) \not\le x_2(t)$. The choice of t ensures that $x_1(t') \le x_2(t')$ for $t' = r + \vartheta$ with $s \le r \le t$ and $r \in \mathcal{I}$ — i.e., $[x_1]^r \le [x_2]^r$ for $s \le r \le t$. Thus, using (6.29), the positivity of $\mathbf{S}$, and (6.30), we have

$$\begin{aligned} x_1(t) &= \mathbf{S}(t-s)x_1(s) + \int_s^t \mathbf{S}(t-r)f_1\left(r, [x_1]^r\right)\, dr \\ &\le \mathbf{S}(t-s)x_2(s) + \int_s^t \mathbf{S}(t-r)f_2\left(r, [x_2]^r\right)\, dr \\ &= x_2(t), \end{aligned}$$

which contradicts the choice of t. ■

Remark 6.2 In an earlier version of this paper, the result of Lemma 3.1 was obtained by applying a scalar version of Theorem th:comp to compare the solution x of (1.1) with the solutions of $\dot{x} + x = f_j(x(t - \tau))$ for $f_1(r) = \min\{r, m\}$ and $f_2(r) = \min\{r, M\}$.

COMMENT: *Briefly indicate the relevant comparisons to get Lemma 3.1. Perhaps also explain the use of decaying exponential solutions in showing, once one has given a fixed interval $\mathcal{I}$ containing the range of $x(\cdot)$, how to show $\overline{\mathcal{I}}(x) \subset \psi(\mathcal{I})$.*

COMMENT: *If this paper would be rewritten as a successor to the version now submitted to JMAA, this section would be omitted — although a further treatment of this generality of comparison and applications might become still another paper.*

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7 What is still missing?

COMMENT: *This section is not meant to appear in the paper, but is for our own benefit to see whether we want to expend the effort to try to obtain any of these results. I have been trying to do that, unsuccessfully, and at this point have run out of ideas. It is not clear to me whether our best choice is seek a complete resolution to the problem or to publish what we have on these issues and turn to consideration of the (much more difficult) situation when one does* not *have convergence to equilibrium.*

Obviously, if the earlier version already submitted were to be accepted for JMAA, it would be important to go through this to see how to adapt it: omitting some duplication (with citation of that paper) and emphasizing the incremental results. Perhaps that eventuality would make it more important *to resolve some of the gaps and conjectures noted here.*

The presentation of our results has taken τ as fixed and asked about conditions on the constitutive function giving global asymptotic stability of the equilibrium solution. Our original conjecture was formulated somewhat differently: for a fixed constitutive function $f(\cdot)$, think of τ as a parameter varying 'up' from 0. The conjecture was that for $\tau \geq 0$ we would have global asymptotic stability precisely for an interval $0 \leq \tau \leq \tau_*$. [There is no suggestion here that this τ_* is related to (2.17), although we might ask how it relates to that onset of instability for the linearized problem $\tau_0(f'(0))$.] Our estimates do have this form when formulated as inequalities for τ, but these are *sufficient* conditions for global asymptotic stability and we do not know that they are sharp. In particular, it is not yet shown that one could not have, for example, a τ-interval of global asymptotic stability, followed by an interval with some other behavior (possibly existence of periodic solutions, possibly chaos, $\ldots$), followed by another interval of global asymptotic stability before the onset of local instability at $\tau_0(f'(0))$.

We need an estimate (even if only asymptotic as $\nu \to -\infty$ and as $\tau \to \tau_0(\nu)$) for $Z_1 = Z_1(\nu, \tau)$ to make Theorem 4.6 more usable.

It looks almost obvious [but seems open!] that *local* stability of the equilibrium cannot imply global asymptotic stability. Specifically, I had conjectured that it is possible to have a constitutive function $f(\cdot)$ (say, with $|f'(0)| < 1$ so one has *local* stability of the equilibrium for arbitrary $\tau > 0$) such that, for some large τ, there is a periodic solution, clearly precluding global asymptotic stability. My idea to show this was to begin with, e.g., a linear $f_0(r) = \nu r$ (taking $\nu < -1$ and $\tau = \tau_0(\nu)$ so one has the periodic solutions $c \cos[\sqrt{\nu^2 - 1}\, t]$) and then perturbing f_0 slightly (with respect to sup-norm) so one gets, e.g., $f'(0) = 0$ for which this τ gives local stability. The hope is that if one restricts perturbation to a small enough neighborhood of 0 and considers large c (so one is in the perturbation neighborhood only for a short time), then one will still have a (perturbed) periodic solution for this perturbed f. So far, I have not succeeded in working out any relevant perturbation argument.

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