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Pointwise and Internal Controllability for the Wave Equation

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Abstract

Problems of internal and pointwise observation and control for the 1-dimensional wave equation arise in the simulation of control and identification processes in electrical engineering, flaw detection, and medical tomography. The generally accepted way of modelling sensors and actuators as pointlike objects leads to results which may make no apparent physical sense: they may depend, for instance, on the rationality or irrationality of the location for a point sensor or actuator. We propose a new formulation of sensor (actuator) action, expressed mathematically by using somewhat unconventional spaces for data presentation and processing. For interaction restricted to an interval of length ε, the limit system of observation (or control) now makes sense when ε tends to zero without a sensitive dependence on the precise location of the limiting point.

This paper is dedicated to the memory of the late J.L.Lions, whose seminal work in applied mathematics led, in particular, to the embedding of control-theoretic problems in the context of the modern theory of partial differential equations and inspired us all.

1. Introduction

We will be considering observation and control of the one-dimensional wave equation on $\mathcal{Q} = [0, \ell] \times [0, T]$ with observation and control restricted to a small

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spatial subinterval $[a, a + \varepsilon] \subset (0, \ell)$. Our principal interest, comparable to that of [4], will be on the asymptotics as $\varepsilon \to 0$, i.e., as the subinterval shrinks to the single point $\{a\}$. We note that [4] obtains dramatically different results for different choices of a — essentially depending on number-theoretic properties of a/ℓ — with a distinction between 'good' points for which the asymptotics behave well and 'bad' points for which they do not. The relevance of such number-theoretic properties was also noted in [2] and, in the somewhat different context of one-dimensional diffusion equations, in [3] and [6]. The principal novelty of the present paper is the introduction of a natural contextual change which makes the asymptotics behave well for *all* choices of the point $a \in (0, \ell)$.

The wave equation is usually presented in the second-order form

$$\zeta_{tt} = \zeta_{xx} \qquad \text{say, with Neumann conditions: } \zeta_x = 0 \text{ at } x = 0, \ell. \tag{1.1}$$

However, we take as particularly 'natural' — especially for consideration of observability — the formulation of the dynamics as a first order system

$$\begin{cases} \varphi_t = \psi_x \\ \psi_t = \varphi_x \end{cases} \qquad \text{or} \qquad \frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{1.2}$$
$$\text{with } \psi = 0 \text{ at } x = 0, \ell.$$

Our viewpoint in this paper will be to work with the problem in this form for the observation problem and then to use the usual duality to construct the corresponding control problem for the corresponding inhomogeneous form

$$\begin{cases} y_t = z_x + f \\ z_t = y_x + g \end{cases} \qquad \text{or} \qquad \frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix} \tag{1.3}$$
$$\text{with } z = 0 \text{ at } x = 0, \ell \text{ and } y = 0 = z \text{ at } t = 0.$$

The physical interpretation of these observation and control problems will be discussed in the next section.

It is standard that (1.1) and (1.2) are related by taking $\psi = \zeta_x$ and $\varphi = \zeta_t$ — corresponding physically to *momentum* and *strain*. Clearly, if ζ is known (observed) in the strip $\mathcal{Q}_\varepsilon = [a, a + \varepsilon] \times [0, T]$, then one also knows both φ and ψ in this strip; the relevant physical energy density is just $\frac{1}{2}[\varphi^2 + \psi^2]$ so the 'natural' topology is in L^2 for this pair (φ, ψ) with (1.2) constituting a Hamiltonian system for this energy.

As a normalization, we may use the change of variables $x = a + s\varepsilon$ $(0 \leq s \leq 1)$ to consider the asymptotics of the optimal controls $(\varphi, \psi)_\varepsilon$ as functions on the fixed space $\mathcal{Q}_* = [0,1] \times [0,T]$, although implemented as functions on $\mathcal{Q}_\varepsilon$. Our principal result is that — with appropriate normalization, but without regard for the choice of $a \in (0,\ell)$ — as $\varepsilon \to 0$ one has convergence on $\mathcal{Q}_*$ of $(\varphi,\psi)_\varepsilon$ to a 'control' associated with the natural limiting point-control problem. For most of our analysis we consider the model problems, (1.2) and (1.3), with $T = 2\ell$, for which explicit computations are available. Although mostly not treated here, we remark that these results generalize to times $T > 2\ell$ (for which we have controllability), to equations with spatially variable coefficients (as we show in Section 6), to certain weighted control norms, etc.

2. Physical interpretation: the wave equation revisited

To understand the relation between (1.2), (1.3) and the usual wave equation, let us briefly sketch the physical derivation. While there are several physical interpretations leading to that equation, we will focus on longitudinal vibrations of an elastic rod.

We begin by taking $q = q(x,t)$ to be the position at time t of the material point labelled by $x \in [0,\ell]$ in the reference configuration; we assume that the rod remains straight with the coordinate system aligned with the rod and with its origin at one end. By Newton's Law we have

$$(2.1)\qquad \begin{aligned} \frac{d}{dt}\,[\text{momentum}] &= [\text{total force}] \\ &= -\,\text{gradient}_{\text{with resp. to } q}\,[\text{PE}] \;+\; [\text{external force}]. \end{aligned}$$

We will assume that we are remaining in the neighborhood of a stable equilibrium state, given by $\overset{\circ}{q}$, so the *potential energy* PE of the system is given approximately quadratically by

$$(2.2)\qquad \text{PE}\,(q) \;\approx\; \frac{1}{2}\int_0^\ell a(x)\left([q-\overset{\circ}{q}]_x\right)^2 dx$$

where $a(\cdot)$ represents some possible variation in the material properties along the rod. Then

$$\begin{aligned} \nabla_q\,[\text{PE}] : h \longmapsto \;\; \frac{d}{d\sigma}[\text{PE}(q+\sigma h)]\Big|_{\sigma=0} &\approx \int_0^\ell a(x)[q-\overset{\circ}{q}]_x\, h_x\, dx \\ &= -\int_0^\ell \left(a(x)[q-\overset{\circ}{q}]_x\right)_x h\, dx + [\text{boundary terms}]. \end{aligned}$$

Taking the boundary conditions so these boundary terms vanish, we make the usual assumption leading to linear dynamics — that the state stays close enough to $\mathring{q}$ for (2.2) to be taken as exact: $-\nabla_q[\mathrm{PE}]= (a[q-\mathring{q}]_x)_x$ so (2.1) becomes

$$(\rho q_t)_t = (a[q-\mathring{q}]_x)_x + F_{\text{ext}} \tag{2.3}$$

For simplicity of exposition, we considering a homogeneous rod for which, with appropriate choices of units, we may take $\rho \equiv 1$ and $a \equiv 1$ in (2.3). Note that the relevant aspects of the state are then

$$[\text{momentum}] = q_t \qquad [\text{stress}] = [q-\mathring{q}]_x$$

and we will assume 'free' endpoints so the boundary conditions for (2.3) are: $[q-\mathring{q}]_x = 0$ at $x = 0, \ell$.

For observation, we will take ζ to be the 'deviation from equilibrium': $\zeta = q-\mathring{q}$ and, since the equilibrium $\mathring{q}$ is here independent of t (i.e., $\mathring{q}_t\equiv 0$ (typically, $\mathring{q}=\mathring{q}(x) = x$) so $\zeta_t = q_t$ and we impose no external forces, we alternatively have (1.1): $\zeta_{tt} = \zeta_{xx}$ or, taking $\varphi = [\text{momentum}] = \zeta_t$ and $\psi = [\text{stress}] = \zeta_x$, have the physical law $\varphi_t = \zeta_x$ coupled with the differentiation identity $\psi_t = \zeta_{xt} = \zeta_{tx} = \varphi_x$ — i.e., (1.2).

As already noted in the Introduction, if q is known in a strip $\mathcal{Q}_\varepsilon$ so $\zeta = q-x$ is known there, then $\zeta_t =: \varphi$ and $\zeta_x =: \psi$ are also known. The problem we will discuss in Section 3 corresponds either to this computation by differentiation of the observation — with $q, \zeta \in H^1(\mathcal{Q}_\varepsilon)$ so $(\zeta_t, \zeta_x) \in V_\varepsilon$ — or to independent measurements of velocity (the same as momentum, here, since $\rho \equiv 1$) and of stress. The latter seems the more meaningful interpretation in a context considering point observation, for which spatial differentiation becomes problematic.

We now consider derivation of the controlled wave equation

$$w_{tt} = w_{xx} + F \qquad \text{with } w_x = 0 \text{ at } x = 0, \ell. \tag{2.4}$$

Traditionally, the only form of control under consideration has been the imposition of the external force $F_{\text{ext}} = f$. We note, however, that the advent of so-called 'smart materials' suggests an additional mode of possible control — by altering $\mathring{q}$. [E.g., this might be accomplished for shape memory alloys by locally adjusting the temperature; we assume that, in our context of small deviation from equilibrium, we would only consider temperature changes small enough as not to make relevant the local structural changes in the PE leading to hysteresis.] It is, of course, $\mathring{p}:=\mathring{q}_t$ which is new here.

We now take $w :=$ [deviation] $= q - \overset{\circ}{q}$, so we get $y :=$ [momentum] $= q_t = w_t + \overset{\circ}{p}$ with $z :=$ [stress] $= w_x$. Then we obtain from (2.3)

$$w_{tt} = \left[q_t - \overset{\circ}{p}\right]_t = w_{xx} + F_{\rm ext} - \overset{\circ}{p}_t$$

— which is just (2.4) with $F := F_{\rm ext} - \overset{\circ}{p}_t$. Similarly,

$$\begin{aligned} y_t &= z_x + F_{\rm ext} \\ z_t &= w_{xt} = w_{tx} = y_x - \overset{\circ}{p}_x \end{aligned}$$

— which is just (4.1) with $f = F_{\rm ext}$ and $g = -\overset{\circ}{p}_x$. We thus have the relation

$$\gamma := F - f = -\overset{\circ}{p}_t = \int^x g_t \qquad \left(\text{or } g = \int^t \gamma_x\right) \tag{2.5}$$

between the controlled wave equation (2.4) and the first order system (1.3). For the control supports to lie in the interval $[a, a+\varepsilon]$ (more precisely, in the strip $\mathcal{Q}_\varepsilon$), we are assuming that $\overset{\circ}{q}$ is independent of t (i.e., $\overset{\circ}{p} = 0$) for $x \notin [a, a+\varepsilon]$ so both g and γ have support in $\mathcal{Q}_\varepsilon$. Note that the initial condition: $w = 0$ at $t = 0$ corresponds to the assumption that the rod is in equilibrium at $t = 0$ with a related interpretation of the velocity condition: $w_t = 0$ at $t = 0$.

From (2.5) we anticipate that γ (and so F in (2.4)) must be considered distributionally. For future reference we will need an adjoint computation for this formulation. Suppose w and ζ are (smooth) solutions, respectively of (2.4) and of its homogeneous form; note that ζ (and so also ζ_t) is periodic in t with period $T = 2\ell$. Then

$$\begin{aligned} \int_0^\ell [w_x\zeta_x + w_t\zeta_t]\, dx\Big|_{t=T} &= \int_0^\ell [w_x\zeta_x + w_t\zeta_t]\, dx\Big|_0^T = \int_{\mathcal{Q}} [w_x\zeta_x + w_t\zeta_t]_t\, dx\, dt \\ &= \int_{\mathcal{Q}} [w_{xt}\zeta_x + w_x\zeta_{xt} + (w_{xx} + F)\,\zeta_t + w_t\,(\zeta_{xx})]\, dx\, dt \\ &= \int_{\mathcal{Q}} [w_x\zeta_t + w_t\zeta_x]_x\, dx + \int_{\mathcal{Q}} F\zeta_t\, dx\, dt \\ &= \int_{\mathcal{Q}} F\zeta_t\, dx\, dt = \int_{\mathcal{Q}} f\zeta_t\, dx\, dt - \int_{\mathcal{Q}} \overset{\circ}{p}_t\, \zeta_t\, dx\, dt\,. \end{aligned}$$

Integrating by parts twice, we now note that

$$\begin{aligned} -\int_{\mathcal{Q}} \overset{\circ}{p}_t\, \zeta_t\, dx\, dt &= -\int_{\mathcal{Q}} \overset{\circ}{p}\, \zeta_{tt}\, dx\, dt + \int_0^\ell \overset{\circ}{p}\, \zeta_t\, dx\Big|_0^T = -\int_{\mathcal{Q}} \overset{\circ}{p}\, \zeta_{xx}\, dx\, dt + \int_0^\ell \overset{\circ}{p}\, \zeta_t\, dx\Big|_0^T \\ &= \int_{\mathcal{Q}} \overset{\circ}{p}_x\, \zeta_x\, dx\, dt + \int_0^T \overset{\circ}{p}\, \zeta_x\, dx\Big|_0^\ell + \int_0^\ell \overset{\circ}{p}\, \zeta_t\, dx\Big|_0^T\,. \end{aligned}$$

Since $\zeta_x = 0$ at $x = 0, \ell$, the second integral on the right vanishes. Assuming $\overset{\circ}{p}$ would also be periodic in t with period $T = 2\ell$ (effectively, that we view g as repeated with this periodicity), we then obtain

$$\int_0^\ell [w_x\zeta_x + w_t\zeta_t]\; dx\Bigg|_{t=T} = \int_{\mathcal{Q}} [f\zeta_t + g\zeta_x]\; dx\, dt \quad . \tag{2.6}$$

Without providing any physical heuristics for this, we will stipulate that the L^2-norm (on $\mathcal{Q}_\varepsilon$) provides an appropriate measure for the 'cost' of implementing the external force $F_{\text{ext}} = f$; similarly, we stipulate that an $H^1(\mathcal{Q}_\varepsilon)$ norm is appropriate for $\overset{\circ}{p}$ — more precisely, the L^2-norm for $\overset{\circ}{p}_x$. Thus, 'optimality' of the control will mean

$$\begin{gathered} \|f\|^2_{L^2(\mathcal{Q}_\varepsilon)} + \|\gamma\|^2_* = \min \\ \text{with} \quad \|\gamma\|_* = \|\, \overset{\circ}{p}_t \,\|_* := \|\, \overset{\circ}{p}_x \,\|_{L^2(\mathcal{Q}_\varepsilon)} = \|g\|_{L^2(\mathcal{Q}_\varepsilon)}. \end{gathered} \tag{2.7}$$

Although (2.4) appears to involve only a single 'control' F, the split into f and γ seems entirely natural in this context.

Remark. One could insert in (2.7) a coefficient μ for $\|g\|_{L^2(\mathcal{Q}_\varepsilon)}$ which would weight the relative difficulty of the two modes of control. This would amount to changing to a new equivalent norm for the space Z below and the evolution operator for the observation system (1.2) would no longer be unitary so the computations corresponding to taking $\mu \neq 1$ would be much more complicated. However, we would expect that any other choice for $0 < \mu < \infty$ would lead to essentially similar results.

The traditional implicit assumption that the equilibrium is constant in t — i.e., that $\overset{\circ}{p} \equiv 0$ so we have simply: $F = F_{\text{ext}}$ — corresponds to taking $\mu = \infty$ in (2.7) and we may view the results of [4] as associated with a double limit: first let $\mu \to \infty$ and *then* let $\varepsilon \to 0$.

3. Formulation and preliminary estimates

We start with the observability problem for the first order system (1.2). It is easy to see that energy is conserved: for any (finite energy) solution one has

$$E(\varphi, \psi) = E(\varphi, \psi; \tau) := \int_0^\ell [\varphi^2(x,\tau) + \psi^2(x,\tau)]\, dx = const \tag{3.1}$$

(independent of τ, since $dE/d\tau = 0$). Thus the system induces a C_0 group of unitary operators $\{G(t)\}$ on the space $W := L^2(0,\,\ell) \times L^2(0,\,\ell)$. We note that the infinitesimal generator of $G(\cdot)$ is

$$
\begin{aligned}
&\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \qquad \text{with} \\
&\mathcal{D}(\mathbf{A}) = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in H^1([0,\ell] \to W) : \psi = 0 \,\mathrm{at}\, 0, \ell \right\}
\end{aligned}
\tag{3.2}
$$

and that $G(\cdot)$ is periodic in t with period 2ℓ — i.e.,

$$G(2\ell) = G(0) = \mathbf{1}_W$$

as may be seen from the representation of the solution to (1.2) in the 'separation of variables' form

$$
\begin{aligned}
\varphi(x,t) &= \sum_{n=0}^{\infty} \left(\alpha_n \sin \frac{\pi n t}{\ell} + \beta_n \cos \frac{\pi n t}{\ell} \right) \cos \frac{\pi n x}{\ell}\,, \\
\psi(x,t) &= \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{\pi n t}{\ell} - \beta_n \sin \frac{\pi n t}{\ell} \right) \sin \frac{\pi n x}{\ell}\,,
\end{aligned}
\tag{3.3}
$$

with $\{\alpha_n\}, \{\beta_n\} \in \ell^2$.

Interchanging the roles of t and x, we note, on the other hand, that the equation in (1.2), considered for $0 < t < T = 2\ell$ with periodicity conditions:

$$\varphi\Big|_{t=0} = \varphi\Big|_{t=T}, \quad \psi\Big|_{t=0} = \psi\Big|_{t=T}, \tag{3.4}$$

induces a C_0 unitary group $\{\hat{G}(x)\}$ on the space $Z := L^2(0,\,T) \times L^2(0,\,T)$ with its usual norm; the infinitesimal generator of $\hat{G}(\cdot)$ is $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}$ with the periodicity conditions (3.4). Note that

$$
\begin{aligned}
G(t) &: \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Bigg|_{t=0} \longmapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Bigg|_{t} \\
\hat{G}(x) &: \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Bigg|_{x=0} \longmapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Bigg|_{x}
\end{aligned}
\qquad \text{for } \begin{cases} \varphi_t = \psi_x \\ \psi_t = \varphi_x \end{cases} \qquad \text{etc.}
$$

We easily verify the unitarity of $\hat{G}$: just integrate by parts to check that

$$\Gamma(\varphi, \psi) = \Gamma(\varphi, \psi; \xi) := \int_0^T [\varphi^2(\xi, t) + \psi^2(\xi, t)]\, dt = const \tag{3.5}$$

(independent of ξ). For consistent choices of initial data, the solutions of (3.4) and of (1.2) coincide in $Q := [0,\, \ell] \times [0,\, T]$. Since

$$T\, E(\varphi, \psi) = \int_Q \left(\varphi^2 + \psi^2\right)\, dx\, dt = \ell\, \Gamma(\varphi, \psi)$$

and we have taken $T = 2\ell$, we have the identity

$$\Gamma(\varphi, \psi) = 2E(\varphi, \psi)\,. \tag{3.6}$$

For $\varepsilon > 0$ small enough that $[a,\, a+\varepsilon] \subset (0,\, \ell)$, we set $\mathcal{Q}_\varepsilon := [a,\, a+\varepsilon] \times [0,\, T]$ and $V_\varepsilon := L^2([a,\, a+\varepsilon] \to Z)$; we introduce the observation operator $\mathcal{O}_\varepsilon$ as follows:

Allowing for time-reversibility, one has a solution operator for (1.2) — with 'initial' condition specified at $t = T$ — followed by restriction to $\mathcal{Q}_\varepsilon$:

$$\mathcal{V}_\varepsilon : W \to [L^2(\mathcal{Q})]^2 \hookrightarrow V_\varepsilon : \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{t=T} \mapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \mapsto \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{\mathcal{Q}_\varepsilon} \tag{3.7}$$

with

$$\begin{aligned} \left\| \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{\mathcal{Q}_\varepsilon} \right\|_{V_\varepsilon}^2 &= \int_a^{a+\varepsilon}\!\!\int_0^T \left[\varphi^2(x,t) + \psi^2(x,t)\right]\, dt\, dx \\ &= \varepsilon\Gamma(\varphi,\psi) = 2\varepsilon E(\varphi,\psi) = 2\varepsilon \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_W^2 . \end{aligned} \tag{3.8}$$

From (3.8) we see that the operator $(1/\sqrt{2\varepsilon})\mathcal{V}_\varepsilon : W \to V_\varepsilon$ is norm-preserving and so is unitary to its range $M_\varepsilon = \{$restrictions to the strip $\mathcal{Q}_\varepsilon$ of solutions of (1.2) with data in $W\}$. Thus, $[(1/\sqrt{2\varepsilon})\mathcal{V}_\varepsilon]^*$ is a left inverse of $(1/\sqrt{2\varepsilon})\mathcal{V}_\varepsilon$ — an actual inverse if we restrict $\mathcal{V}_\varepsilon$ to the (necessarily closed) subspace $M_\varepsilon \subset V_\varepsilon$. We may then set $\mathcal{O}_\varepsilon := (1/2\varepsilon)[\mathcal{V}_\varepsilon]^* : V_\varepsilon \to W$ and, for solutions of the system (1.2), one has

$$\mathcal{O}_\varepsilon := \frac{1}{2\varepsilon}\mathcal{V}_\varepsilon^* : \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{\mathcal{Q}_\varepsilon} \mapsto \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{t=T} \qquad \text{since} \quad \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{\mathcal{Q}_\varepsilon} \in M_\varepsilon \tag{3.9}$$

so $\mathcal{O}_\varepsilon\big|_{M_\varepsilon}$ is the desired observation operator — $\mathcal{O}_\varepsilon\big|_{M_\varepsilon} = [\mathcal{V}_\varepsilon : W \to M_\varepsilon]^{-1}$ — for the system with observation on $\mathcal{Q}_\varepsilon$. Note that $\sqrt{2\varepsilon}\,\mathcal{O}_\varepsilon$ is unitary from M_ε to W.

From the continuity in x of $\begin{pmatrix} \varphi(x,\cdot) \\ \psi(x,\cdot) \end{pmatrix}$ as an element of Z — i.e., the continuity of $x \mapsto \hat{G}(x)$, taking for $\hat{G}$ the strong topology of operators on Z — we note that we can also consider pointwise observation at $x = a$, defining

$$\tag{3.10} \begin{aligned} \mathcal{V}_0 \ &: W \to C([0,\ell] \to Z) \to Z \\ &: \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{t=T} \mapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{x=a} \end{aligned}$$

for arbitrary $a \in (0,\ell)$ with

$$\tag{3.11} \left\| \mathcal{V}_0 \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_Z^2 = \Gamma(\varphi,\psi) = 2E(\varphi,\psi) = 2 \left\| \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix} \right\|_W^2 .$$

We see from (3.11) that $(1/\sqrt{2})\mathcal{V}_0$ is norm-preserving: $W \to Z$, so we may set $\mathcal{O}_0 := \frac{1}{2}\mathcal{V}_0^* : Z \to W$ and have

$$\mathcal{O}_0\Big|_{M_0} = [\mathcal{V}_0 : W \to M_0]^{-1} : M_0 \subset Z \to W : \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{x=a} \mapsto \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{t=T} .$$

Thus, $\mathcal{O}_0|_{M_0}$ is the observation operator for the system with point observation at $x = a$. Note that $\sqrt{2}\,\mathcal{O}_0$ is unitary from M_0 to W.

4. Control problems

We turn now to control problems and first consider the inhomogeneous boundary value problem on $\mathcal{Q} = [0,\ell] \times [0,T]$

$$\tag{4.1} \frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}$$

with $z = 0$ at $x = 0,\ell$ and $y = 0 = z$ at $t = 0$.

It is well known that (4.1) has a unique solution $\begin{pmatrix} y \\ z \end{pmatrix}$ in $C([0,T] \to W)$ for each $\begin{pmatrix} f \\ g \end{pmatrix}$ in V_ε so we can evaluate at $t = T$ and associate with this system the operator

$$\tag{4.2} \mathcal{U}_\varepsilon : V_\varepsilon \hookrightarrow L^2(\mathcal{Q}) \to W : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \end{pmatrix}\Big|_{t=T} .$$

We must also consider the pointwise problem:

$$\frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} y \\ z \end{pmatrix} + \delta(x-a)\begin{pmatrix} f_0(t) \\ g_0(t) \end{pmatrix} \tag{4.3}$$
$$\text{with } z=0 \text{ at } x=0,\ell \text{ and } y=0=z \text{ at } t=0.$$

It is easily seen that (4.3) has a unique solution $\begin{pmatrix} y \\ z \end{pmatrix}$ in $C([0,T]\to W)$ for each $\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$ in Z so, as with (4.2), we can introduce the associated solution operator

$$\mathcal{U}_0 : Z \to W : \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \end{pmatrix}\Bigg|_{t=T}. \tag{4.4}$$

We next wish to show controllability: that, for an arbitrary target state $\begin{pmatrix} y^T \\ z^T \end{pmatrix} \in W$, there is some forcing term for (4.1) (or for (4.3), respectively) for which this target is attained at time $t=T$, i.e., to show that $\mathcal{V}_\varepsilon : V_\varepsilon \to W$ and $\mathcal{V}_0 : Z \to W$ are surjective. We will then define control operators by selecting this forcing term (control), for each target in W, to have minimal norm.

A standard computation [3] shows that $\mathcal{U}_\varepsilon$ and $\mathcal{V}_\varepsilon$ are adjoints. To see this, let $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and $\begin{pmatrix} y \\ z \end{pmatrix}$ be smooth (classical) solutions of (1.2) and of (4.1), respectively. Using integration by parts we have

$$\begin{aligned}
0 &= \int_Q \left\langle \frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle dx\,dt \\
&= -\int_Q \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \frac{\partial}{\partial t}\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} y \\ z \end{pmatrix} \right\rangle dx\,dt \\
&\quad + \int_0^\ell \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \Bigg|_{t=0}^{t=T} dx - \int_0^T \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \Bigg|_{x=0}^{x=\ell} dt \\
&= -\int_Q \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle dx\,dt + \int_0^\ell \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \Bigg|_{t=T} dx
\end{aligned}$$

[3]Notationally, we will use here $\langle\cdot,\cdot\rangle$ for the $\mathbb{R}^2$ inner product and $(\cdot,\cdot)$ — with a subscript — for the inner products of W and of V_ε.

so

$$(4.5)\qquad \int_0^\ell \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle \Bigg|_{t=T} dx = \int_a^{a+\varepsilon}\int_0^T \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle dt\,dx.$$

Recalling the definitions of the solution maps $\mathcal{V}_\varepsilon$ and $\mathcal{U}_\varepsilon$, this identity becomes

$$(4.6)\qquad \left(\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}, \mathcal{U}_\varepsilon \begin{pmatrix} f \\ g \end{pmatrix} \right)_W = \left(\mathcal{V}_\varepsilon \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right)_{V_\varepsilon}$$

and, since smooth solutions are dense, this extends by continuity to all functions from W, V_ε — i.e., $[\mathcal{U}_\varepsilon]^* = \mathcal{V}_\varepsilon$, as asserted above. An essentially identical argument applies to (4.3) to give

$$\left(\begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}, \mathcal{U}_0 \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \right)_W = \left(\mathcal{V}_0 \begin{pmatrix} \varphi^T \\ \psi^T \end{pmatrix}, \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \right)_Z$$

for functions from W, Z. Combining, we have shown that

$$(4.7)\quad [\mathcal{U}_\varepsilon : V_\varepsilon \to W]^* = \mathcal{V}_\varepsilon : W \to V_\varepsilon \quad \text{and} \quad [\mathcal{U}_0 : Z \to W]^* = \mathcal{V}_0 : W \to Z\,.$$

Since we already know that $(1/\sqrt{2\varepsilon})\mathcal{V}_\varepsilon$ is norm-preserving, hence injective, it follows that $\mathcal{U}_\varepsilon$ must be surjective to W and that $(1/\sqrt{2\varepsilon})\mathcal{U}_\varepsilon$ is also norm-preserving from $[\mathcal{N}(\mathcal{U}_\varepsilon)]^\perp = \mathcal{R}(\mathcal{U}_\varepsilon^*) = M_\varepsilon$, which selects the control of minimum norm. Much as for $\mathcal{O}_\varepsilon$ earlier, we obtain the desired right inverse of $\mathcal{U}_\varepsilon$ by using the control operator

$$(4.8)\qquad \begin{aligned} \mathcal{C}_\varepsilon := (1/2\varepsilon)\mathcal{V}_\varepsilon \quad &: W \to M_\varepsilon \quad [\subset V_\varepsilon \hookrightarrow [L^2(\mathcal{Q})]^2] \\ &: \begin{pmatrix} y^T \\ z^T \end{pmatrix} \longmapsto \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Bigg|_{V_\varepsilon} \end{aligned}$$

where $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is given by the homogeneous equation (1.2) with $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} y^T \\ z^T \end{pmatrix}$ at $t = T$. This inverts (4.2) so $\mathcal{U}_\varepsilon\,\mathcal{C}_\varepsilon = \mathbf{1}_W$, i.e., the control $\mathcal{C}_\varepsilon \begin{pmatrix} y^T \\ z^T \end{pmatrix}$ does determine a solution satisfying

$$(4.9)\qquad \begin{pmatrix} y \\ z \end{pmatrix}\Bigg|_{t=T} = \begin{pmatrix} y^T \\ z^T \end{pmatrix}.$$

This proves controllability of the system (4.1) and, indeed, provides the control with minimal V_ε norm: for a given target state $\begin{pmatrix} y^T \\ z^T \end{pmatrix} \in W$, we solve the initial boundary value problem (1.2) with the 'initial' condition

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{t=T} = \begin{pmatrix} y^T \\ z^T \end{pmatrix} \tag{4.10}$$

and then use the resulting

$$\begin{pmatrix} f_\varepsilon \\ g_\varepsilon \end{pmatrix} := \frac{1}{2\varepsilon}\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{\mathcal{Q}_\varepsilon} \tag{4.11}$$

as a control in the system (4.1) to get (4.9).

Similarly, for the point control problem we have

$$\begin{aligned} \mathcal{C}_0 := (1/2)\mathcal{V}_0 \quad &: W \to M_0 \ [\subset Z\,] \\ &: \begin{pmatrix} y^T \\ z^T \end{pmatrix} \longmapsto \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} := \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\Big|_{x=a} \end{aligned} \tag{4.12}$$

inverting (4.4) so $\mathcal{U}_0\,\mathcal{C}_0 = \mathbf{1}_W$ with $\mathcal{C}_0$ giving the control of minimum norm in Z. In view of (3.8), (3.11), we note that these minimum norm controls satisfy

$$\left\| \mathcal{C}_\varepsilon \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_{V_\varepsilon} = \frac{1}{2\varepsilon}\left\| \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_W \quad \text{and} \quad \left\| \mathcal{C}_0 \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_Z = \frac{1}{2}\left\| \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_W \tag{4.13}$$

for each target state in W.

5. Convergence as $\varepsilon \to 0$

For any fixed target $\begin{pmatrix} y^T \\ z^T \end{pmatrix} \in W$, we let $\begin{pmatrix} f_\varepsilon \\ g_\varepsilon \end{pmatrix}$ be the optimal control pair $\mathcal{C}_\varepsilon \begin{pmatrix} y^T \\ z^T \end{pmatrix} \in V_\varepsilon \hookrightarrow L^2(\mathcal{Q})$ and let $\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}$ be the corresponding controlled solution of (4.1) using this control; similarly, we let $\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} := \mathcal{C}_0 \begin{pmatrix} y^T \\ z^T \end{pmatrix} \in Z$ and let $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ be the solution of (4.3) corresponding to that control. In this section we wish to show convergence of these optimal controls — when appropriately scaled in the fixed space $V_* := C([0,1] \to Z)$ — and also to show the strong convergence in $C([0,T] \to W)$ of the corresponding controlled solutions.

For the first, we begin by introducing a correspondence $\mathcal{T}_\varepsilon$ between functions on $\mathcal{Q}_\varepsilon = [a, a+\varepsilon] \times [0,T]$ and functions on the fixed domain $\mathcal{Q}_* = [0,1] \times [0,T]$. Thus we set

$$[\mathcal{T}_\varepsilon f]\,(s) := \sqrt{\varepsilon}\, f(a+\varepsilon s) \qquad \text{for } s \in [0,1]$$

when $[x \mapsto f(x)]$ is given for $x \in [a, a+\varepsilon]$. Overloading notation somewhat, we use the same symbol $\mathcal{T}_\varepsilon$ for several maps: considering this as an operator: $L^2(a, a+\varepsilon) \to L^2(0,1)$ and also as an operator: $C[a, a+\varepsilon] \to C[0,1]$ — as well as operators for corresponding Z-valued functions:

$$\begin{array}{l} L^2(\mathcal{Q}_\varepsilon \to \mathbb{R}^2) = L^2([a,a+\varepsilon] \to Z) =: V_\varepsilon \\ \qquad\qquad\qquad\qquad \longrightarrow \;\; L^2(\mathcal{Q}_* \to \mathbb{R}^2) = L^2([0,1] \to Z), \\ [\, C([0,\ell] \to Z) \;\; \overset{\text{restriction}}{\longrightarrow} \;]\quad C([a,a+\varepsilon] \to Z) \\ \qquad\qquad\qquad\qquad \longrightarrow \;\; C([0,1] \to Z) =: \mathcal{V}_*\,. \end{array}$$

Similarly, we define $\mathcal{T}_0 : \mathbb{R} \to C[0,1] \hookrightarrow L^2(0,1)$ by

$$[\mathcal{T}_0 r]\,(s) := r \quad \text{(independent of } s\text{) for } s \in [0,1]$$

and again overload notation to consider this also as denoting operators:

$$Z \;\longrightarrow\; \mathcal{V}_* := C([0,1] \to Z) \hookrightarrow L^2(\mathcal{Q}_* \to \mathbb{R}^2) = L^2([0,1] \to Z).$$

Note that one has $\|\mathcal{T}_\varepsilon f\| = \|f\|$ and $\|\mathcal{T}_0 f\| = \|f\|$ for the L^2-norms so, in view of (4.13), we have

$$\left\| \mathcal{T}_\varepsilon \mathcal{C}_\varepsilon \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_{L^2([0,1]\to Z)} = \left\| \mathcal{T}_0 \mathcal{C}_0 \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_{L^2([0,1]\to Z)} = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} y^T \\ z^T \end{pmatrix} \right\|_W .$$

This is already sufficient to show weak convergence (for a subsequence; to some limit) in $L^2(\mathcal{Q}_* \to \mathbb{R}^2)$.

However, it is now easy to obtain the desired strong convergence in $\mathcal{V}_*$ from our characterizations of the control operators $\mathcal{C}_\varepsilon = (1/2\varepsilon)\mathcal{V}_\varepsilon$, $\mathcal{C}_0 = (1/2)\mathcal{V}_0$ in terms of the solution operators for the homogeneous equation (1.2). We need simply note that

$$
\begin{aligned}
\left[\mathcal{T}_\varepsilon \begin{pmatrix} f_\varepsilon \\ g_\varepsilon \end{pmatrix}\right](s,\cdot) &= \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(a+\varepsilon s,\cdot) \\
&\xrightarrow{\varepsilon\to 0} \begin{pmatrix} \varphi(a,\cdot) \\ \psi(a,\cdot) \end{pmatrix} = \left[\mathcal{T}_\varepsilon \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}\right](s,\cdot)
\end{aligned} \tag{5.1}
$$

with the convergence (uniform: $[0,1] \to Z$) following here from the continuity: $[0,\ell] \to Z$ of the solution $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ of (1.2) with $\left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{t=T} = \begin{pmatrix} y^T \\ z^T \end{pmatrix}$, i.e., from the strong continuity in x of $[\hat{G}(x) : Z \to Z]$.

We remark in passing that it is now not difficult, with this in hand, to show the convergence, as support shrinks, of the controls themselves to the limit $\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}\delta(x-a)$ with convergence in the sense of distributions — more precisely, in the weak-* topology of the dual space to $C([0,\ell] \to Z)$. We omit this argument, since it is closely related to the argument we are about to present.

We now proceed to show the strong convergence in $C([0,T] \to W)$ of the controlled solutions. To this end, taking any $\tau \in (0,T)$, we let $\begin{pmatrix} p \\ q \end{pmatrix}$, $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ be the solutions of the homogeneous equation (1.2) such that

$$
\left.\begin{pmatrix} p \\ q \end{pmatrix}\right|_{t=\tau} = \left[\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix} - \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}\right]_{t=\tau} \quad \text{and} \quad \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_{t=T} = \begin{pmatrix} y^T \\ z^T \end{pmatrix}.
$$

Much as in obtaining (4.5) and using (4.8), (4.12), we then get

$$
\begin{aligned}
&\left\| \left[\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix} - \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}\right]_{t=\tau} \right\|_W^2 = E(p,q) \\
&\quad = \int_0^\tau \left[\frac{1}{2\varepsilon}\int_a^{a+\varepsilon} \left\langle \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle dx - \frac{1}{2}\left.\left\langle \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle\right|_{x=a} \right] dt \\
&\quad = \frac{1}{2\varepsilon}\int_a^{a+\varepsilon} \left[\left(\chi_\tau \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)_Z - \left.\left(\chi_\tau \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)_Z\right|_{x=a} \right] dx
\end{aligned}
$$

where χ_τ is the operator on Z setting values to 0 for $t > \tau$. Now observe that

$$\left.\begin{pmatrix} p \\ q \end{pmatrix}\right|_x = \hat{G}(x-a)\left.\begin{pmatrix} p \\ q \end{pmatrix}\right|_{x=a}, \text{ etc.,}$$

and that the continuity of $[r \mapsto \hat{G}(r)]$ (with $[\hat{G}(r)]^* = \hat{G}(-r)$ so also continuous in r to the strong operator topology) gives

$$\left\| \left[[\hat{G}(r)]^* \chi_\tau \hat{G}(r) - \chi_\tau \right] \left.\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right|_a \right\|_Z \leq \delta(\varepsilon)$$

with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and from this we have

$$\left\| \left[\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix} - \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \right]_{t=\tau} \right\|_W \leq \frac{\delta(\varepsilon)}{\sqrt{2}}.$$

This, of course, gives the desired convergence in $C([0,T] \to W)$ of the controlled solutions.

It is interesting also to consider an alternative argument for this convergence. Note that

$$\begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix} = \sqrt{\frac{1}{\ell}} \begin{pmatrix} \cos nx \\ i \sin nx \end{pmatrix}, \quad \lambda_n = in\,, \quad n \in \mathbb{Z}$$

are the eigenfunctions and eigenvalues of the operator $\mathbf{A}$ of (3.2). This family forms an orthonormal basis for W so the functions $\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}$, $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ can be expanded as

$$\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix} = \sum_n c_n^\varepsilon(t) \begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \sum_n c_n^0(t) \begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix}, \tag{5.2}$$

where, using (4.8), (4.12),

$$\begin{aligned} c_n^\varepsilon(t) &= \int_0^t e^{\lambda_n(t-\tau)} \frac{1}{2\varepsilon} \int_a^{a+\varepsilon} \left\langle \begin{pmatrix} \varphi(x,\tau) \\ \psi(x,\tau) \end{pmatrix}, \begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix} \right\rangle dx\, d\tau, \\ c_n^0(t) &= \int_0^t e^{\lambda_n(t-\tau)} \frac{1}{2} \left.\left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \xi_n \\ \zeta_n \end{pmatrix} \right\rangle\right|_{x=a} d\tau. \end{aligned} \tag{5.3}$$

Noting that $\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix}$ and $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$ are in $C([0,T] \to W)$, the expansions (5.2) give

$$\sum_n |c_n^\varepsilon(t)|^2 = \left\| \begin{pmatrix} y_\varepsilon(\cdot,t) \\ z_\varepsilon(\cdot,t) \end{pmatrix} \right\|_W^2 , \qquad \sum_n |c_n^0(t)|^2 = \left\| \begin{pmatrix} y_0(\cdot,t) \\ z_0(\cdot,t) \end{pmatrix} \right\|_W^2 \tag{5.4}$$

while from (5.3) we have

$$\left\| \begin{pmatrix} y_\varepsilon(\cdot,t) \\ z_\varepsilon(\cdot,t) \end{pmatrix} - \begin{pmatrix} y_0(\cdot,t) \\ z_0(\cdot,t) \end{pmatrix} \right\|_W^2 = \sum_n |c_n^\varepsilon(t) - c_n^0(t)|^2 = \sum_{|n|>N} + \sum_{|n|\le N} .$$

Choosing N big enough, we can make the first sum small; then choosing ε small enough, we can make each term of the (finite) second sum small, uniformly in $t \in [0,T]$, since

$$\begin{aligned} &c_n^\varepsilon(t) - c_n^0(t) \\ &\quad = \frac{1}{2}\int_0^t e^{\lambda_n(t-\tau)} \int_0^1 \left[\left\langle \begin{pmatrix} \hat\varphi_\varepsilon \\ \hat\psi_\varepsilon \end{pmatrix}, \begin{pmatrix} \hat\xi_n \\ \hat\zeta_n \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} \hat\varphi_0 \\ \hat\psi_0 \end{pmatrix}, \begin{pmatrix} \hat\xi_n \\ \hat\zeta_n \end{pmatrix}\Bigg|_{s=0} \right\rangle \right] ds\, d\tau \end{aligned}$$

and we can again use the fact that $x \mapsto (\varphi, \psi)$ is continuous to Z. As with the previous argument, this shows that

$$\sup_{0\le t\le T} \left\| \begin{pmatrix} y_\varepsilon(\cdot,t) \\ z_\varepsilon(\cdot,t) \end{pmatrix} - \begin{pmatrix} y_0(\cdot,t) \\ z_0(\cdot,t) \end{pmatrix} \right\|_W^2 \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

6. Another approach to the wave equation

Let us start with the observation problem for the equation

$$v_{tt} = v_{xx}\,, v = 0 \text{ at } x = 0, \ell$$

$$v\Big|_{t=T} = v_0 \in L^2(0,\ell), \quad v_t\Big|_{t=T} = v_1 \in H^{-1}(0,\ell)\,.$$

Then

$$\{v, v_t\} \in C([0,T] \to W) \quad \text{with } W = L^2(0,\ell) \times H^{-1}(0,\ell)$$

and

$$\{v, v_x\} \in C([0,\ell] \to Z) \quad \text{with } Z = L^2(0,T) \times \left(H^1(0,T)\right)'\,.$$

Introduce the observation operator $\mathcal{O}_\varepsilon$ acting from W to $V_\varepsilon := L^2([a, a+\varepsilon] \to Z)$ by the rule

$$\mathcal{O}_\varepsilon\{v_0, v_1\} = \{v, v_x\}|_{\mathcal{Q}_\varepsilon}\,.$$

Let us set

$$\omega_n = n\pi/\ell\,, \ \ \varphi_n(x) = \sqrt{2/\ell}\,\sin\omega_n x\,, \ \ n\in \mathbb{N}\,.$$

and

$$v_0(x) = \sum_n \alpha_n\varphi_n(x)\,, \ \ v_1(x) = \sum_n \beta_n\varphi_n(x)\,.$$

Then

$$v(x,t) = \sum_n \left(\alpha_n \cos\omega_n(T-t) + \beta_n \frac{\sin\omega_n(T-t)}{\omega_n}\right)\varphi_n(x)\,,$$

$$v(x,t) = \sum_n \left(\alpha_n \cos\omega_n(T-t) + \beta_n \frac{\sin\omega_n(T-t)}{\omega_n}\right)\varphi_n'(x)$$

and

$$\begin{aligned}\|\mathcal{O}_\varepsilon\{v_0,v_1\}\|^2 &= \int_a^{a+\varepsilon} dx\left[\|v(\cdot,x)\|^2_{L^2(0,T)} + \|v_x(\cdot,x)\|^2_{(H^1(0,T))'}\right]\\ &\asymp \int_a^{a+\varepsilon} dx \sum_n \left(|\alpha_n|^2 + |\beta_n/\omega_n|^2\right)\left(\sin^2\omega_n x + \cos^2\omega_n x\right)\\ &\asymp \varepsilon\left[\|v_0\|^2_{L^2(0,\ell)} + \|v_1\|^2_{H^{-1}(0,\ell)}\right]\,.\end{aligned}$$

Consider now the control problem for the equation

$$w_{tt} = w_{xx} + F^\varepsilon\,; w = 0 \,\text{at}\ \ x = 0,\ell$$

$$w = w_t = 0\,\text{at}\ \ t = 0\,.$$

If $F^\varepsilon \in L^2(\mathcal{Q}_\varepsilon)$ then

$$\{w,w_t\}\in C([0,T]\to W')\ \ \text{with}\ \ W' = H^1_0(0,\ell)\times L^2(0,\ell)\,.$$

Let us represent F^ε in the form $f^\varepsilon(x,t) + g^\varepsilon(x,t)$ and note that

$$\begin{aligned}\int_0^\ell g^\varepsilon(x,t)v(x,t)\,dx \ &= G^\varepsilon(x,t)v(x,t)\Big|_{x=0}^{x=\ell} - \int_0^\ell G^\varepsilon(x,t)v_x(x,t)\,dx\\ &= -\int_0^\ell G^\varepsilon(x,t)v_x(x,t)\,dx\end{aligned}$$

where $G^\varepsilon(x,t) = \int_0^t g^\varepsilon(x,\tau)\,d\tau$. Using integration by parts we have

$$\int_0^\ell \left[wv_t - w_t v\right]_{t=T}\,dx = \int_a^{a+\varepsilon}\int_0^T \left[f^\varepsilon v - G^\varepsilon v_x\right]\,dtdx\,.$$

Then for

$$\{f^\varepsilon, G^\varepsilon\}\in L^2\left([a,a+\varepsilon]; L^2(0,T)\times H^1(0,T)\right) = V_\varepsilon{}'$$

we get

$$\langle w(\cdot,T),v_1\rangle - \langle w_t(\cdot,T),v_0\rangle \tag{6.1}$$

where $\langle \cdot , \cdot \rangle$ here denotes the duality between $H^{-1}(0,\ell)$ and $H_0^1(0,\ell)$.

Introduce the operator $\mathcal{U}_\varepsilon$ acting from ${V_\varepsilon}'$ to W' by the rule

$$\begin{pmatrix} f^\varepsilon \\ -G^\varepsilon \end{pmatrix} \mapsto \begin{pmatrix} w(\cdot, T) \\ -w_t(\cdot, T) \end{pmatrix} .$$

Then $\mathcal{U}_\varepsilon = (\mathcal{O}_\varepsilon)^*$ and

$$\mathcal{C}_\varepsilon := \left(\mathcal{U}_\varepsilon \Big|_{M_\varepsilon} \right)^{-1} = \left({\mathcal{O}_\varepsilon}^* \Big|_{M_\varepsilon} \right)^{-1} \quad \text{and} \quad \|\mathcal{C}_\varepsilon\| \asymp 1/\sqrt{\varepsilon}.$$

Let

$$w_0(x) = \sum_n a_n \varphi_n(x) \, , \quad w_1(x) = \sum_n b_n \varphi_n(x) \, .$$

Then, using (6.1), we obtain the solution of the control problem

$$w(\cdot, T) = w_0 \, , \quad w_t(\cdot, T) = w_1$$

in the form

$$f^\varepsilon(x,t) = \sum_n \varepsilon^{-1} \left[b_n \cos \omega_n (T-t) + a_n \omega_n \sin \omega_n (T-t) \right] \varphi_n(x)|_{[a,a+\varepsilon]} \ ,$$

$$G^\varepsilon(x,t) = \sum_n \varepsilon^{-1} \left[b_n \cos \omega_n (T-t) + a_n \omega_n \sin \omega_n (T-t) \right] \varphi_n'(x)|_{[a,a+\varepsilon]} \ .$$

We can prove, as we have done for the first order system, that the solution w^ε tends to the solution of the pointwise optimal control problem in the norm of $C([0,T] \to W')$ as $\varepsilon \to 0$.

Remark: All the results of this section continue to be valid for equations with x-dependent coefficients

$$\text{(6.2)} \qquad \frac{\partial}{\partial t} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} y \\ z \end{pmatrix} + Q(x) \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}$$

where $Q(x)$ is a continuous 2×2 matrix-valued function on $[0, \ell]$; we again consider the boundary conditions

$$\text{(6.3)} \qquad z\Big|_{x=0} = z\Big|_{x=\ell} = 0 \, ,$$

and initial conditions

$$\text{(6.4)} \qquad y\Big|_{t=0} = z\Big|_{t=0} = 0.$$

Since the situation is quite similar to those above for $Q \equiv 0$, we need only sketch the arguments.

The corresponding dual system is then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}+Q^*(x)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\tag{6.5}$$

with

$$\psi\Big|_{x=0}=\psi\Big|_{x=\ell}=0.\tag{6.6}$$

The corresponding family $\left\{\begin{pmatrix}\xi_n\\ \zeta_n\end{pmatrix}\right\}$ forms a Riesz basis in W with a dual basis $\left\{\begin{pmatrix}\tilde{\xi}_n\\ \tilde{\zeta}_n\end{pmatrix}\right\}$; the corresponding family $\left\{e^{\lambda_n t}\right\}$ is a Riesz basis in $L^2(0,T)$. If

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\Bigg|_{t=T}=\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\in W$$

then

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in C([0,T]\to W)\bigcap C([0,T]\to Z)$$

as can easily be proved using, for instance, the Fourier method and the representation

$$\begin{pmatrix}\varphi(x,t)\\ \psi(x,t)\end{pmatrix}=\sum_n \alpha_n e^{\lambda_n(t-T)}\begin{pmatrix}\xi_n(x)\\ \zeta_n(x)\end{pmatrix}.$$

As before, we can introduce the solution operator for (6.5)

$$\mathcal{V}_\varepsilon:\ \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\Bigg|_{t=T}\mapsto\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\Bigg|_{\mathcal{Q}_\varepsilon}.$$

Since $0<\inf_{x,n}|\xi_n(x)|^2\le\sup_{x,n}|\xi_n(x)|^2<\infty$, one has

$$\begin{aligned}\left\|\mathcal{V}_\varepsilon\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|^2&=\int_a^{a+\varepsilon}\int_0^T\left(|\varphi(x,t)|^2+|\psi(x,t)|^2\right)dt\,dx\\&\asymp\int_a^{a+\varepsilon}\sum_n\left(|\xi_n(x)|^2+|\zeta_n(x)|^2\right)|\alpha_n|^2\,dx\\&\asymp\varepsilon\sum_n|\alpha_n|^2\asymp\varepsilon\left\|\begin{pmatrix}\varphi^T\\ \psi^T\end{pmatrix}\right\|^2.\end{aligned}$$

Next, introduce the operator

$$\mathcal{U}_\varepsilon : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} y \\ z \end{pmatrix}\Bigg|_{t=T}$$

from $V_\varepsilon = L^2([a, a+\varepsilon]; Z)$ to W. As with (3.9), (4.6), we can prove that $\mathcal{U}_\varepsilon = (\mathcal{O}_\varepsilon)^*$. Then

$$\mathcal{C}_\varepsilon := \left(\mathcal{U}_\varepsilon\Big|_{M_\varepsilon}\right)^{-1} = \left({\mathcal{O}_\varepsilon}^*\Big|_{M_\varepsilon}\right)^{-1} \text{ and } \|\mathcal{C}_\varepsilon\| \asymp 1/\sqrt{\varepsilon}.$$

In terms of the representation

$$\begin{pmatrix} y \\ z \end{pmatrix}\Bigg|_{t=T} = \begin{pmatrix} y^T(x) \\ z^T(x) \end{pmatrix} = \sum_n \beta_n \begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix} \in W,$$

for the prescribed terminal state we have

$$\mathcal{C}_\varepsilon \begin{pmatrix} y^T \\ z^T \end{pmatrix} =: \begin{pmatrix} f_\varepsilon \\ g_\varepsilon \end{pmatrix} = \sum_n \frac{\beta_n \theta_n(t)}{\int_a^{a+\varepsilon} \left(|\xi_n(x)|^2 + |\zeta_n(x)|^2\right)} \begin{pmatrix} \xi_n(x) \\ \zeta_n(x) \end{pmatrix}\Bigg|_{\mathcal{Q}_\varepsilon},$$

where $\{\theta_n(t)\}$ is the Riesz basis in $L^2(0,T)$ dual to $\left\{e^{\lambda_n(t-T)}\right\}$. Similarly, the optimal pointwise control now has the form

$$\mathcal{C}_0 \begin{pmatrix} y^T \\ z^T \end{pmatrix} =: \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = \sum_n \frac{\beta_n \theta_n(t)}{|\xi_n(a)|^2 + |\zeta_n(a)|^2} \begin{pmatrix} \xi_n(a) \\ \zeta_n(a) \end{pmatrix}\Bigg|_{\mathcal{Q}_\varepsilon}.$$

Inserting these explicit formulas in the equations (and noting the continuity in x of $|\xi_n(x)|^2 + |\zeta_n(x)|^2$) shows that

$$\begin{pmatrix} y_\varepsilon \\ z_\varepsilon \end{pmatrix} \to \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \text{ in } C([0,T] \to W) \text{ as } \varepsilon \to 0.$$

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