# HOW VIOLENT ARE FAST CONTROLS?[1]

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ABSTRACT:

For any controllable linear system it is clear that the minimum control energy must increase unboundedly as the available time for exact control decreases to 0. This is made precise, obtaining asymptotically $\mathcal{O}(T^{-(K+1/2)})$ behavior for the norm of the control operator where $K$ is the order of the 'least controllable' modes (the minimal exponent for the rank condition).

KEY WORDS: linear system, LQ optimal control, asymptotic behavior.

[1]This research was partially supported under grant AFOSR-82-0271. Portions of this were done while the author was visiting at the Systems Research Center (U. of Md.) with NSF support under CDR-85-00108 and at the Centre for Mathematical Analysis (Australian National University). This paper appeared in Math. of Control, Signals, Syst. **1**, pp. 89–95 (1988).

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# 1 Introduction

Consider a linear control system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \qquad , \qquad x(0) = 0 \tag{1.1}$$

with $\mathbf{A}, \mathbf{B}$ constant matrices ($n \times n$ and $n \times m$, respectively, so $x(\cdot)$ is $I\!R^n$-valued and the control $u(\cdot)$ is $I\!R^m$-valued). Assuming this is controllable, we know that for each terminal time $T > 0$ and each target $\xi \in I\!R^n$ there exist controls $u(\cdot)$ giving $x(T) = \xi$ and, indeed, that there is unique such control

$$u_{opt} = u_{opt}(\cdot; T, \xi) \in L^2([0,T] \to I\!R^m) =: \mathcal{U} = \mathcal{U}_T \tag{1.2}$$

minimizing $\|u\|$. The norm to be minimized is, of course, taken to be that of $\mathcal{U} = L^2([0,T] \to I\!R^m)$ so $\|u_{opt}\|^2$ gives the least control energy needed to reach the target $\xi$ at time $T$.

It is to be expected that more violent control would be needed as the time $T$ available becomes shorter[3]. Our object in this paper is to give a precise (asymptotic) answer to the question of the title. Since the optimal control $u_{opt}$ is given by a linear operator

$$\mathbf{C}_T : \xi \mapsto u_{opt}(\cdot; T, \xi) : I\!R^n \longrightarrow \mathcal{U} = \mathcal{U}_T \ , \tag{1.3}$$

the principal result can be stated as

$$\|\mathbf{C}_T\| \sim \gamma T^{-(K+1/2)} \qquad \text{as } T \to 0 \tag{1.4}$$

where $K$ is the minimal exponent giving the well known *rank condition* for controllability:

$$rank \ [\mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^K\mathbf{B}] = n \tag{1.5}$$

and $\gamma \neq 0$ is also computable from $\mathbf{A}, \mathbf{B}$.

It is worth noting that this also estimates sensitivity for observation of the adjoint problem. If one can observe $y := \mathbf{B}^* z$ for a solution $z$ of the equation $\dot{z} = -\mathbf{A}^* z$, then one easily sees that one recovers the state through $z(T) = \mathbf{C}_T^* y(\cdot)$. This means that the uncertainty in the recovered state due to noise or measurement error $e(\cdot)$ in the observation can be estimated by

[3] The uniqueness of $u_{opt}$ and the linearity of the map: $\xi \mapsto u_{opt}$ follow from the Hilbert space projection theorem under more general conditions than here. From uniqueness it follows that $\|u_{opt}\|$ is strictly decreasing in $T$ for each $\xi \neq 0$.

$\|\mathbf{C}_T^* e(\cdot)\| \leq \|\mathbf{C}_T\| \; \|e(\cdot)\|$. The same result (1.4) shows how sensitivity to error increases as the observation time shrinks: since, plausibly, we might anticipate

$$Exp[\int_0^T |e|^2 dt] \sim \sigma^2 T, \quad \text{i.e.,} \quad Exp[\|e(\cdot)\|] \sim \sigma T^{1/2},$$

the sensitivity estimate becomes

$$\text{(1.6)} \qquad \text{expected } \| \text{ uncertainty in } z(T)\| \sim \gamma\sigma T^{-K} \;\text{ as } T \to 0.$$

The formula (2.3) is classical but it is interesting to observe historically that the question of the title seems to have been considered first for distributed parameter systems[4] although it was posed for the present finite dimensional case at least as far back as 1975 [3].

# 2 Formulation

Treatment of (1.1) is expressible in terms of the matrix exponential, given by the convergent series

$$\text{(2.1)} \qquad e^{s\mathbf{A}} := \sum_0^\infty (s^k/k!)\mathbf{A}^k.$$

The solution $x$ of (1) is then given by

$$x(t) = \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}u(s)ds$$

so, in particular, one has $x(T) = \mathbf{V}u(\cdot)$ where

$$\text{(2.2)} \qquad \mathbf{V} = \mathbf{V}_T : \mathcal{U} \to I\!R^n : u(\cdot) \mapsto \int_0^T e^{(T-s)\mathbf{A}}\mathbf{B}\; u(s)ds.$$

A standard argument shows that $\|u(\cdot)\|_{\mathcal{U}}$ is minimized, subject to the condition $\mathbf{V}u = \xi$, by taking $u \in \mathcal{R}(\mathbf{V}^*)$ whence

$$u_{opt}(\cdot\,; T, \xi) = \mathbf{V}^*\omega \;\text{ with, necessarily, }\; \mathbf{V}\mathbf{V}^*\omega = \xi;$$

[4]One has $\log\|\mathbf{C}_T\| = \mathcal{O}(1/T)$ (sharply) for the known infinite-dimensional cases [4], [1], [2].

so one has $\mathbf{C}_T : \xi \mapsto u_{opt}$ given by

$$\mathbf{C}_T = \mathbf{V}_T^*(\mathbf{V}_T\mathbf{V}_T^*)^{-1} \tag{2.3}$$

where controllability gives, as we see, invertibility of $\mathbf{V}_T\mathbf{V}_T^* =: \mathbf{W}_T = \mathbf{W}$. We easily see that $\mathbf{V}^* : I\!R^n \to \mathcal{U}$ is given by

$$[\mathbf{V}^*\omega](t) = [e^{(T-t)\mathbf{A}}\mathbf{B}]^*\omega \qquad \text{for } \ t \in [0,T] \tag{2.4}$$

so the $n \times n$ matrix $\mathbf{W} := \mathbf{V}\,\mathbf{V}^*$ is given by

$$\mathbf{W} = \mathbf{W}_T = \int_0^T e^{s\mathbf{A}}\mathbf{B}[e^{s\mathbf{A}}\mathbf{B}]^* ds. \tag{2.5}$$

Clearly, $\mathbf{W}$ is self-adjoint and (at least) semidefinite from its form. We have the identity

$$\begin{aligned} \|u_{opt}\|^2 &= \|\mathbf{C}_T\xi\|^2 = \langle \mathbf{V}^*\mathbf{W}^{-1}\xi,\ \mathbf{V}^*\mathbf{W}^{-1}\,\xi\rangle \\ &= (\mathbf{W}^{-1}\xi)\cdot(\mathbf{V}\,\mathbf{V}^*\mathbf{W}^{-1}\xi) = (\mathbf{W}_T^{-1}\xi)\cdot\xi \end{aligned} \tag{2.6}$$

which makes it clear that our object must be to compute $\mathbf{W}_T^{-1}$ asymptotically.

The key to our approach is the invertibility of

$$\mathbf{Q} := \lim_{T\to 0} T^{-(2K+1)}\mathbf{\Gamma}_T\mathbf{W}_T\mathbf{\Gamma}_T, \tag{2.7}$$

using a suitable family of operators $\mathbf{\Gamma} = \mathbf{\Gamma}_T$ such that

$$\mathbf{\Gamma}_T \ \text{ invertible for } \ T \neq 0\ , \qquad \mathbf{\Gamma}_T = \mathbf{\Gamma}_0 + \mathcal{O}(T); \tag{2.8}$$

see (2.12), below.

Given the matrices $\mathbf{A}, \mathbf{B}$ we consider the nested sequence $(\mathcal{S}_0, \mathcal{S}_1, \ldots)$ of subspaces of $I\!R^n$ given recursively by

$$\begin{aligned} \mathcal{S}_k &= \mathcal{S}_{k-1} + \mathcal{R}(\mathbf{A}^k\mathbf{B}) \qquad \text{with } \ \mathcal{S}_{-1} := \{0\}, \\ \mathcal{S}_0 &= \mathcal{R}(\mathbf{B}),\ \ \mathcal{S}_1 = \mathcal{R}(\mathbf{B}) + \mathcal{R}(\mathbf{A}\,\mathbf{B}),\ \ \ldots \end{aligned} \tag{2.9}$$

so each $\mathcal{S}_k$ is the column space (range) of the composite matrix $[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^k\mathbf{B}]$. The assumption of controllability means that $\mathcal{S}_K = I\!R^n$ for large enough $K$ (i.e., (1.5)) and we fix $K$ as the *minimal* exponent/index giving this.

For each $k \quad (0 \leq k \leq K)$ we can find the orthogonal complement of $\mathcal{S}_{k-1}$ in $\mathcal{S}_k$ and let $\mathbf{E}_k$ be the orthogonal projection on this subspace. This gives the important fact that

$$\mathbf{E}_k \mathbf{A}^j \mathbf{B} = \mathbf{0} \ \text{ for } \ j < k \leq K \tag{2.10}$$

since $j < k$ gives $\mathcal{R}(\mathbf{A}^j \mathbf{B}) \subset \mathcal{S}_{k-1} \subset \mathcal{N}(\mathbf{E}_k)$. We observe, although we do not need the fact, that

$$m \geq \dim \mathcal{R}(\mathbf{B}) = \dim \mathcal{R}(\mathbf{E}_0) \geq \dim \mathcal{R}(\mathbf{E}_1) \geq \ldots \geq \dim \mathcal{R}(\mathbf{E}_K);$$

we *will* need the fact that $\mathcal{S}_{K-1} \neq \mathcal{S}_K = I\!R^n$ by the definition of $K$ so $\dim \mathcal{R}(\mathbf{E}_K) \neq 0$ and $\mathbf{E}_K \neq \mathbf{0}$. The construction of $\{\mathbf{E}_k\}$ gives a direct sum decomposition

$$\begin{aligned} \mathbf{1} &= \mathbf{E}_0 + \ldots + \mathbf{E}_K \ , \qquad I\!R^n = \oplus_0^K \mathcal{R}(\mathbf{E}_k), \\ \mathcal{S}_k &= \mathcal{R}(\mathbf{E}_0) \oplus \ldots \oplus \mathcal{R}(\mathbf{E}_k) \ \text{ for } \ k = 0, \ldots, K. \end{aligned} \tag{2.11}$$

Thus, introducing

$$\mathbf{\Gamma} = \mathbf{\Gamma}_T := \sum_0^K k!\, T^{K-k} \mathbf{E}_k \tag{2.12}$$

we see that (2.8) holds with $\mathbf{\Gamma}_0 = K!\, \mathbf{E}_K \neq \mathbf{0}$.

# 3 Principal Computation

Our object in this section is to obtain (2.7), with $\mathbf{\Gamma}_T$ as in (2.12), computing $\mathbf{Q}$ and showing it is invertible.

The integral expression (2.5) gives, on substituting $s = T\sigma$,

$$T^{-(2K+1)} \mathbf{\Gamma W \Gamma} = \int_0^1 [T^{-K} \mathbf{\Gamma} e^{T\sigma \mathbf{A}} \mathbf{B}][T^{-K} \mathbf{\Gamma} e^{T\sigma \mathbf{A}} \mathbf{B}]^* d\sigma.$$

Using (2.12) and (2.1), we have (for $T > 0$)

$$\begin{aligned} T^{-K} \mathbf{\Gamma} e^{T\sigma \mathbf{A}} \mathbf{B} &= \sum_{k=0}^K k! T^{-k} \mathbf{E}_k \quad \sum_{j=0}^\infty \frac{\sigma^j}{j!} T^j \mathbf{A}^j \mathbf{B} \\ &= \sum_{k=0}^K \sum_{j=0}^\infty \frac{k! \sigma^j}{j!} T^{j-k} \mathbf{E}_k \mathbf{A}^j \mathbf{B}. \end{aligned}$$

By (2.10), the terms with $j < k$ vanish so no negative powers of $T$ actually appear on the right; we then split the sum into the terms with $j = k$ and those with $j \geq k+1$ for which we set $i = j-(k+1) = 0, 1, \ldots$. Thus,

$$\begin{aligned} T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B} &= \sum_{k=0}^{K}\sigma^k\mathbf{E}_k\mathbf{A}^k\mathbf{B} \\ &\quad +T\left[\sum_{k=0}^{K}\sum_{i=0}^{\infty}\frac{k!\sigma^{i+k+1}T^i}{(i+k+1)!}\mathbf{E}_k\mathbf{A}^j\mathbf{B}\right] \\ &= \mathbf{P}(\sigma)+T\mathbf{R}_1(T,\sigma). \end{aligned} \tag{3.1}$$

Restricting our attention to $T \leq 1$, which is certainly permissible as we are only interested in the limit $T \to 0$, an easy estimation gives the uniform bound

$$\begin{aligned} \|\mathbf{R}_1(T,\sigma)\| &\leq \sum_{k=0}^{K}\sum_{i=0}^{\infty}\|\mathbf{A}\|^{i+k+1}\|\mathbf{B}\|/(i+1)! \\ &= (1+\ldots+\|\mathbf{A}\|^K)\|\mathbf{B}\|(e^{\|\mathbf{A}\|}-1) \end{aligned}$$

since $(i+k+1)! \geq (i+1)!\,k!$. Hence, (3.1) gives[5]

$$T^{-K}\mathbf{\Gamma}\,e^{T\sigma\mathbf{A}}\mathbf{B} = \mathbf{P}(\sigma)+\mathcal{O}(T)$$

and

$$(T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B})\;(T^{-K}\mathbf{\Gamma}e^{T\sigma\mathbf{A}}\mathbf{B})^* \tag{3.2}$$

$$\begin{aligned} &= [\mathbf{P}(\sigma)+T\mathbf{R}_1(T,\sigma)]\;[\mathbf{P}(\sigma)+T\mathbf{R}_1(T,\sigma)]^* \\ &= \mathbf{P}(\sigma)\mathbf{P}^*(\sigma)+T\mathbf{R}_2(T,\sigma), \end{aligned}$$

with $\mathbf{R}_2(T,\sigma) := (\mathbf{R}_1\mathbf{P}^*+\mathbf{P}\mathbf{R}_1^*+T\mathbf{R}_1\mathbf{R}_1^*)$ uniformly bounded. Thus, integrating,

$$T^{-(2K+1)}\mathbf{\Gamma}\mathbf{W}\mathbf{\Gamma} = \mathbf{Q}+\mathcal{O}(T) \tag{3.3}$$

with $\mathcal{O}(T) = T\int\mathbf{R}_2 d\sigma =: T\mathbf{R}_3(T)$ and

$$\begin{aligned} \mathbf{Q} &:= \int_0^1\mathbf{P}(\sigma)\mathbf{P}^*(\sigma)d\sigma \\ &= \sum_{j,k=0}^{K}(j+k+1)^{-1}\mathbf{E}_j\mathbf{A}^j\mathbf{B}\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k. \end{aligned} \tag{3.4}$$

[5]Note that our estimation of $\mathbf{R}_1$ precisely legitimates the use of the $\mathcal{O}(T)$ notation.

We must show that $\mathbf{Q}$ is invertible.

**Lemma** : $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi = 0 \Longrightarrow \mathbf{E}_k\xi = 0$.

PROOF : For any $\xi \in I\!R^n$ we have $\mathbf{E}_k\xi \in \mathcal{S}_k := \mathcal{S}_{k-1} + \mathcal{R}(\mathbf{A}^k\mathbf{B})$ by definition so we may write

$$\mathbf{E}_k\xi = \mathbf{A}^k\mathbf{B}\eta + \xi' \qquad\qquad (\xi' \in \mathcal{S}_{k-1})$$

for some $\eta \in I\!R^m$. Then, assuming $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi = 0$, we would have

$$\begin{aligned} \|\mathbf{E}_k\xi\|^2 &= (\mathbf{A}^k\mathbf{B}\eta + \xi') \cdot (\mathbf{E}_k\xi) \\ &= \eta \cdot (\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi) + (\mathbf{E}_k\xi') \cdot \xi = 0 \end{aligned}$$

since $\mathbf{E}_k\xi' = 0$ for $\xi' \in \mathcal{S}_{k-1}$. ■

From (3.4) we see that

$$\xi \cdot \mathbf{Q}\xi = \int_0^1 \xi \cdot [\mathbf{P}(\sigma)\mathbf{P}^*(\sigma)\xi] d\tau = \int_0^1 \|\mathbf{P}^*(\sigma)\xi\|^2 d\tau$$

so $\mathbf{Q}\xi = 0$ only if $\mathbf{P}^*(\sigma)\xi \equiv 0$. From the definition of $\mathbf{P}(\cdot)$, this would mean that each term $\sigma^k\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi$ would have to vanish and, by the Lemma, this would imply $\mathbf{E}_k\xi = 0$ for each $k$.

Hence, from (2.11), $\mathbf{Q}\xi = 0$ would give $\xi = 0$ and we have thus shown that $\mathbf{Q}\xi = 0$ only for $\xi = 0$. For an $n \times n$ matrix $\mathbf{Q}$, this ensures invertibility.

# 4 Results

We must draw the desired conclusions from (3.3).

It is clear from the bound on $\mathbf{R}_1(T,\sigma)$ and the obvious fact that $\mathbf{P}(\sigma)$ is bounded uniformly on $[0,1]$ that $\mathbf{R}_2(T,\sigma)$ is uniformly bounded so $\mathbf{R}_3(T)$ is uniformly bounded – say, $\|\mathbf{R}_3(T)\| \leq M_3$ for $0 \leq T \leq 1$. Restricting attention to $T \leq 1/2M_3\|\mathbf{Q}^{-1}\| =: \tau$, we have

$$\begin{aligned} (\mathbf{Q} + T\mathbf{R}_3)^{-1} &= \mathbf{Q}^{-1}(\mathbf{1} + T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1} \\ &= \mathbf{Q}^{-1} + T\mathbf{R}_4(T) \end{aligned}$$

with

$$\begin{aligned} \|\mathbf{R}_4\| &\leq \|\mathbf{Q}^{-1}\| \; \|(\mathbf{1} + T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1} - \mathbf{1}\|/T \\ &= \|\mathbf{Q}^{-1}\| \; \|\mathbf{Q}^{-1}\mathbf{R}_3(\mathbf{1} + T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1}\| \\ &\leq \|\mathbf{Q}^{-1}\| \; \|\mathbf{Q}^{-1}\mathbf{R}_3\|/(1 - T\|\mathbf{Q}^{-1}\mathbf{R}_3\|) \\ &\leq 2M_3\|\mathbf{Q}^{-1}\|^2 \qquad\qquad \text{for } \; 0 \leq T \leq \tau, \end{aligned}$$

Now, inverting each side of (3.3) is legitimate for $T \neq 0$ and gives

$$\begin{aligned} T^{2K+1}\mathbf{\Gamma}_T^{-1}\mathbf{W}_T\mathbf{\Gamma}_T^{-1} &= \mathbf{Q}^{-1} + T\mathbf{R}_4, \\ T^{2K+1}\mathbf{W}_T^{-1} &= \mathbf{\Gamma}_T(\mathbf{Q}^{-1} + T\mathbf{R}_4)\mathbf{\Gamma}_T \\ &= \mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0 + T\mathbf{R}_5(T) \end{aligned} \tag{4.1}$$

with, obviously, $\|\mathbf{R}_5(T)\|$ uniformly bounded on $0 \leq T \leq \tau$. In particular, this proves (independently of the standard controllability arguments) the invertibility of $\mathbf{W}_T$ — at least for small $T > 0$ and so *a fortiori* for all $T > 0$ by the non-negativity of the integrand in (2.5).

From (2.6) and the positivity of $\mathbf{W}, \mathbf{W}^{-1}$ we then have

$$\begin{aligned} \|\mathbf{C}_T\|^2 &:= \max\{\|\mathbf{C}_T\xi\|^2 : \|\xi\| = 1\} \\ &= \max\{\xi \cdot \mathbf{W}_T^{-1}\xi : \|\xi\| = 1\} = \|\mathbf{W}_T^{-1}\| \\ &= T^{-(2K+1)}[\|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0\| + \mathcal{O}(T)]. \end{aligned} \tag{4.2}$$

This, of course, is just (1.4) with, from (2.12),

$$\gamma := \|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0\|^{1/2} = K!\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\|^{1/2} \tag{4.3}$$

once one shows $\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K \neq \mathbf{0}$ so $\gamma \neq 0$. Note that the positivity of $\mathbf{Q}$, hence of $\mathbf{Q}^{-1}$, gives

$$\begin{aligned} \|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\| &= \max\{\xi \cdot (\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\xi) : \|\xi\| = 1\} \\ &= \max\{\xi \cdot \mathbf{Q}^{-1}\xi : \|\xi\| = 1,\ \xi = \mathbf{E}_K\xi\} \\ &= \|\mathbf{Q}^{-1}|_{\mathcal{R}(\mathbf{E}_K)}\|. \end{aligned}$$

Since $\mathcal{R}(\mathbf{E}_K) \neq \{0\}$ by the minimality of $K$, this is clearly non-zero.

At this point we work out in somewhat greater detail the case of scalar control ($m = 1$). The $n \times 1$ matrix $\mathbf{B}$ is now just a vector and we set

$$\beta_0 = \mathbf{B}, \quad \beta_k = \mathbf{A}^k\beta_0 \qquad \text{for } k = 0, \ldots, n-1. \tag{4.4}$$

Note that controllability gives $K = n - 1$ in this case so, for scalar control, our result (4.2) becomes

$$\|\mathbf{C}_T\| \sim \gamma T^{-(n+1/2)} + \mathcal{O}(T^{-(n-1/2)}). \tag{4.5}$$

To compute $\gamma$ here, note first that $(\beta_0, \ldots, \beta_{n-1})$ is a basis for $I\!R^n$ and let $(\varepsilon_0, \ldots, \varepsilon_{n-1})$ be the orthonormal basis obtained from that by the Gram-Schmidt procedure so $\mathbf{E}_k : \xi \mapsto (\xi \cdot \varepsilon_k)\varepsilon_k$ and $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi$ becomes $(\beta_k \cdot \varepsilon_k)\ (\xi_k \cdot \varepsilon_k)$. Then (3.4) becomes

$$\mathbf{Q}\xi = \sum_{j=0}^{K-1} \left[ \sum_{k=0}^{K-1} \frac{(\beta_j \cdot \varepsilon_j)(\beta_k \cdot \varepsilon_k)}{j+k+1} (\xi \cdot \varepsilon_k) \right] \varepsilon_j. \tag{4.6}$$

This shows that, *re-written in terms of the orthonormal basis* $(\varepsilon_0, \ldots, \varepsilon_{n-1})$, the new matrix for $\mathbf{Q}$ is just $\mathbf{DHD}$ where $\mathbf{D} :=$ diag $[\beta_j \cdot \varepsilon_j]$ and $\mathbf{H}$ is the $n \times n$ Hilbert matrix. Then

$$\begin{aligned} \|\mathbf{E}_K \mathbf{Q}^{-1} \mathbf{E}_K\| &= \varepsilon_K \cdot \mathbf{Q}^{-1} \varepsilon_K \\ &= [\text{lower right corner element of } \mathbf{D}^{-1}\mathbf{H}^{-1}\mathbf{D}^{-1}] \\ &= (\beta_{n-1} \cdot \varepsilon_{n-1})^{-2}\ [\text{lower right corner element of } \mathbf{H}^{-1}] \end{aligned}$$

whence

$$\gamma = C_n (\beta_{n-1} \cdot \varepsilon_{n-1})^{-1} \tag{4.7}$$

with $(C_n := (n-1)!)$ [lower right corner element of the inverse of $\mathbf{H}]^{1/2}$. The coefficient $C_n$ grows extremely rapidly with $n$ but, of course, is fixed for any given dimensionality. Thus, $\beta_{n-1} \cdot \varepsilon_{n-1}$ provides the only dependence on the *particular* system (1.1); it is just the norm of the component of $\mathbf{A}^{n-1}\beta_0$ orthogonal to *span* $\{\beta_0, \ldots, \mathbf{A}^{n-2}\beta_0\}$.

Returning to the general case, we now consider the asymptotics for a particular target $\xi$ (rather than the 'worst case' treatment above). We have, from (2.6),

$$x(T; u(\cdot)) = \xi \Longrightarrow \|u(\cdot)\| \geq \|\mathbf{C}_T \xi\| = (\xi \cdot \mathbf{W}_T^{-1} \xi)^{1/2}.$$

From (4.1) we have

$$\|\mathbf{C}_T \xi\| = T^{-(K+1/2)} K! (\xi_K \cdot \mathbf{Q}^{-1} \xi_K)^{1/2} + \mathcal{O}(T^{-(K-1/2)})$$

where we have abbreviated $\xi_K := \mathbf{E}_K \xi$. We may write this, assuming[6] $\xi_K \neq 0$, as

$$\|\mathbf{C}_T \xi\| \sim (K! \|\mathbf{Q}^{-1/2} \xi_K\|) T^{-(K+1/2)}, \tag{4.8}$$

which gives the same asymptotic growth rate for (almost all) targets.

[6] This is 'almost always' true — it fails only when $\xi$ happens to lie exactly in the (proper) subspace $\mathcal{N}(\mathbf{E}_K)$ in which case one has slower blowup. Even in that case slight perturbations would, almost inevitably, give *some* component in $\mathcal{R}(\mathbf{E}_K)$ so this analysis would dominate.

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