How violent are fast controls, III*

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Abstract

For (coupled systems of) partial differential equations for which nullcontrol is possible in arbitrarily short time, the typical blowup rate for the control cost is exponential in $1/T$. It is shown how to derive this rate for a variety of systems, including the thermoelastic system with control restricted to a small patch in the domain and to a single component (thermal, displacement, or velocity).

Key words: Distributed parameter system, optimal control, nullcontrol, blowup rate, energy functional, partial differential equation, thermoelastic.

AMS subject classifications. ...

1 Introduction

We consider linear autonomous control systems of the form

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}\,u \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.1}$$

where $\mathcal{A}$ generates a C_0 semigroup $\mathbf{S}(\cdot)$ on the reflexive state space $\mathcal{X}$ and $\mathcal{B} : \mathcal{U} \to \mathcal{X}$ is a suitable control operator $\mathbf{x} = \mathbf{x}(\cdot;\mathbf{x}_0, u)$ will denote the (mild) solution of (1.1):

$$\mathbf{x}(t;\mathbf{x}_0, u) = \mathbf{S}(t)\mathbf{x}_0 + \int_0^t \mathbf{S}(t-\tau)\,\mathcal{B}u(\tau)\,d\tau \tag{1.2}$$

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for $0 \le t \le T$ and[1] $u \in \mathcal{U}_T$. Our concern is with blowup as $T \to 0$ of the *nullcontrol cost* $\mathfrak{C}(T) = \mathfrak{C}(T; \mathcal{A}, \mathcal{B})$ for the linear autonomous control system (1.1) — i.e., the minimal constant such that

$$\begin{cases} \text{For each } \mathbf{x}_0 \in \mathcal{X} \text{ there exists } u(\cdot) \in \mathcal{U}_T \text{ with} \\ \quad \|u(\cdot)\| \le \mathfrak{C}(T)\,|\mathbf{x}_0| \qquad \mathbf{x}(T; \mathbf{x}_0, u) = 0 \end{cases} \tag{1.3}$$

The blowup rate of controls as one attempts this with $T \to 0$ was treated for the finite dimensional case $\mathcal{X} = \mathbb{R}^K$ in [13, 15]; our interest here is in the infinite dimensional case.

For reference, we note without proof the standard duality.

Theorem 1.1. *For* (1.1) *as above,* (1.3) *is equivalent to the* observation inequality:

$$|y(0)| \le \mathfrak{C}(T) \left[\int_0^T |\mathcal{B}^* y(t)|^2\, dt\right]^{1/2} \tag{1.4}$$

for every X^**-valued solution* $\mathbf{y}$ *on* $[0, T)$ *of the adjoint equation:*

$$-\dot{\mathbf{y}} = \mathcal{A}^* \mathbf{y} \qquad \text{(i.e., } \mathbf{y}(t) = \mathbf{S}^*(s)\mathbf{y}(t+s)) \tag{1.5}$$

[If X_* *is an* $\mathbf{S}^*$*-invariant subspace of* X^**, then* (1.4) *for* X_**-valued solutions of* (1.5) *is equivalent to* (1.3) *holding for all* $\mathbf{x}_0 \perp X_*$*.]* ■

It was shown in [12], for the one-dimensional heat equation, that

$$\mathfrak{C}(T) \le e^{\mathfrak{c}/T} \qquad \text{for small } T > 0. \tag{1.6}$$

for some constant $\mathfrak{c}$, i.e., $\ln \mathfrak{C}(T) = \mathcal{O}(1/T)$ as $T \to 0$; that this is the best possible in that setting was already known from [5]. We refer to [14, 8] and their references for some previous work on blowup rates for infinite dimensional problems. We do note that several papers (e.g., [7, 4], etc.) have obtained (1.6) for a variety of problems so, with this in mind, we take the blowup rate of (1.6) as 'standard' and define the nullcontrollability rate

[1] We will consistently take $\mathcal{U}_T = L^2([0, T] \to \mathcal{U})$, while noting that, as in [15], other norms could be handled similarly.

constant[2]

$$\mathfrak{c}_* = \mathfrak{c}_*(\mathcal{A},\mathcal{B}) := \limsup_{T\to 0}\ T\ln \mathfrak{C}(T;\mathcal{A},\mathcal{B}) \tag{1.7}$$

Our objective, then, is to show for various $(\mathcal{A},\mathcal{B})$ in infinite dimensional settings that we have $\mathfrak{c}_*(\mathcal{A},\mathcal{B}) < \infty$ — better, to estimate $\mathfrak{c}_*(\mathcal{A},\mathcal{B})$.

For coupled systems with control restricted to one component, blowup estimates of the form (1.6) have not been available in the case of local control. We do note that blowup estimates have recently been obtained [11, 3] for nullcontrollability with local control (equivalently, for local observation) in a single component in the context of the thermoelastic system

$$w_{tt} + \Delta^2 w - \alpha\Delta\vartheta = 0, \qquad \vartheta_t - \Delta\vartheta + \alpha\Delta w_t = 0 \tag{1.8}$$

which was also treated[3] in [8]. These papers, each following the approach of the sequence [6, 9, 2], showed for this system that

$$\left\{\begin{array}{c}\text{For any } \beta > 1 \text{ there is } \mathfrak{c}_\beta \text{ such that (for small } T > 0\text{):}\\ \mathfrak{C}(T;\mathcal{A},\mathcal{B}) \le \exp\left[\mathfrak{c}_\beta T^{-\beta}\right]\end{array}\right. \tag{1.10}$$

This $\exp\left[\mathcal{O}(1/T^\beta)\right]$ estimate is, of course, only mildly weaker than the expected asymptotics (1.6), which corresponds to the limit case $\beta = 1$, but does not show that $\mathfrak{c}_*(\mathcal{A},\mathcal{B}) < \infty$.

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[2] If (1.1) is not rapidly nullcontrollable (i.e., if there is a minimum nullcontrol time so $\mathfrak{C}(T)$ would not be defined for small $T > 0$), then we take $\mathfrak{c}_* = \infty$; also — although no such cases are known — (1.7) might conceivably give $\mathfrak{c}_* = \infty$ even for a rapidly nullcontrollable example. On the other hand, if $\ln \mathfrak{C}(T;\mathcal{A},\mathcal{B}) = o(1/T)$ — as, e.g., in the finite dimensional case: c.f., [13, 15] where $\mathfrak{C}(T;\mathcal{A},\mathcal{B}) = \mathcal{O}((1/T)^\beta)$ — we say that $\mathfrak{c}_* = 0+$.

[3] Actually, [8] treated the more general system

$$w_{tt} - \gamma[\Delta w]_{tt} + \Delta^2 w - \alpha\Delta\vartheta = 0, \qquad \vartheta_t - \Delta\vartheta + \alpha\Delta w_t = 0 \tag{1.9}$$

for $\gamma \ge 0$ — i.e., a Kirchhoff plate model when $\gamma > 0$ rather than the Euler-Bernoulli model of (1.8) for $\gamma = 0$. While it was possible to handle global control/observation ($\omega = \Omega$) for (1.9), we note that (1.9) with $\gamma > 0$ is not rapidly nullcontrollable using local control restricted to $\omega \subsetneq \Omega$. [It should be noted that [14, 8] do not treat *local* control — taking (1.1) to be a partial differential equation with $\mathcal{X}$ a space of functions on a region Ω with the range of $\mathcal{B}$ restricted to support in some small patch $\omega \subset \Omega$; to show nullcontrollability at all with control restricted to general (small, open) ω seems to demand some use of Carleman estimates.] Since, in contrast to [8], we are now concerned with this localization of control as well as with rapid nullcontrollability, we restrict our attention here to the Euler-Bernoulli model (1.8) with 'hinged' boundary conditions.

Stimulated by these two papers, our objective here is to show that a modification of the [6, 9, 2] approach can, indeed, give a finite nullcontrollability rate $\mathfrak{c}_*$ for this thermoelastic problem (1.8) with control restricted to a single component and with support in a patch ω and can even estimate $\mathfrak{c}_*(\mathcal{A}, \mathcal{B})$ in this case. Rather than organizing the exposition to follow the shortest route to that result, we take the opportunity to present related results which appear to be of some independent interest.

2 Principal results

This section might be considered a single theorem, but it is split for presentation here in two parts. The first part, the heart of our analysis, compares the nullcontrollability results for a pair of systems like (1.1) with the same system operator $\mathcal{A}$ but with different control operators $\mathcal{B}_1$ and $\mathcal{B}_2$ as a more sophisticated version of the obvious comparison

$$\mathfrak{C}(T; \mathcal{A}, \mathcal{B}\mathcal{K})) \leq \|\mathcal{K}^{-1}\| \, \mathfrak{C}(T; \mathcal{A}, \mathcal{B}) \tag{2.1}$$

available when $\mathcal{K}$ is boundedly invertible. The second part, somewhat more technically oriented, considers an apparently weaker property of (1.1) — seeking only to approximate the nullcontrollability: for some $d > 0$ and every $\mathbf{x}_0 \in \mathcal{X}$ (and small $T > 0$), there exists $u(\cdot) \in \mathcal{U}_T$ such that

$$|\mathbf{x}(T; \mathbf{x}_0, u)| \leq e^{-d/T} |\mathbf{x}_0| \tag{2.2}$$

with not too great control cost, i.e., there is some $c > 0$ giving $(c, d) \in \Gamma$ where

$$\Gamma = \Gamma(\mathcal{A}, \mathcal{B}) := \left\{ (c, d) : \begin{array}{c} \text{For small } T > 0 \text{ and all } \mathbf{x}_0 \in \mathcal{X} \\ \text{one has (2.2) with } \|u\| \leq e^{c/T} |\mathbf{x}_0|. \end{array} \right\} \tag{2.3}$$

— and then shows that $\mathfrak{c}_*(\mathcal{A}, \mathcal{B}) < \infty$ if $\Gamma(\mathcal{A}, \mathcal{B})$ is nonempty.

We now turn to the relation between two systems $(1.1)_1$ and $(1.1)_2$ — with the same $\mathcal{A}$, but different control operators $\mathcal{B}_1$ and $\mathcal{B}_2$ so we are comparing

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}_j \, u \qquad \mathbf{x}(0) = \mathbf{x}_0 \qquad\quad (j = 1, 2)$$

Our goal will be to use a comparison problem $(1.1)_1$, for which we presumably already have information, to obtain information about the 'real problem' $(1.1)_2$ and the key assumption here will be a controlled form of (2.1) when restricted to certain subspaces: we will postulate the existence of a family of subspaces $\mathcal{X}_\sigma \subset \mathcal{X}^*$ such that

$$[\mathbf{H}] \qquad \begin{cases} i] & \text{each } \mathcal{X}_\sigma \text{ is invariant under } \mathbf{S}^*(\cdot) \qquad \text{and} \\ ii] & \mathbf{v} \perp \mathcal{X}_\sigma \quad \Longrightarrow \quad |\mathbf{S}(t)\mathbf{v}| \;\le\; C\, e^{-\alpha\sigma t}\, |\mathbf{v}| \\ iii] & \mathbf{w} \in \mathcal{X}_\sigma \quad \Longrightarrow \quad |\mathcal{B}_1^*\mathbf{w}| \;\le\; C\, e^{\gamma\sqrt{\sigma}}\, |\mathcal{B}_2^*\,\mathbf{w}| \\ \multicolumn{2}{l}{\text{for some constants } C, \alpha, \gamma > 0 \text{ and large } \sigma > 0.} \end{cases} \tag{2.4}$$

Theorem 2.1. *Suppose we already know that* $\mathfrak{c}_* = \mathfrak{c}_*(\mathcal{A}, \mathcal{B}_1) < \infty$ *and the control operators* $\mathcal{B}_1, \mathcal{B}_2$ *are related by the existence of a family of subspaces* $\mathcal{X}_\sigma \subset \mathcal{X}^*$ *satisfying* $[\mathbf{H}]$. *Then* $\Gamma(\mathcal{A}, \mathcal{B}_2)$ *is also nonempty: more specifically, one has* $(c, d) \in \Gamma(\mathcal{A}, \mathcal{B}_2)$ *whenever*

$$c > \underline{c}(s),\ \ 0 < d < \underline{d}(s) \ \textit{for some}\ s > \underline{s} \tag{2.5}$$

with $\underline{c}(s) = 2\mathfrak{c}_* + \gamma s,\ \underline{d}(s) = 2\alpha s^2 - \underline{c}(s),\ \underline{s} := (\gamma/4\alpha)\left[1 + \sqrt{1 + 8\alpha\mathfrak{c}_*/\gamma^2}\right]$.

Proof: Given $\mathfrak{c} > \mathfrak{c}_*(\mathcal{A}, \mathcal{B}_1)$, Theorem 1.1 gives the observation inequality:

$$|\mathbf{S}^*(\tau)\eta|^2 \le e^{2\mathfrak{c}/\tau} \int_0^\tau |\mathcal{B}_1^*\,\mathbf{S}^*(t)\,\eta|^2\,dt \tag{2.6}$$

for the adjoint system. The invariance [**H**-i] gives $\mathbf{w} = \mathbf{S}^*(t)\eta \in \mathcal{X}_\sigma$ for any $\eta \in \mathcal{X}_\sigma$, so [**H**-$iii$] then gives the observation inequality:

$$|\mathbf{S}^*(\tau)\,\eta|^2 \le e^{2\mathfrak{c}/\tau} C^2\, e^{2\gamma\sqrt{\sigma}} \int_0^\tau |\mathcal{B}_2^*\,\mathbf{S}^*(t)\,\eta|^2\,dt \tag{2.7}$$

for the new adjoint system. We have (2.7) for all $\eta \in \mathcal{X}_\sigma$ and, using Theorem 1.1 in reverse, this observation inequality in turn gives

$$\begin{cases} \text{For } \mathbf{x}_0 \in \mathcal{X} \text{ there exists } u(\cdot) \in \mathcal{U}_\tau \text{ with} \\ \qquad \|u(\cdot)\| \le e^\mu\, |\mathbf{x}_0| \qquad\qquad \hat{\mathbf{x}}(\tau; \mathbf{x}_0, u) \perp \mathcal{X}_\sigma \\ \ \text{where} \quad \mu = \mu(\sigma, \tau) = \mathfrak{c}/\tau + \ln C + \gamma\sqrt{\sigma} \end{cases} \tag{2.8}$$

[Here $\hat{\mathbf{x}}(\cdot)$ is the solution of $(1.1)_2$ and so is given by the analogue $(1.2)_2$ of (1.2) (i.e., replacement $\mathcal{B} \leftarrow\!\shortmid \mathcal{B}_2$) — which then gives

$$|\hat{\mathbf{x}}(\tau; \mathbf{x}_0, u)| \le K_1 |\mathbf{x}_0| + K_2\, e^\mu\, |\mathbf{x}_0| \le K\, e^\mu\, |\mathbf{x}_0| \tag{2.9}$$

where K_1 bounds $\mathbf{S}(\cdot)$ on $[0,\tau]$ and K_2 similarly[4] bounds $\mathbf{S}(\cdot)\,\mathcal{B}_2$, giving $K = K_1 + K_2$.] Setting $\tau = T/2$ for a given small $T > 0$ and extending the control $u(\cdot)$ of (2.8) to $\mathcal{U}_T$ as 0 on $(\tau, 2\tau] = (T/2, T]$, we may use (2.8) and [**H**-*ii*] to get

$$\begin{aligned} \|u\| &\le \exp[\mu(\sigma, T/2)]\,|\mathbf{x}_0|, \\ |\hat{\mathbf{x}}(T; \mathbf{x}_0, u)| &\le e^{-\alpha\sigma T/2}|\hat{\mathbf{x}}(\tau; \mathbf{x}_0, u)| \le \exp[-\lambda(\sigma, T)]\,|\mathbf{x}_0| \\ \text{with} \qquad & \lambda(\sigma, T) = \alpha\,\sigma\,T/2 - \mu(\sigma, T/2) - \ln K \end{aligned} \tag{2.10}$$

For any $s > 0$, setting $\sqrt{\sigma} = s/T$ then gives

$$\begin{aligned} \mu(\sigma, T/2) &= \quad (2\mathfrak{c}_* + \gamma s)/T + o(1/T) &&= \underline{c}(s)/T + o(1/T), \\ \lambda(\sigma, T) &= \left(2\alpha s^2 - \underline{c}(s)\right)/T + o(1/T) &&= \underline{d}(s)/T + o(1/T). \end{aligned} \tag{2.11}$$

The requirement $s > \underline{s}$ just ensures that $\underline{d}(s) > 0$ and then (2.5) gives $\mu(\sigma, T/2) < c/T$ and $\lambda(\sigma, T) > d/T$ for small enough T, so (2.10) shows that $(c, d) \in \Gamma(\mathcal{A}, \mathcal{B}_2)$. ■

Remark 2.2. The single relevant parameter in [**H**] is $\gamma/\sqrt{\alpha}$ since we may reparametrize the subspaces by $\sigma \leftarrow\!\!\dashv \alpha\sigma$; it is a minor exercise to see that this gives the same $(c, d) \in \Gamma(\mathcal{A}, \mathcal{B}_2)$ in (2.5). Somewhat more interesting is the observation that the key hypothesis [**H**-*iii*] is only used to obtain (2.7) from (2.6), i.e., to have

$$\int_0^\tau |\mathcal{B}_1^*\,\mathbf{S}^*(t)\,\eta|^2\,dt \le C^2\,e^{2\gamma\sqrt{\sigma}} \int_0^\tau |\mathcal{B}_2^*\,\mathbf{S}^*(t)\,\eta|^2\,dt$$

We note that, if we replaced [**H**-*iii*] by the weaker condition

$$\int_0^\tau |\mathcal{B}_1^*\,\mathbf{S}^*(t)\,\eta|^2\,dt \le C^2\,e^{2\gamma\sqrt{\sigma}}\,e^{2\tilde{\mathfrak{c}}/\tau} \int_0^\tau |\mathcal{B}_2^*\,\mathbf{S}^*(t)\,\eta|^2\,dt, \tag{2.12}$$

this would only have the effect of replacing $\mathfrak{c}$ by $(\mathfrak{c} + \tilde{\mathfrak{c}})$ in (2.8): one would still have $\mathfrak{c}_*(\mathcal{A}, \mathcal{B}_2) < \infty$ in the Corollary — and, indeed, this last would still be unaffected by a further weakening of (2.12) in replacing the upper limit τ of the integral on the left by, e.g., $\tau/2$. ■

[4]Strictly speaking, we only need K_2 to bound the related integral maps from $\mathcal{U}$-valued to $\mathcal{X}$-valued functions.

We now consider the significance of the set $\Gamma(\mathcal{A},\mathcal{B})$. In general, we cannot expect, for an arbitrary equation of the form (1.1), that there would be any possibility of (2.2) for any d and any control cost when T is small — certainly this is the case for the wave equation, where one has a finite speed of propagation, if control is spatially limited. [On the other hand, if the system is rapidly nullcontrollable (i.e., nullcontrollable for arbitrarily small $T>0$), then (2.2) is possible for every d by using a nullcontrol; if $\mathfrak{c}_*(\mathcal{A},\mathcal{B})<c<\infty$, we say that $(c,\infty)\in\Gamma(\mathcal{A},\mathcal{B})$.] It is immediate from its definition that $(c,d)\in\Gamma$ implies $(c',d')\in\Gamma$ whenever $c'\geq c,\ 0<d'\leq d$ so the interior of Γ is $\{(c,d):c>c_-(d)\}$ for some nondecreasing function[5] $c_-(\cdot)$ on $(0,d_+)\subset\mathbb{R}_+$.

Theorem 2.3. *If $\Gamma(\mathcal{A},\mathcal{B})\neq\emptyset$, then (1.1) is rapidly nullcontrollable and*

$$\mathfrak{c}_*(\mathcal{A},\mathcal{B})\leq(c+d)c/d \tag{2.13}$$

for any $(c,d)\in\Gamma(\mathcal{A},\mathcal{B})$.

Proof: We partition $[0,T)=\bigcup_k\mathcal{I}_k$ with $\mathcal{I}_k=[t_{k-1},t_k]$, setting

$$t_k-t_{k-1}=\tau_k:=(1-\vartheta)\,\vartheta^{k-1}\,T \qquad \text{so}\ \ \tau_{k+1}=\vartheta\tau_k \qquad \text{and}$$

$$\sum_1^\infty\tau_k=T,\qquad S_K:=\sum_1^K\frac{1}{\tau_k}=\frac{1}{(1-\vartheta)\,T}\sum_1^K\vartheta^{-(k-1)}=\frac{(\vartheta^{-K}-1)}{(1-\vartheta)^2\,T}$$

with $t_0=0$ and with $\vartheta\in(0,1)$ to be determined — except that, once we are given $(c,d)\in\Gamma(\mathcal{A},\mathcal{B})$, we will require

$$\vartheta>\frac{c}{c+d}\qquad\text{whence}\qquad\varepsilon:=1-\frac{c}{d}\left(\frac{1}{\vartheta}-1\right)>0. \tag{2.14}$$

For such (c,d), given $\mathbf{x}_0\in\mathcal{X}$, the definition of Γ ensures existence of some u_1 on $\mathcal{I}_1=[0,\tau_1]$ such that $\|u_1\|\leq e^{c/\tau_1}|\mathbf{x}_0|$ and $|\mathbf{x}_1|\leq e^{-d/\tau_1}|\mathbf{x}_0|$ where $\mathbf{x}_1=\mathbf{x}(\tau_1;\mathbf{x}_0,u_1)$. Similarly, proceeding recursively to construct controls u_k on $\mathcal{I}_k$ (noting the autonomy of the system to use the definition of Γ again), we then have

$$\mathbf{x}(t_k;\mathbf{x}_0,u)=\mathbf{x}(\tau_k;\mathbf{x}_{k-1},u_k)=:\mathbf{x}_k\quad\text{with}\ \left\{\begin{array}{ll}|\mathbf{x}_k| & \leq e^{-d/\tau_k}\,|\mathbf{x}_{k-1}|\\ \|u_k\| & \leq e^{c/\tau_k}\,|\mathbf{x}_{k-1}|\end{array}\right. \tag{2.15}$$

[5]From Theorem 2.3 below, we see that $d_+=\infty$, so $c_-(\cdot)$ is actually defined and bounded on all of $\mathbb{R}_+$, whenever Γ is nonempty. At this point it is not clear when $(c_-(d),d)\in\Gamma$ or whether $c_-(\cdot)$ must be a concave function.

where the control $u \in \mathcal{U}_T$ is defined as u_k on each $\mathcal{I}_k$ so[6] $\|u\| \le \sum_k \|u_k\|$. By induction, we easily get from (2.15) that

$$\begin{aligned} |\mathbf{x}_k| &\le \exp\left[-d\, S_k\right]\, |\mathbf{x}_0| \\ \|u_k\| &\le e^{-\nu_k}\, |\mathbf{x}_0| \qquad \text{with} \quad \nu_k := [d\, S_{k-1} - c/\tau_k] \end{aligned} \tag{2.16}$$

Note that, recalling (2.14),

$$\begin{aligned} \nu_{k+1} - \nu_k &= d\left[(S_k - S_{k-1}) - \frac{c}{d}\left(\frac{1}{\tau_{k+1}} - \frac{1}{\tau_k}\right)\right] \\ &= \varepsilon d/\tau_k \qquad \ge\ \varepsilon d/\tau_1 \end{aligned}$$

By induction, $\nu_k \ge (k-1)\varepsilon d/\tau_1 + \nu_1$ and, setting $\mathfrak{c} = c/(1-\vartheta)$, we note that $-\nu_1 = c/\tau_1 = \mathfrak{c}/T$. Thus we have

$$\begin{aligned} \|u_k\| &\le\ e^{-(k-1)\varepsilon d/\tau_1}\, e^{\mathfrak{c}/T}\, |\mathbf{x}_0| \\ \|u\| &\le\ \sum_1^\infty \|u_k\| = C e^{\mathfrak{c}/T}\, |\mathbf{x}_0| \qquad \text{with } C = \sum_0^\infty e^{-k\varepsilon d/\tau_1} \end{aligned} \tag{2.17}$$

[We are not concerned with this constant C, for which $\ln C = o(1/T)$ in any case.] Note that (1.2) gives, for any k,

$$|\mathbf{x}(T)| \le |\mathbf{S}(T-t_k)\mathbf{x}_k| + \left|\int_{t_k}^T |\mathbf{S}(T-t)\, u(t)|\; dt\right| \le K_1|\mathbf{x}_k| + K_2 \left\| u\Big|_{(t_k,T)} \right\|$$

going to 0 as $k \to \infty, t_k \to T$ — which verifies that this u is, indeed, a nullcontrol. Finally, subject to (2.14), which requires $(1-\vartheta) > d/(c+d)$, we minimize $\mathfrak{c} = c/(1-\vartheta)$ in the estimate (2.17) to obtain (2.13). ∎

We combine these theorems to obtain:

Theorem 2.4. *Under the hypotheses of Theorem 2.1,* $(1.1)_2$ *will blow up at the 'standard' rate* $\exp[\mathcal{O}(1/T)]$ *as* $T \to 0$, *i.e.,* $\hat{\mathfrak{c}}_* = \mathfrak{c}_*(\mathcal{A}, \mathcal{B}_2)$ *will be finite. In particular,*

$$\mathfrak{c}_*(\mathcal{A}, \mathcal{B}_1) = 0+ \quad \Longrightarrow \quad \mathfrak{c}_*(\mathcal{A}, \mathcal{B}_2) \le 2\gamma^2/\alpha \tag{2.18}$$

[6] Actually, we have $\|u\|^2 = \sum_k \|u_k\|^2$ since the $\{u_k\}$ are supported on disjoint intervals and so are orthogonal.

PROOF: From Theorem 2.3 we will have $\mathfrak{c}_*(\mathcal{A},\mathcal{B}_2) \le (c+d)c/d$ for all $(c,d) \in \Gamma(\mathcal{A},\mathcal{B}_2)$ as in (2.5) so

$$\hat{\mathfrak{c}}_* = \mathfrak{c}_*(\mathcal{A},\mathcal{B}_2) \;\le\; \inf_{s>\underline{s}} \left\{ \frac{[\underline{c}(s)+\underline{d}(s)]\ \underline{c}(s)}{\underline{d}(s)} \right\} < \infty$$

When $\mathfrak{c}_* = 0+$ we have, simply, $\underline{c} = \gamma s$, $\underline{d} = 2\alpha s^2 - \gamma s$ and can optimize by taking $s = \gamma/\alpha$ to get (2.18). ■

3 Applications: the thermoelastic system

The essential tools which now make it possible for us to obtain the expected $\exp[\mathcal{O}(1/T)]$ blowup rate for a variety of systems involving the Dirichlet Laplacian are Theorem 2.4 and a deep result due to Jerison and Lebeau:

Theorem 3.1. *For a bounded connected region Ω, let $\{(z_j,\lambda_j)\}$ be the eigenpairs of $L = -\Delta$ on Ω with Dirichlet boundary conditions so $0 < \lambda_0 \le \lambda_j \to \infty$. Then for any nonempty open $\omega \subset \Omega$ there is a constant $\gamma > 0$ such that, for all $\sigma > 0$ and for every function $w \in Z_\sigma = \mathrm{span}\{z_j : \lambda_j \le \sigma\}$, one has*

$$\int_\Omega |w(s)|^2\,ds \le C^2 e^{2\gamma\sqrt{\sigma}} \int_\omega |w(s)|^2\,ds \tag{3.1}$$

PROOF: This is immediate from Theorem 14.6 of [6]. ■

Throughout this section we will let $\Omega, \omega, L, \{z_j\}, Z_\sigma$ be as above.

As a first application, we consider the heat equation.

$$\begin{array}{rlc} x_t = \Delta x + u & \text{on } \mathcal{Q}_T = (0,T] \times \Omega \\ x(0) = x_0 & \qquad x\Big|_{\partial\Omega} = 0 \end{array} \tag{3.2}$$

with the control u required to have support in the patch ω. Although the result we obtain here is already known (cf., e.g., [4]), it shows how the present approach applies.

Theorem 3.2. *The heat equation* (3.2) *with control restricted to a patch ω is rapidly nullcontrollable with* $\exp[\mathcal{O}(1/T)]$ *blowup rate as $T \to 0$.*

PROOF: We take $\mathcal{U} = \mathcal{X} = L^2(\Omega)$ and $\mathcal{A} = \Delta = -L$. For $\mathcal{B}_1 = I$, without the control restriction to ω, it is quite easy to obtain exact nullcontrollability with $\mathfrak{C}(T; \mathcal{A}, \mathcal{B}_1) \sim 1/\sqrt{T}$ so $\mathfrak{c}_*(\alpha, \mathcal{B}_1) = 0+$. This, then, provides a suitable comparison for the 'real' problem with patch control, corresponding to the use of $\mathcal{B}_2 = \mathbf{\Pi}_\omega = \mathcal{B}_1 \mathbf{\Pi}$ — enabling the use of Theorems 2.4, 3.1.

Taking $\mathcal{X}_\sigma = \mathcal{Z}_\sigma$ as in Theorem 3.1, we immediately have [**H-**i] and Theorem 3.1 just gives the key comparison estimate [**H-**iii]. On the other hand, since $\{z_j\}$ is orthonormal, the orthocomplement $\mathcal{X}_\sigma^\perp$ is just $\mathrm{span}\{z_j : \lambda_j > \sigma\}$ so for $x \perp X_\sigma$ one has [**H-**ii] with $\alpha = 1$. We then apply Theorem 2.4 to obtain the desired result with $\mathfrak{c}_*(\mathcal{A}, \mathcal{B}_2)$ as in (2.18), taking γ from the application of Theorem 3.1. ∎

Finally, we consider local control in a single component for the thermoelastic system

$$\begin{gathered} w_{tt} + \Delta^2 w - \alpha \Delta \vartheta = 0, \qquad \vartheta_t - \Delta \vartheta + \alpha \Delta w_t = 0, \\ \vartheta = w = \Delta w = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.3}$$

It is significant for our analysis that we must consider this with the same boundary conditions for $w, \Delta w, \vartheta$ so, absorbing these in the domain specification, Δ is the same operator at each of its occurrences in (3.4). For definiteness we have imposed hinged boundary condition for the first equation and Dirichlet conditions for ϑ so, throughout, Δ is the Dirichlet Laplacian on the region Ω to which Theorem 3.1 applies.

Theorem 3.3. *The controlled thermoelastic system* (3.3) *using local control, restricted to an arbitrary patch $\omega \in \Omega$ and a single component, is rapidly nullcontrollable with* $\exp[\mathcal{O}(1/T)]$ *blowup rate as $T \to 0$.*

The proof will be almost identical to the proof above of Theorem 3.2; the only new idea is the observation that the problem factors so one can deduce the relevant spectral decomposition here from the decomposition for the scalar operator L.

PROOF: Setting $y = -\Delta w$ and $z = w_t$, we can write (3.3) — with control — as a first order system on the state space $X = L^2(\Omega \to \mathbb{R}^3) = \mathbb{R}^3 \otimes L^2(\Omega)$ with $\mathcal{A} = M \otimes L$:

$$\dot{\mathbf{x}} = [M \otimes L]\mathbf{x} + \mathcal{B}\, u \qquad \mathbf{x} = \begin{pmatrix} \vartheta \\ y \\ z \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 0 & \alpha \\ 0 & 0 & 1 \\ -\alpha & -1 & 0 \end{pmatrix}. \tag{3.4}$$

For control in a single component, we take $[\mathcal{B}_1 u](s) = u(s)\mathbf{b}$ ($s \in \Omega$) for $u(\cdot) \in \mathcal{U} = L^2(\Omega)$ with three possibilities for $\mathbf{b} \in \mathbb{R}^3$: taking $\mathbf{b} = \mathbf{e}_1$ or $= \mathbf{e}_2$ or $= \mathbf{e}_3$ to select the desired component: thermal, displacement, or velocity. For consideration of local control, we then take $\mathcal{B}_2 = \mathbf{b\Pi}$ where $\mathbf{\Pi} = \mathbf{\Pi}_\omega$ is restriction to the patch $\omega \subset \Omega$ (i.e., multiplication of $u(\cdot)$ by the characteristic function χ_ω).

With this $\mathcal{B}_1$, $(1.1)_1$ is just the thermoelastic system with global control ($\omega = \Omega$) in a single component, for which it is already known — cf., e.g., [1, 8] and references there — that $\mathfrak{C}(T; \mathcal{A}, \mathcal{B}_1) \sim T^{-5/2}$ so $\mathfrak{c}_*(\mathcal{A}, \mathcal{B}_1) = 0+$. We now take $\mathcal{X}_\sigma = \mathbb{R}^3 \otimes \mathcal{Z}_\sigma$ so $\mathcal{X}_\sigma$ consists of all $x \in \mathcal{X}$ of the form $x(s) = \sum_j z_j(s) y_j$ with vectors $y_j \in \mathbb{R}^3$ and with j such that $\lambda_j \le \sigma$. For any such x we may set $w(s) = \mathbf{b} \cdot x = \sum_j (\mathbf{b} \cdot y_j) z_j \in \mathcal{Z}_\sigma$ and will have $|\mathcal{B}_1 x|^2 = |w|^2$ and also $|\mathcal{B}_2 x|^2 = |\mathbf{\Pi} w|^2$ so Theorem 3.1 gives the key comparison estimate [**H-**iii]. On the other hand, since $\{z_j\}$ is orthonormal, the orthocomplement $\mathcal{X}_\sigma^\perp$ is just $\mathbb{R}^3 \otimes \mathcal{Z}_\sigma^\perp$ with $\mathcal{Z}_\sigma^\perp = \mathrm{span}\{z_j : \lambda_j > \sigma\}$. Thus, for $x \perp X_\sigma$ one has

$$x = \sum_{\{\lambda_j > \sigma\}} y_j z_j \qquad \mathbf{S}(t)x = \sum_{\{\lambda_j > \sigma\}} e^{tM} y_j e^{-\lambda_j t} z_j$$

which gives [**H-**i, ii] with $\alpha = 1$. We then apply Theorem 2.4 to obtain the desired result with $\mathfrak{c}_*(\alpha, \mathcal{B}_2)$ as in (2.18), taking γ from the application of Theorem 3.1. ■

Remark 3.4. We have, as promised, shown the desired blowup rate (1.6) for the thermoelastic problem with local control in a single component with an estimate based on Theorem 3.1 for the constant $\mathfrak{c}_*$ of (1.7), thus improving the results (1.10) of [11, 3]. The method of analysis clearly is applicable to other coupled systems as well. However, it is worth noting here what is *not* covered by this analysis. First, we note situations with the Dirichlet Laplacian replaced by some other operator, e.g., with variable coefficients or different boundary conditions: all that would be needed to extend the present analysis to such settings would be a corresponding extension of the Jerison-Lebeau estimate of Theorem 3.1. Second, we have only considered here interior patches; the standard trick[7] for boundary patch control then works for scalar equations, but is unavailable for systems.

[7]... artificially perturbing Ω by a bulge at the boundary patch containing a small 'interior' control patch and then using the trace.

The core of this approach has been the use of Theorem 3.1 to obtain [**H**]. It was noted in Remark 2.2 that the key estimate [**H**-*iii*] can be substantially weakened without losing the conclusion of Theorem 2.1 and, noting that the nature of [**H**] ensures a close connection with a spectral expansion for the spatial operator, it seems conceivable that, as with the 'observability resolvent estimates' of [10], it might be possible to deduce some useful form of [**H**-*iii*] from nullcontrollability, using $\mathcal{B}_2$, of the corresponding scalar wave equation.

The major area left completely untouched by this analysis involves systems not quite as separable as here — e.g., (cf., e.g., [1]) the thermoelastic system with different boundary conditions for ϑ or other than hinged boundary conditions for the elastic subproblem. ■

References

[1] G. Avalos and I. Lasiecka, *On the null-controllability of thermoelastic plates and singularity of the associated minimal energy function*, J. Math. Anal. Appl. **10**, pp. 34–61, 2004.

[2] A. Benabdallah and G. Naso, *Nullcontrollability of a thermoelastic plate*, Abstract Applied Analysis, pp. 585-601, 2002.

[3] P. Cokeley, *Localized null controllability and corresponding minimal norm control blowup rates of thermoelastic systems,* J. Math. Anal. Appl., to appear.

[4] E. Fernndez-Cara and E. Zuazua, *The cost of approximate controllability for heat equations: The linear case,* Advances in Differential Equations **5**, pp. 465–514, 2000.

[5] E. Güichal, *A lower bound of the norm of the control operator for the heat equation*, J. Math. Anal. Appl. **110**, no.2, pp. 519–527, 1985.

[6] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions,* Chapter 14 in Harmonic Analysis and Partial Differential Equations, *Essays in honor of Alberto P. Calderón,* (edited by M. Christ, C.E. Kenig, C. Sadosky), Univ. Chicago Press, Chicago, (1999), pp. 223–239.

[7] W. Krabs, G. Leugering, and T.I. Seidman, *On boundary controllability of a vibrating plate,* Appl. Math. Opt. **13**, pp. 205–229, 1985.

[8] I. Lasiecka and T.I. Seidman, *Blowup estimates for observability of a thermoelastic coupled system*, Asymptotic Analysis **50**, pp. 93–120, 2006.

[9] G. Lebeau and L. Robbiano, *Contrôle exacte de l'équation de la chaleur,* Comm. PDEs **20**, pp.335–356, 1995.

[10] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Functional Anal. **218**, pp. 425–444, 2005.

[11] L. Miller, *On the cost of fast controls for thermoelastic plates,* Asymptot. Anal. **51**, pp. 93–100, 2007.

[12] T.I. Seidman, *Two results on exact boundary control of parabolic equations,* Appl. Math. and Opt. **11**, pp. 145–152, 1984.

[13] T.I. Seidman, *How violent are fast controls?*, Math. of Control, Signals Syst. **1**, pp. 89-95, 1988.

[14] T.I. Seidman, *On uniform null-controllability and blowup estimates,* Chapter 15 in *Control Theory of Partial Differential Equations*, (edited by O. Imanuvilov, G. Leugering, R. Triggiani, B.-Y. Zhang), Chapman & Hall/CRC, Boca Raton, (2005), pp. 215–227.

[15] T.I. Seidman and J. Yong, *How violent are fast controls, II*, Math. of Control, Signals, Syst. **9**, pp. 327–340, 1997.