An introduction to control theory for PI

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Abstract: A brief introduction to distributed parameter syste
erned by linear partial differential equations:
*abstract ODEs and semigroups, duality of observation and nullc
nullcontrolability and stabilization, geometry and the wave equat
trol of the heat equation.*

Much is like standard system theory (with Linear Algebra replaced
Analysis), but we also emphasize the role of geometry.

An outline:

1. Some examples and questions
2. Reformulation as abstract ODEs; Semigroups
3. A basic Duality Theorem
4. Some results

Example 1: an optimal control problem

Suppose we are given a region $\Omega \subset \mathbb{R}^2$ and consider the heat equa
equation)

$$
(1) \qquad \begin{array}{ll} u_t = u_{xx} + u_{yy} + \varphi & \text{on } \mathcal{Q} = (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u = u_0 & \text{on } \Omega \text{ at } t = 0 \end{array}
$$

Fix a subregion $\omega \subseteq \Omega$.

Problem 1: Given u_0 and a target $\bar{u}$ in $L^2(\Omega)$, choose $\varphi \in I$
minimize

$$
(2) \qquad \mathcal{J}(\varphi) = \int_0^T \|\varphi(t,\cdot)\|^2 \, dt \quad + \lambda \|u(T,\cdot) - \bar{u}\|^2
$$

subject to the condition that $\varphi(t,x,y) = 0$ when $(x,y) \notin \omega$.

What would we like to know?

1. Is (1) well-posed?
2. Is the minimum attained?
   [How does this depend on u_0? on $\bar{u}$? on T? on ω?]
3. Which targets $\bar{u}$ can be reached exactly? are they dense? H
   depend on T? on ω?
4. How can we compute the optimal control?
   Can we characterize the optimal control (first order optimality c
5. What happens as $\lambda \to \infty$? [nullcontrol: $\bar{u} = 0$]
   What are the asymptotics as $T \to 0$?

Example 2: an observation problem

We again consider

$$(3)\qquad \begin{aligned} u_t &= u_{xx} + u_{yy} && \text{on } \mathcal{Q} = (0,T) \times \Omega \\ u &= 0 && \text{on } (0,T) \times \partial\Omega \\ u &= ? && \text{on } \Omega \text{ at } t = 0 \end{aligned}$$

Problem 2: Given $\omega \subset \Omega$, we observe u on $\mathcal{Q}_\omega = (0,T) \times$
estimate $\bar{u}$ for $u(T, \cdot)$ — the state on all of Ω.

What would we like to know?

1. If $y = u\Big|_\omega \equiv 0$ on $\mathcal{Q}_\omega$, does that imply $u \equiv 0$ on all of $\mathcal{Q}$?
2. What would be the effect of observation noise?
3. What would be an 'optimal' map: $y \mapsto \bar{u}$
   [Is this linear? continuous?]
4. How is this problem related to the previous one?

Example 3: another optimal control problem

Suppose we are given a region $\Omega \subset \mathbb{R}^2$ and consider the wave equa

$$
\begin{array}{lcl}
 & w_{tt} = w_{xx} + w_{yy} + \varphi & \text{on } \mathcal{Q} = (0,T) \times \Omega \\
(4) & w = 0 & \text{on } (0,T) \times \partial\Omega \\
 & w = w_0 \quad w_t = w_1 & \text{on } \Omega \text{ at } t = 0
\end{array}
$$

Fix a subregion $\omega \subseteq \Omega$.

Problem 3: Given $\mathbf{u}_0 = [w_0, w_1]$ and a target state $\bar{\mathbf{u}} = [\bar{w}_0, \bar{w}$
$L^2(\Omega)$, choose $\varphi \in L^2(\mathcal{Q})$ so as to minimize

$$\mathcal{J}(\varphi) = \int_0^T \|\varphi(t,\cdot)\|^2 \, dt \quad + \lambda \|\mathbf{u}(T,\cdot) - \bar{\mathbf{u}}\|^2 \tag{5}$$

subject to the condition that $\varphi(t, x, y) = 0$ when $(x, y) \notin \omega$.

What would we like to know?

1. All the same questions as for Example 1

$\ldots$

2. How are these problems (Examples 1, 3) similar? different?

Semigroups and abstract ODEs:

Let $\mathbf{S}(t) : X \to X : u_0 \mapsto u(t)$ for the solution of the (linear,
abstract ODE

$$u' = \mathbf{A}u \qquad u(0) = u_0 \tag{6}$$

Assuming wellposedness, we have

$$\begin{array}{ll} \mathbf{S}(t+s) = \mathbf{S}(t)\mathbf{S}(s) \quad \mathbf{S}(0) = \mathbf{I} & \dfrac{d\mathbf{S}(t)}{dt} = \mathbf{A}\mathbf{S}(t) = \mathbf{S}(t)A \\ \mathbf{S}(t)u_0 \to u_0 \text{ (all } u_0 \in \mathcal{X}) & \|\mathbf{S}(t)\| \le Me^{\omega t} \quad \text{(some} \end{array} \tag{7}$$

The semigroup $\mathbf{S}(t) = e^{t\mathbf{A}}$ is defined on X if:

$$\mathbf{A} \text{ closed, } \mathcal{D}(\mathbf{A}) \text{ dense, } (\lambda - \omega)^n \|(\lambda - \mathbf{A})^{-n}\| \text{ bounded.}$$

[If $t \mapsto \mathbf{S}(t)$ is analytic on a sector, then: $\|[-\mathbf{A}]^\alpha \mathbf{S}(t)\| \le Mt^{-\alpha}e^{\omega$

Dirichlet Laplacian:

We consider the *Dirichlet Laplacian* to be the (unbounded) linea
$X_0 = L^2(\Omega)$ given by

$$(8) \qquad \begin{aligned} \Delta : X_0 &= L^2(\Omega) \supset \mathcal{D}(\mathbf{A}) \to X_0 : u \mapsto u_{xx} + u_{yy} \\ \text{with} \quad \mathcal{D}(\Delta) &= \left\{ u \in H^2(\Omega) : u\Big|_{\partial\Omega} = 0 \right\} \end{aligned}$$

This satisfies the conditions to generate a semigroup $\mathbf{S}(\cdot)$ so, for Ex

$$(9) \qquad \begin{aligned} (1) &\iff u' = \Delta u + \mathbf{B}\varphi, \quad u(0) = u_0 \\ &\iff u(t) = \mathbf{S}(t)u_0 + \int_0^t \mathbf{S}(t-s)\mathbf{B}\varphi(s)\,ds \end{aligned}$$

where $\mathbf{B} : U = L^2(\omega) \hookrightarrow X$.
For this Example the semigroup $\mathbf{S}(\cdot)$ is analytic.

Two other examples:

The wave equation (4) $w_{tt} = \Delta w + \varphi$ can be written as a first
$\mathbf{u}' = \mathbf{A}\mathbf{u} + \boldsymbol{\varphi}$ by setting

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} [-\Delta]^{1/2} w \\ w_t \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 0 & [-\Delta]^{1/2} \\ [-\Delta]^{1/2} & 0 \end{pmatrix}$$

Here $\mathbf{S}(\cdot)$ is *not* analytic.

[Alternatively, $\mathbf{u} = (\nabla w,\ w_t)^{\mathsf{T}}$ gives $u_t = \nabla v,\ v_t = \nabla \cdot u$.]

The system for a linear thermoelastic plate

$$(10) \qquad w_{tt} + \Delta^2 w - \alpha\Delta\vartheta = 0 \qquad \vartheta_t - \Delta\vartheta + \alpha\Delta w_t = \varphi$$

(control in the thermal component) can be put in first order form b

$$\mathbf{u} = \begin{pmatrix} \vartheta \\ u \\ v \end{pmatrix} = \begin{pmatrix} \vartheta \\ \Delta w \\ w_t \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} \Delta & 0 & -\alpha\Delta \\ 0 & 0 & \Delta \\ \alpha\Delta & -\Delta & 0 \end{pmatrix} \qquad \boldsymbol{\varphi}$$

Some inverse problems:

Problem 4: We observe $y(\cdot) = \mathbf{C}u(\cdot)$ on $[0, T]$ with u satisfyi
f, $u(0) = 0$ — f an unknown input which we would like to determin
treat this as an optimal control problem for the equation $v' = \mathbf{A}v$
choosing the control φ to minimize $\mathcal{J}(\varphi) = \varepsilon\|\varphi\|^2_{\mathbb{U}} + \lambda\|\mathbf{C}v - y\|^2_{\hat{\mathbb{U}}}$ a
φ_{opt} as the estimate for f.

Problem 5: Longitudinal vibration of a straight viscoelastic ro
linearized equation $w_{tt} = w_{xx} + \varepsilon w_{xxt}$. Assume this is fixed at one en
and an unknown contact force $f(t)$ at the other end $([w_x + \varepsilon w_{xt}]\Big|_{x=\ell}$
an observed motion $\bar{y}(\cdot) = w(\cdot, \ell)$. We wish to find $f(\cdot)$ by replacing
which minimizes the observation error $\|w(\cdot, \ell) - \bar{y}\|$.

Problem 6: Let $u_t = u_{xx} - qu$ with $u(\cdot, 0) = 0$ and $u_x(\cdot, \ell) =$
$a(\cdot)$ for an experiment designed to determine the unknown function
observation of $y(\cdot) = u(\cdot, \ell)$.

Abstract control problem:

The abstract version of the optimal control problem is then

$$\textit{Approximate } \bar{\mathbf{u}} \textit{ by } \mathbf{S}_T\mathbf{u}_0 + \mathfrak{B}\boldsymbol{\varphi} \textit{ with } \|\boldsymbol{\varphi}\|_{\mathbb{U}} \textit{ small.}$$

with $\mathbf{S}_T = \mathbf{S}(T)$ and, in view of (9),

$$\mathfrak{B} : \mathbb{U} \to X : \boldsymbol{\varphi} \mapsto \int_0^T \mathbf{S}(T-s)\mathbf{B}\boldsymbol{\varphi}(s)\,ds$$

where $\mathbb{U}$ is a Banach space of U-valued functions on $[0,T]$.

For the Hilbert space case $\mathbb{U} = L^2([0,T] \to U)$ of (2), minimizing
$\mathcal{J}(\boldsymbol{\varphi}) = \|\boldsymbol{\varphi}\|_{\mathbb{U}}^2 + \lambda\|\mathbf{S}_T\mathbf{u}_0 + \mathfrak{B}\boldsymbol{\varphi} - \bar{\mathbf{u}}\|^2$ gives the first order optima
$\boldsymbol{\varphi} + \mathfrak{B}^*\lambda(\mathbf{S}_T\mathbf{u}_0 + \mathfrak{B}\boldsymbol{\varphi} - \bar{\mathbf{u}}) = 0$

$$\text{so} \qquad \boldsymbol{\varphi} = -\lambda(\mathbf{I} + \lambda\mathfrak{B}^*\mathfrak{B})^{-1}\mathfrak{B}^*(\mathbf{S}_T\mathbf{u}_0 - \bar{\mathbf{u}})$$

The adjoint problem:

We must compute the adjoint $\mathfrak{B}^*$: consider

$$-v' = \mathbf{A}^* v, \quad v(T) = \eta \qquad \text{so } v(t) = \mathbf{S}^*(T-t)\eta \tag{11}$$

(referred to as the *adjoint equation*). Then

$$\begin{aligned} \langle \mathfrak{B}\boldsymbol{\varphi}, \eta \rangle &= \langle \mathbf{u}(T), v(T) \rangle = \qquad \langle \mathbf{u}(t), v(t) \rangle \Big|_0^T = \int_0^T \langle \mathbf{u} \\ &= \int_0^T [\langle \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\varphi}, v \rangle + \langle u, -\mathbf{A}^* v \rangle] \, dt \\ &= \int_0^T \langle \boldsymbol{\varphi}, \mathbf{B}^* v \rangle \, dt \end{aligned}$$

$$\text{so} \qquad \mathfrak{B}^* : \ X^* \to \mathbb{U}^* : \eta \mapsto y(\cdot) = \mathbf{B}^* v(\cdot)$$

[Note that the map: $y(\cdot) \mapsto v(0) = \mathbf{S}^*(T)\eta$ — if it is defined — give
estimation of Example 2 (although there is a time reversal involved)

The Duality Theorem:

What happens if $\lambda \to \infty$ (exact nullcontrol: $\bar{\mathbf{u}} = 0$)? We assu
$\mathbf{S}_T : X \to \tilde{X}$ and $\mathfrak{B} : \mathbb{U} \to \tilde{X}$ are continuous.

Theorem 1. *The following are equivalent:*

1. *For each* $\mathbf{u}_0 \in X$ *there is a nullcontrol* $\boldsymbol{\varphi} \in \mathbb{U}$ *so* $\mathbf{S}_T\mathbf{u}_0 + \mathfrak{B}\boldsymbol{\varphi}$
2. *One has the range containment* $\mathcal{R}(\mathbf{S}_T) \subset \mathcal{R}(\mathfrak{B})$ *in* $\tilde{X}$
3. *There is a continuous map:* $\boldsymbol{\Gamma} : X \to \mathbb{U}$ *such that* $\mathbf{S}_T + \mathfrak{B}\boldsymbol{\Gamma}$
4. *For solutions of the adjoint equation:* $-v' = \mathbf{A}^* v$, *one has* inequality:

   (12) $$\|v(0)\|_X \leq \mathfrak{c}\|\mathbf{B}^* v(\cdot)\|_{\mathbb{U}^*}$$

   so $\boldsymbol{\Gamma}_1 : \mathcal{R}(\mathfrak{B}^*) \to X^* : \mathbf{B}^* v(\cdot) \mapsto -v(0)$ *is defined and continu*

[If $\|\boldsymbol{\varphi}\|_{\mathbb{U}} \leq \mathfrak{C}\|\mathbf{u}_0\|$ in 1., then this $\mathfrak{C}$ is the same as in (12) and $\|\boldsymbol{\Gamma}_1$

Proof of the Duality Theorem:

Clearly 1.$\Longleftrightarrow$2. and 3.$\Rightarrow$1. To show 1.$\Rightarrow$3., let $\tilde{\mathbb{U}}$ be the quotient s
and define an operator

$$\mathbf{M} : X \times \tilde{\mathbb{U}} \to X : (\mathbf{u}_0, [\boldsymbol{\varphi}]) \mapsto \mathbf{S}_T \mathbf{u}_0 + \mathfrak{B}\boldsymbol{\varphi}$$

This is continuous so $\mathcal{N}(\mathbf{M})$ is closed — and is the graph of a l
$\boldsymbol{\Gamma}_0 : X \to \tilde{\mathbb{U}} : \mathbf{u}_0 \mapsto [\boldsymbol{\varphi}]$ since $[\boldsymbol{\varphi}]$ is unique in $\mathcal{N}(\mathbf{M})$. As 1. me
everywhere defined on the Banach space X, its continuity follows
Graph Theorem; we can then appeal to the Michael Selection Theo

Note that we may identify $\tilde{\mathbb{U}}^*$ with $\overline{\mathcal{R}(\mathfrak{B}^*)}$ and, by the earlie
of $\mathfrak{B}^*$, the operators $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$ are indeed adjoints; hence 3.$\Longleftrightarrow$4.

[In a Hilbert space setting ($\mathbb{U}^* = \mathbb{U}$), we can identify the quotient sp
subspace $\mathcal{N}(\mathfrak{B})^\perp = \overline{\mathcal{R}(\mathfrak{B}^*)} \hookrightarrow \mathbb{U}$ and $\boldsymbol{\Gamma} : X \to \mathbb{U}$ is linear; then $\|\boldsymbol{\Gamma}$
This theorem is the heart of Lions' *Hilbert Uniqueness Method* (H

A boundary control problem:

Unlike ODEs, one can have control which apparently does not differential equation itself. For example, consider

$$(13) \qquad \begin{aligned} u_t = u_{xx} + u_{yy} \qquad & \text{on } \mathcal{Q} = (0,T) \times \Omega \\ u = \varphi \qquad & \text{on } (0,T) \times \partial\Omega \\ u = u_0 \qquad & \text{on } \Omega \text{ at } t = 0 \end{aligned}$$

Problem 7: Much as for Problem 1 — except that the control choice of boundary data, constrained so $\varphi = 0$ except on a specifie with the control space $\mathbb{U}$ a space of functions on $\Sigma_\omega = [0,T] \times \omega$. [Similar problems of boundary control (or observation: Example 2) for other equations, such as the wave equation.]

[$\mathfrak{B}\varphi$ is defined as the value at T of the solution of the equation with initial data $\mathbf{u}_0 = 0$. If this map is continuous: $\mathbb{U} \to \tilde{X}$, then the Du continues to apply. One must be very careful (especially for the w with appropriately specifying the spaces involved and checking regu

A trick:

For a boundary control patch ω open in $\partial\Omega$, there is a simple trick boundary problem above to Example 1, involving an interior contro

Artificially, add a 'bump' to Ω such that the interface between the lies within $\omega \subset \partial\Omega$; call the new region Ω'. Artificially define a cont the interior of the bump, hence in Ω'. Embed X in a space of func e.g., extending each $\mathbf{u}_0$ as 0 on the bump. If one can then control t control on ω', then the trace on $[0, T] \times \omega$ of the resulting solution a nullcontrol for the original boundary control problem.
[Essentially the same trick works for the observation problem of Ex

A damping inequality and stabilization:

Our first *damping inequality* is: there are K, ϑ with $\vartheta < 1$ for wh

for every u_0 *in* X *there is some* $\hat{\varphi}$ *on* $[0, T]$ *such that*

$$\|\mathbf{S}_T u_0 + \mathfrak{B}_T \hat{\varphi}\| \leq \vartheta \|u_0\| \qquad \|\hat{\varphi}\|_{\mathbb{U}} \leq K \|u_0\| \tag{14}$$

Given (14), recursively set $u_j = u(jT) = \mathbf{S}_T u_{j-1} + \mathfrak{B}_T \hat{\varphi}_{j-1}$ and t
$[jT, (j+1)T]$ from it ($u_0 \hookleftarrow u_j$). Concatenating these intervals giv
such that (for any ($\alpha < |\ln \vartheta|/T$)

$$\mathcal{J}_\alpha(\varphi) = \int_0^\infty [\|\varphi\|_U^2 + \|u\|^2] e^{2\alpha t}\, dt < \infty$$

The map $\mathbf{\Gamma} : u_0 \mapsto$ the $\mathcal{J}_\alpha$-minimizing control is linear and continu
$\mathbf{K} : u \mapsto [\mathbf{\Gamma} u](0)$ is continuous (evaluation at the initial point), a
$\varphi(\cdot)$ in feedback form: $\varphi(t) = \mathbf{K}u(t)$ so $\mathbf{A} + \mathbf{BK}$ generates an expon
semigroup.

[We only mention the considerable work on use of the *algebraic Ri*
(ARE for operators on infinite dimensional Hilbert spaces) to obtai

Two abstract results:

We know two abstract results to obtain nullcontrolability from da

Theorem 2. *Suppose, much like* (14)*, one has* T, K, ϑ *with* $\vartheta <$ *for each initial state* u *there exists* $\hat{u}$ *and a control* φ *on* $[0,$ $\|\varphi\|_{\mathbb{U}} \leq K\|u\|$ *and* $\|\mathbf{S}_T(u - \hat{u}_1) + \mathfrak{B}_T\hat{\varphi}\| \leq \vartheta\|u\|$*. Then for each* *nullcontrol* φ *such that* $\|\varphi\|_{\mathbb{U}_T} \leq \mathfrak{C}\|u_0\|$ *with* $\mathfrak{C} \leq K/(1 - \vartheta)$*.*

Proof: Recursively obtain $\hat{\varphi}_j$ on $[0, T]$ and $u_{j+1} = \hat{u}_j$ such that

$$\|\hat{\varphi}_j\| \leq K\|u_j\| \qquad \|u_{j+1}\| \leq \vartheta\|u_j\| \qquad \mathbf{S}_T(u_j - u_{j+1}) + \mathfrak{B}_T$$

Take $\varphi = \sum_0^\infty \hat{\varphi}_j$ and summing the telescopic series $\sum \mathbf{S}_T(u_j - u_{j+1}$ shows φ is a nullcontrol for u_0.

Theorem 3. *Suppose there exist $c, d > 0$ such that, for any τ for each u_0 there exists $\hat{\varphi}$ on $[0, \tau]$ for which*

$$\|\varphi\|_{\mathbb{U}} \leq e^{c/\tau}\|u_0\|, \qquad \|\mathbf{S}_\tau u_0 + \mathfrak{B}_\tau \hat{\varphi}\| \leq e^{-d/\tau}\|u_0\|.$$

Then one has 'rapid nullcontrolability' (all $T > 0$): for each T, nullcontrol φ on $[0, T]$ with $\|\varphi\|_{\mathbb{U}_T} \leq \mathfrak{C}\|u_0\|$ for $\mathfrak{C} = e^{\mathfrak{c}/T}$ if $\mathfrak{c} > ($

Proof: Choose $c/(c + d) < r < 1$ (so $\varepsilon = 1 - (c/d)(1 - \vartheta)/\vartheta$ $\tau_0 = (1 - \vartheta)T$ so, with $t_0 = 0$, $t_{j+1} = t_j + \tau_j$, one has the partiti for $[0, T)$. Recursively obtain $\hat{\varphi}_j$ on $[t_j, t_{j+1}]$ and set $u_{j+1} = \mathbf{S}$ Concatenating gives φ on $[0, T]$. It is easy to see that $\|u_j\| \to 0$ nullcontrol. A slightly messy estimation shows $\|\varphi\|_{\mathbb{U}} \leq \Sigma\|\hat{\varphi}_j\| \leq$ $\mathfrak{c} = c/(1 - r)$ and optimizing the choice of r concludes the proof.

Geometry and the wave equation:

For the classical wave equation disturbances/information propaga
so there is necessarily a minimum observation time to 'see' any pa
outside the observation patch ω: the key to understanding the ge
wave equation is ray tracing with this propagation speed.

Consider the 1-D wave equation $w_{tt} = w_{xx}$ corresponding to sm
vibrations of a string — which we fix at one end $(w\Big|_{x=0} \equiv 0)$ with
of the other end $(x = \ell)$ as control. Now think of this as a semi-
without control and with data vanishing outside $[0, \ell]$ Ray tracing –
$w(t, x) = f(x + t) + g(x - t)$ with f, g matching the data — sho
vanish within $[0, \ell]$ for $t > 2\ell$ so we can take $w(\cdot, \ell)$ for this solution
Note that the longest 'ray' within $\Omega = (0, \ell)$, allowing for the reflec
has length 2ℓ: exactly the control time found.

Geometry and the wave equation, continued:

Let L be the sup of lengths of rays (geodesics), allowing for s
tion at the noncontroling part of the boundary, until entering the
It was shown by Bardos-Lebeau-Rauch (1992) that the minimum
controlability is, indeed, L. The necessity of this for nullcontrolabi
tion/prediction seems obvious (and *trapped rays* should preclude nu
for any T at all), but one needs sufficiency.

Neglecting many technical difficulties, one notes from Scat
that (e.g., for "star-complemented regions") in suitable time T a
fraction of the initial energy will exit through the control portion o
so, using this trace as $\hat{\varphi}$, one obtains a damping inequality as in T
nullcontrolability follows.

Geometry and the heat equation:

Boundary nullcontrolability had been known for the 1-D case via
monic analysis and spectral expansion — separately for observatio
control as Theorem 1 was then unknown.

Essentially the result for the wave equation noted above was
(1973) and he also used this to show, by a harmonic analysis argu
Miller's (2004) related use of the "transmutation method") corresp
ary nullcontrolability for the heat equation; Seidman (1976) showed
boundary nullcontrolability (using all of $\partial\Omega$) directly from the 1-dim
by embedding Ω in a cylinder. It was shown in (1984) that Russ
gave $\mathfrak{C} = e^{\mathcal{O}(1/T)}$ for the heat equation.

Nullcontrolability from an arbitrary open control patch ω
shown (Lebeau-Robbiano, 1995) using *Carleman inequalities*, a tec
obtain a quantitative form of uniqueness from data on a patch. [T
enough technicalities not to describe it here.]

Rapid patch nullcontrolability of the heat equation:

We use instead an argument based on a deep theorem of Jerison-
eigenfunctions of Δ (itself obtained using Carleman inequalities):

Theorem 4. (JL) *Let* $\{(z_j, \lambda_j)\}$ *be the eigenpairs of* Δ *so* $0 < \lambda$
Then for open $\omega \subset \Omega$ *there is* $\gamma > 0$ *such that, for all* $\sigma > 0$ *and e*
$w \in Z_\sigma = \mathrm{span}\{z_j : \lambda_j \leq \sigma\}$, *one has*

$$\int_\Omega |w(s)|^2\, ds \leq C^2 e^{2\gamma\sqrt{\sigma}} \int_\omega |w(s)|^2\, ds \tag{15}$$

To show patch nullcontrolability of the heat equation, note that
with control on all of Ω ($\mathfrak{C} = \mathcal{O}(1/\sqrt{T})$). Split $u_0 = v_\sigma + w_\sigma$ with w_σ
the observation inequality for this whence Theorem 4 gives a similar i
$\mathcal{Q}_\omega$ for Z_σ with a factor $e^{\gamma\sqrt{\sigma}\tau/2}$ whence nullcontrol with that estim
control spillover to v_σ, is dominated, for large $\sigma = (s/\tau)^2$, by the e
the uncontroled second half of $[0, \tau]$. Combining gives a damping ineq
to apply Theorem 3which shows that the heat equation is rapidly
on ω with control norm blowup $\mathfrak{C} = e^{\mathcal{O}(1/T)}$ as $T \to 0$.

A couple of open problems:

We would be incomplete without noting at least a couple of open

- The same argument as was used just above for the heat equati rapid nullcontrolability for the thermoelastic plate with control patch ω in just one component. However, the trick noted earlie boundary nullcontrolability from this does not work for coupled wants control only in one component; a similar difficulty arises if conditions are different for w, ϑ. How can one then show rapid nu here with $\mathfrak{C} = e^{\mathcal{O}(1/T)}$?

- For all of this we have implicitly assumed ω to be an *open* subs $\partial\Omega$). For open ω, what are the asymptotics as ω shrinks to a p obtain similar results if control is restricted instead to a subset having positive measure? [This kind of question arises, e.g., in c bang-bang theorems for constrained control.]

- And of course we have not touched yet on problems involving cons
or nonlinear equations or non-additive control or shape control o