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Blowup estimates for observability of a thermoelastic system

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Abstract. With observation restricted to a single component: displacement, velocity, or temperature, we consider observability of the nonscalar thermoelastic system

$$[1-\gamma\Delta]w_{tt}+\Delta^2 w-\alpha\Delta\theta=0, \qquad \theta_t-\Delta\theta+\alpha\Delta w_t=0,$$

coupling heat conduction with a Kirkhoff or Euler–Bernoulli plate model. One does have observability in arbitrarily short time here, but necessarily has blowup of the sensitivity as the observation time $T \to 0$ and also as the coupling coefficient $\alpha \to 0$. In this paper we are able to examine the asymptotics of this blowup for two situations: global observation (i.e., on all of Ω) and, with significant restrictions, boundary observation. The blowup rates obtained are of optimal order: $\mathcal{O}(T^{-5/2})$ for global observation, corresponding to what is known for 3-dimensional systems, and exponential in $\mathcal{O}(1/T)$ for boundary observation, corresponding to what is known for scalar PDE problems. Our methods permit us also to obtain asymptotics as $\alpha \to 0$ – a question which can only arise for systems.

Keywords: parametric asymptotics, minimal energy, blowup, thermoelastic system, observation time, coupling parameter, system components, distributed parameter systems, partial differential equations

1. Introduction

Beginning with [20], blowup estimates for the parametric dependence of observation and control have been considered in a variety of contexts, for both finite and infinite-dimensional dynamics: cf., e.g., [21, 24,13,23,8,22,2,3,25,17–19] – obtaining the asymptotic dependence as $T \to 0$ of the coefficient $\mathfrak{C}$ in observability estimates

$$\|\text{state at time } T\| \leqslant \mathfrak{C}\|\text{observation during } [0,T]\|, \tag{1.1}$$

so $\mathfrak{C}$ bounds the norm of the observability map: observation $\mapsto$ state.

Remark 1.1. It is well-known in the control-theoretic literature that this observability is equivalent to nullcontrollability for a control problem: the adjoint of the observation map has the interpretation of providing the minimal energy control in the related nullcontrol problem, taking an initial state to 0 so the function $T \mapsto \mathfrak{C}(T)$ whose asymptotics we are investigating is also referred to as a *minimal energy*

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function for nullcontrollability and has dual interest in that context: it indicates the energy needed for fast control as in the titles of [21,17].

We remark also that interest in study of the asymptotics of minimal energy functions is motivated not only by such control theoretic considerations, but also by an interesting connection with stochastic analysis: we note, e.g., Proposition 2.28 in [6] (cf. also [7]):

The transition semigroup $R_t : \phi \mapsto E(\phi(X(t,\cdot))$ on $C_b(H)$ associated with the Ornstein–Uhlenbeck process $\mathrm{d}X_t = AX(t)\,\mathrm{d}t + B\,\mathrm{d}W(t)$ (where W is Wiener cylindrical noise of trace class) satisfies the estimate:

$$\left|\frac{\partial}{\partial t} R_t\phi(x)\right| \leqslant \|\mathfrak{C}(t)\|\|\phi\|, \quad x \in H.$$

Here $T \mapsto \mathfrak{C}$ is the minimal energy function associated with the nullcontrollability problem for the deterministic system

$$y_t = Ay + Bu, \qquad y(0) = x \in H$$

so the singularity at the origin of this transition semigroup is determined by the asymptotic blowup rate of $\mathfrak{C}$ for the nullcontrollability map.

Primarily focusing on scalar equations (with the exception of [2,3,25]), the papers cited earlier have been specifically concerned with the blowup rate as the observation/control time $T \to 0$ for systems which permit this, i.e., for which one has observability/nullcontrollability in arbitrarily short time. In contrast, our concern in this paper is with parametric asymptotics for parametrized families of *nonscalar distributed parameter systems for which observation/control is restricted to a single component.* In this we are concerned not only with the asymptotics as $T \to 0$ – again requiring observability/nullcontrollability in arbitrarily short time so $\mathfrak{C}(T) < \infty$ for all $T > 0$ – but also with the dependence on a new parameter, only meaningful in this context: a *coupling parameter* α such that the system decouples if $\alpha = 0$. In particular, our exposition will be in the context of the thermoelastic system

$$\begin{aligned} &w_{tt} - \gamma[\Delta w]_{tt} + \Delta^2 w - \alpha\Delta\theta = 0, && \\ &\theta_t - \Delta\theta + \alpha\Delta w_t = 0, && \text{on } \mathcal{Q} = \mathcal{Q}_T = [0,T]\times\Omega, \\ &w = 0, \qquad \Delta w = 0, \qquad \theta = 0 && \text{on } \Sigma = [0,T]\times\partial\Omega, \end{aligned} \tag{1.2}$$

coupling the mechanical components w, w_t with the temperature θ. The coupling parameter $\alpha > 0$ appearing in (1.2) reflects the interaction between the two constituent parts of the system: the plate equation and the heat equation. If we think of Ω as a subset of $\mathbb{R}^2$, then the first equation of (1.2) – say, decoupled by taking $\alpha = 0$, is a standard plate equation with "simply supported" boundary conditions; for $\gamma = 0$ we have an Euler–Bernoulli plate.

This may be put in operator form by taking $\mathbf{A}$ to be the Laplace operator on the bounded region Ω with homogeneous Dirichlet conditions:

$$\begin{aligned} &\mathbf{A} : z \mapsto -\Delta z : L^2(\Omega) \supset \mathcal{D} \to L^2(\Omega), \\ &\mathcal{D} = H^2(\Omega) \cap H_0^1(\Omega) = \{z \in H^2(\Omega)\colon\ z|_{\partial\Omega} = 0\} \subset L^2(\Omega). \end{aligned} \tag{1.3}$$

The system (1.2) can then equivalently be written as

$$\mathbf{M}w_{tt} + \mathbf{A}^2 w + \alpha \mathbf{A}\theta = 0, \qquad \theta_t + \mathbf{A}\theta - \alpha \mathbf{A} w = 0, \tag{1.4}$$

where $\mathbf{M} = \mathbf{M}_\gamma = \mathbf{1} + \gamma\mathbf{A}$. Note that the Laplace operator $\mathbf{A}$ of (1.3) is self-adjoint, positive definite on $L^2(\Omega)$ and invertible with $\mathbf{A}^{-1}$ continuous: $L^2(\Omega) \to \mathcal{D} \subset H^2(\Omega)$; most of our analysis would be unchanged if we considered (1.4) with more general $\mathbf{A}$ having these properties. It is known that the system given by (1.4) written in the 3-component variable (w, w_t, θ) generates a strongly continuous semigroup of contractions on the energy space $\mathcal{H}_* \equiv \mathcal{D}(\mathbf{A}) \times \mathcal{D}(\mathbf{M}^{1/2}) \times L^2(\Omega)$.

This system is of interest in its own right and also serves here as an exemplar of our present concerns: we will examine the asymptotics both as $T \to 0$ and as $\alpha \to 0$. [Note for the latter that when $\alpha = 0$ no observation of θ can provide any information whatever about w and vice versa so the coefficient $\mathfrak{C}$ must necessarily blow up when $\alpha \to 0$ just as it necessarily blows up when $T \to 0$ – cf., e.g., [22].] The system (1.2) further involves the additional parameter γ, related to the significance of rotational forces, and we note that the nature of the system dynamics is entirely different when $\gamma = 0$: the solution semigroup is then analytic [15,16], while the system has a predominantly hyperbolic character for $\gamma > 0$ [14].

To the best of our knowledge, all the available results on parametric asymptotics for nonscalar systems of partial differential equations with observation/control restricted to a single component have been in the context of the restricted thermoelastic system: (1.2) with $\gamma = 0$, and have restricted attention to the asymptotics as $T \to 0$, keeping $\alpha > 0$ fixed [2,3,25]. Thus, our treatment here breaks new ground in its treatment of the more general system, allowing $\gamma > 0$, and in its consideration of the effect of decoupling while also obtaining new results for the asymptotics as $T \to 0$.

Our treatment splits into two, somewhat disjoint, parts. As the first of these parts, we will be considering in Sections 2 and 3 the case of

- global observation of a single component, i.e., over the entire region Ω.

Since the ‘entire’ component is observed in this case, and positive results on nullcontrollability of (1.4) with just one component are available [16,1], the situation seems quite comparable to the finite-dimensional situation treated in [21,24]. The relevant asymptotics there – $\mathcal{O}(T^{-[N-1/2]})$ as in Theorem 3.3 – suggest for this 3-component case that we should have

$$\mathfrak{C}(T) \sim T^{-5/2} \tag{1.5}$$

as $T \to 0$. [Indeed, if one restricts consideration to a single eigenspace, the system does become a first-order 3-dimensional system to which the results of [21] apply so the $\mathcal{O}(T^{-5/2})$ asymptotics would then be exact so this blowup rate is necessarily optimal.] In these sections we will independently present two approaches:

- weighted energy estimates,
- spectral expansion – based on [21]

by which to obtain the optimal[1] $\mathcal{O}(T^{-5/2})$ asymptotics for (1.2) in this case and to consider the asymptotics as $\alpha \to 0$ as well. [Rather than seeking blowup asymptotics for (1.2) as $\gamma \to 0$, we will obtain a treatment uniform in γ by using a γ-dependent norm for the estimation.]

[1]Optimal $\mathcal{O}(T^{-5/2})$ asymptotics with global controls (supported in the entire region Ω) dual to these global observation problems have recently been derived in [2] for nullcontrollability of thermoelastic plates (1.2) with $\gamma = 0$ and subject to a variety of physically relevant boundary conditions – including canonical models of simply supported, clamped and free boundary conditions. The methods used in [2] rely on weighted energy estimates which are intrinsically nonspectral. These are very flexible in dealing with the variety of unstructured problems which arise in noncommutative cases of the operator matrix

For more localized observation the situation is more complicated and more difficult. In comparison with the known results for *scalar* problems (see [20,8,17] for the heat equation and [13,18] for plate and Schrödinger equations) it would seem plausible to seek results for observation on an arbitrary patch, either within the region Ω or in the boundary $\partial\Omega$, but our available tools at present are inadequate to obtain general patch observability/nullcontrollability and related estimates[2] when observation is restricted to a single component of the system. What we *are* able to treat is the case of

- observation of a single component on a base of a Cartesian product region where separation of variables then permits reduction to a family of spatially one-dimensional problems.

We will also restrict our attention here to the Euler–Bernoulli plate since one loses observability/nullcontrollability for small T when $\gamma \neq 0$. Indeed, according to Theorem 1.1.2 in [14], the principal part of the differential operator describing the Kirkhoff plate dynamics of (w, w_t) has speed of propagation $\gamma^{-1/2}$; for the thermoelastic dynamics with $\gamma > 0$ one then expects a finite speed of propagation for the singular support.

Comparing with the prior results obtained for scalar problems, we anticipate, for the system (1.2) with $\gamma = 0$, the same asymptotics as for those problems: that

$$\ln \mathfrak{C}(T) = \mathcal{O}(1/T) \quad \text{as } T \to 0. \tag{1.6}$$

[Again, restricting to a suitable subspace, the results of, e.g., [9,17] apply directly and we see that this blowup rate must be optimal.] In Section 4 we will use the approach

- spectral expansion – based on [23]

to obtain those optimal $\exp[\mathcal{O}(1/T)]$ asymptotics for our system and also to consider the asymptotics as $\alpha \to 0$.

Note added in proof: The above is a severe geometric limitation, but these methods do obtain the desired $\exp[\mathcal{O}(T^{-1})]$ asymptotics as $T \to 0$. Since acceptance of this paper we have become aware of two forthcoming treatments of patch control in one component for this problem. We note, however, that each obtains the asymptotics $\exp[\mathcal{O}(T^{-\beta})]$ $(\beta > 1)$.

P. Cokeley, Localized and boundary null controllability properties, and associated minimal norm asymptotics, of two nonstandard parabolic partial differential systems, Ph.D thesis, Univ. Nebraska, to appear.

L. Miller, On the cost of fast controls for thermoelastic plates, *Asymptotic Analysis*, to appear.

1.1. An abstract formulation

While our exposition will be specifically for the thermoelastic system (1.2), it will be convenient to view this in the abstract form

$$\dot{U} = \mathbf{L}_* U, \qquad \varphi = \mathbf{B} U \tag{1.7}$$

L in (1.11) (clamped or free boundary conditions) or problems with controlled time variation of the coefficients in **A**. On the other hand, we note that these prior analyses [2,3] did not consider the asymptotics with respect to variation of the coupling constant α or the rotational force parameter γ.

[2] In the case when $\gamma > 0$, a preliminary question of just nullcontrollability of (1.4) in arbitrarily short time $T > 0$ with controls restricted to a patch or a boundary has a negative answer. With $\gamma = 0$, the one prior positive nullcontrollability result from a patch in an arbitrary short time T is for the case of thermal controls [4].

with the state $U(t)$ and observation $\varphi(t)$ taking values in Hilbert spaces $\mathcal{H}$, $\mathcal{X}$, respectively; $\mathbf{L}_*$ is the infinitesimal generator of a C_0 semigroup on $\mathcal{H}$. [For the system (1.4) under our consideration here we may consider, as convenient, the semigroup formulation, the obvious expansion using the eigenpairs of $\mathbf{A}$, or the weak formulation of the system.]

Of course, our interest here is in nonscalar systems for which $\mathcal{H}$ is a product space and $\mathbf{L}_*$ has a corresponding block matrix structure[3] with the observation operator $\mathbf{B}$ involving only one of the components of $\mathcal{H}$. We then refer to $U(t)$ as the "full state" with $\mathbf{B}$ involving the "observed state component." A new consideration here is that the block matrix structure of $\mathbf{L}_*$ may involve a *coupling parameter* α – decoupling the observed state component from others when $\alpha = 0$ – and we are also interested in the asymptotics as $\alpha \to 0$, for which we will also have blowup. The *observation estimates* (1.1) under consideration have here the form

$$\|U(T)\|_{\mathcal{H}} \leqslant \mathfrak{C}\|\varphi\|_{L^2([0,T]\to\mathcal{X})}, \tag{1.8}$$

where $L^2([0,T] \to \mathcal{X})$ is the Hilbert space of $\mathcal{X}$-valued functions on $[0,T]$ with the obvious inner product and norm; note that $\mathfrak{C}$ is just (a bound on) the norm of the operator:

$$\varphi \mapsto U(T) : L^2\bigl([0,T] \to \mathcal{X}\bigr) \to \mathcal{H}.$$

We will be considering situations in which we have a parametrized family of such problems – if nothing else, the observation time T is already a significant parameter, but the operator $\mathbf{L}_*$ may also vary parametrically – and we are interested in the asymptotics of the dependence of the estimation constant $\mathfrak{C}$ in (1.8) on the parameters of the problem.

There are many ways of putting our particular system (1.2) in the first-order form of (1.7). We choose to take the state to consist of the three components θ, $u = \mathbf{A}w$, and $v = \mathbf{M}^{1/2}w_t$, corresponding to the natural energy

$$\mathcal{E} = \frac{1}{2}\bigl\|\mathbf{M}^{1/2}w_t\bigr\|^2 + \frac{1}{2}\|\mathbf{A}w\|^2 + \frac{1}{2}\|\theta\|^2 = \frac{1}{2}\bigl[\|u\|^2 + \|v\|^2 + \|\theta\|^2\bigr]. \tag{1.9}$$

[For future reference we note that

$$\mathcal{E}(t) = \mathcal{E}(s) - \int_s^t \langle\theta, \mathbf{A}\theta\rangle\,\mathrm{d}\hat{t} \quad \text{for } s \leqslant t, \tag{1.10}$$

i.e., dissipation at the rate $\langle\theta, \mathbf{A}\theta\rangle$.] Thus, introducing the 3×3 operator matrix $\mathbf{L}$, the system (1.4) becomes

$$U = \begin{pmatrix}\theta\\ u\\ v\end{pmatrix} = \begin{pmatrix}\theta\\ \mathbf{A}w\\ \mathbf{M}^{1/2}w_t\end{pmatrix} \quad\text{and}$$

$$\dot{U} = \mathbf{L}\mathbf{A}\,U \quad \text{with} \tag{1.11}$$

$$\mathbf{L} = \begin{pmatrix} -\mathbf{1} & 0 & \alpha\mathbf{M}^{-1/2}\\ 0 & 0 & \mathbf{M}^{-1/2}\\ -\alpha\mathbf{M}^{-1/2} & -\mathbf{M}^{-1/2} & 0\end{pmatrix}$$

[3] It is significant for us that the 'blocks' of $\mathbf{L}_*$ are commuting operators for (1.2).

so $\mathbf{L}_* = \mathbf{LA}$. Our desired *observation inequality* then takes the form

$$2\mathcal{E}(T) \leqslant \mathfrak{C}^2 \|\varphi\|^2_{L^2([0,T]\to\mathcal{X})} \tag{1.12}$$

for solutions of (1.4), where, recalling (1.7), $\varphi = \mathbf{B}U$. The constant $\mathfrak{C}$ is, of course, to be independent of the particular solution, but will necessarily depend on the particular construction of U, the choice of observation φ, and on $\Omega, \mathbf{A}$ and the various parameters: in particular, we are considering asymptotics in T, α for $\mathfrak{C} = \mathfrak{C}(T; \alpha)$.

Notation. Henceforth we will let $\langle \cdot, \cdot \rangle$ denote both the scalar product on $L^2(\Omega)$ and, without confusion, the scalar product on $\mathcal{H} = L^2(\Omega \to \mathbb{R}^3)$; similarly, $|\cdot|$ is both the absolute value in $\mathbb{R}$ or $\mathbb{C}$ and the usual Euclidean norm of $\mathbb{R}^N$ while $\|\cdot\|$, without any subscript, denotes the norms of $L^2(\Omega)$ and of $\mathcal{H}$ as well as matrix and operator norms; where needed, we write $\|\cdot\|_{\mathcal{Q}}$ for the norm of $L^2(\mathcal{Q}_T) = L^2([0,T] \to L^2(\Omega))$, etc., so, e.g., $\|f\|^2_{\mathcal{Q}} = \int_0^T \|f\|^2 \, \mathrm{d}t$ with $\|\varphi\|$ the $L^2(\Omega)$-norm.

2. Global observation: weighted energy

As noted above, we are considering the linear thermoelastic system (1.2)

$$\begin{aligned} &[1 - \gamma\Delta]w_{tt} + \Delta^2 w - \alpha\Delta\theta = 0, \\ &\theta_t - \Delta\theta + \alpha\Delta w_t = 0 \end{aligned}$$

on $\mathcal{Q}_T = (0,T) \times \Omega$ with $\Omega \subset \mathbb{R}^n$ bounded and with $\alpha > 0$; the constant $\gamma \geqslant 0$ represents effects of rotational forces. Although the dimensionality is not significant for this analysis, we may think of $n = 2$ so the first equation here becomes a standard (Kirkhoff) plate equation on decoupling the system; an Euler–Bernoulli plate when $\gamma = 0$. We are imposing the simply supported boundary conditions of (1.2):

$$w = 0, \qquad \Delta w = 0, \qquad \theta = 0 \quad \text{on } \Sigma = [0,T] \times \partial\Omega. \tag{2.1}$$

The abstract version of this thermoelastic system (1.2), (2.1) is (1.4) with (1.3) and $\mathbf{M} = \mathbf{M}_\gamma = \mathbf{1} + \gamma\mathbf{A}$.

The traditional state for (1.2) is

$$(w, w_t, \theta) \in \mathcal{H}_* \equiv \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times H_\gamma^1(\Omega) \times L_2(\Omega),$$

where $H_\gamma^1(\Omega) = \{z\colon \mathbf{M}^{1/2} z \in L^2(\Omega)\}$. As noted, it will be more convenient, however, for us to take the state as $U = (\theta, u, v) \equiv (\theta, \mathbf{A}w, \mathbf{M}^{1/2} w_t)$ in $\mathcal{H} = [L^2(\Omega)]^3$ with the energy norm

$$\|U\|^2_{\mathcal{H}} = \|U\|^2 = \|\theta\|^2 + \|\mathbf{A}w\|^2 + \left\|\mathbf{M}^{1/2} w_t\right\|^2 \tag{2.2}$$

recalling (1.9). It is well known [15] that the C_0 contraction semigroup on $\mathcal{H}$ generated by the system (1.2), (2.1) is analytic when $\gamma = 0$; in this case we have $U(t) \in \mathcal{D} \times \mathcal{D} \times \mathcal{D}$ for $t > 0$. We also note from (1.10) that one has the *energy identity*:

$$\left\|U(s)\right\|^2 + 2\int_t^s \left\|\nabla\theta(r)\right\|^2 \mathrm{d}r = \left\|U(t)\right\|^2 \quad \text{for } 0 \leqslant t \leqslant s. \tag{2.3}$$

Recall that we are considering $\|U(T)\|_{\mathcal{H}} \leqslant \mathfrak{C}\|\varphi\|$ and, comparing with (1.5), for this 3-component system, we expect $\mathfrak{C} = \mathfrak{C}(T;\alpha) \sim T^{-5/2}$ as $T \to 0$ with $\alpha > 0$ fixed for this global observation.

Our principal result for this section is then the following:

Theorem 2.1. *Let $0 < T \leqslant T_0$, $\gamma \geqslant 0$ and $0 < \alpha$. In each of the cases:* [$\varphi = \mathbf{M}^{1/2}\theta$] *or* [$\varphi = \mathbf{M}u = \mathbf{MA}w$] *or* [$\varphi = \mathbf{M}^{1/2}v = \mathbf{M}w_t$]*, it is possible to observe the indicated state component φ and from this to determine all the components of the full state (w, w_t, θ) at the final time T for the linear thermoelastic system* (1.2)*. The linear operator: $\varphi \mapsto U(T) = (\theta, u, v)|_{t=T}$ is continuous from $L^2(\mathcal{Q}_T)$ to $\mathcal{H} = (L^2(\Omega))^3$ for each of the cases:*

- Thermal observation ($\varphi = \mathbf{M}^{1/2}\theta$) *or*
- Displacement observation ($\varphi = \mathbf{M}u = \mathbf{MA}w$*, corresponding to observation with w topologized in* $L^2([0,T] \to H^2(\Omega))$) *or*
- Velocity observation ($\varphi = \mathbf{M}^{1/2}v = \mathbf{M}w_t$).

One further has bounds for the operator norms $\mathfrak{C} = \mathfrak{C}_{\theta,u,v} = \mathfrak{C}(T;\alpha)$, see (1.8)*,*

$$\begin{aligned}
&\mathfrak{C}_\theta \leqslant \Psi_\theta\big(\alpha T^{-1/2} + \alpha^{-1}T^{-5/2}\big),\\
&\mathfrak{C}_u \leqslant \Psi_u\big(\alpha T^{-1/2} + \alpha^{-1}T^{-5/2}\big),\\
&\mathfrak{C}_v \leqslant \Psi_v\big(\alpha^2 T^{-1/2} + \big(1+\alpha^{-1}\big)T^{-5/2}\big),
\end{aligned} \tag{2.4}$$

where each constant c and $\Psi_{\theta,u,v}$ is to be independent of T, α over the range $(0, T_0] \times (0, \infty)$ – although necessarily dependent on T_0 and $\mathbf{A}$.

Note that, while our estimates in (2.4) are, in each case, also independent of $0 \leqslant \gamma \leqslant \bar{\gamma}$, the interpretation varies significantly since the norms involved are γ-dependent: the second component of the space $\mathcal{H}_*$ is just $L^2(\Omega)$ when $\gamma = 0$, but immediately becomes $H_0^1(\Omega)$ when we take γ at all positive. Similar comments apply to the choices of topology for the observations (e.g., observing $\mathbf{M}^{1/2}\theta$ in $L^2(\mathcal{Q}_T)$ rather than $\theta \in L^2(\mathcal{Q}_T)$), but one should also note that these are entirely relative: since $\mathbf{M}^{-1/2}U(\cdot)$ also satisfies the same system, we could, e.g., observe $\theta \in L^2(\mathcal{Q}_T)$ to determine $w_t(T) \in L^2(\Omega)$.

We emphasize again that our primary interest here is with the asymptotics presented in (2.4), rather than with mere existence and continuity of each map: $\varphi \mapsto (\theta, w, w_t)|_{t=T}$ for (arbitrary) fixed $T, \alpha > 0$.

Remark 2.2. We remark that (2.4) matches (1.5) and (3.1) as $T \to 0$ for fixed $\alpha > 0$, but here gives more information as T and α vary jointly. E.g., for thermal observation or displacement observation, if we could arrange that $\alpha \sim T^{-1}$ as $T \to 0$, we would then have the slower asymptotic blowup: $C_\theta = \mathcal{O}(T^{-3/2})$.

2.1. Some identities

To prepare for the proof of Theorem 2.1 we introduce some notation and identities. Using a positive C^2 weight function $h = h(t)$ with $h = 0 = \dot{h}$ at $0, T$, we define

$$\langle f, g\rangle_h := \int_0^T h(t)\langle f, g\rangle\,\mathrm{d}t, \quad \|f\|_h^2 := \langle f, f\rangle_h = \int_0^T h(t)\,\|f\|^2\,\mathrm{d}t \tag{2.5}$$

for f, g in $L^2(\mathcal{Q}_T)$. Using the self-adjoint positive operator $\mathbf{M} = \mathbf{M}_\gamma = \mathbf{1} + \gamma\mathbf{A}$ on $L_2(\Omega)$, we recall that with $v = \mathbf{M}^{1/2} w_t$ we have

$$\langle u, v\rangle_h \equiv \langle \mathbf{M}^{1/2} u, w_t\rangle_h, \qquad \|v\|_h \equiv \|\mathbf{M}^{1/2} w_t\|_h. \tag{2.6}$$

As usual, we then have $|\langle f, g\rangle_h| \leqslant \|f\|_h\|g\|_h \leqslant \varepsilon\|g\|_h^2 + (1/4\varepsilon)\|f\|_h^2$. Writing $\langle \dot{h} f, g\rangle = \langle \sqrt{\varepsilon h} f, (\dot{h}/2\sqrt{\varepsilon h}) g\rangle$, we have

$$\left| c\int_0^T \dot{h}\langle f, g\rangle\, \mathrm{d}t\right| \leqslant \varepsilon\|f\|_h^2 + \left(\frac{c^2}{4\varepsilon}\|\dot{h}^2/h\|_{L^\infty(0,T)}\right)\|g\|_{\mathcal{Q}}^2 \tag{2.7}$$

and similarly for integrals with $\ddot{h}$. There are no boundary terms in integration by parts so, for suitably t-differentiable f, g, one has

$$\begin{aligned} \langle \dot{f}, g\rangle_h &= -\langle f, \dot{g}\rangle_h - \int_0^T \dot{h}\langle f, g\rangle\, \mathrm{d}t, \\ \int_0^T \dot{h}\langle \dot{f}, g\rangle\, \mathrm{d}t &= -\int_0^T \dot{h}\langle f, \dot{g}\rangle\, \mathrm{d}t - \int_0^T \ddot{h}\langle f, g\rangle\, \mathrm{d}t. \end{aligned} \tag{2.8}$$

It will be convenient to take $h(t) = \eta(t/T)$ on $[0, T]$ where $\eta(\cdot)$ is a C^2 positive function on $[0, 1]$ with $\eta = \eta' = 0$ at $0, 1$ and $\int_0^1 \eta(s)\, \mathrm{d}s = 1$; we further assume that $(\eta')^2/\eta$ and $(\eta'')^2/\eta$ are each in $L^\infty(0, 1)$. [One example is $\eta(s) = \frac{280}{57}(s - s^2)^4$.] For such h we have

$$\begin{aligned} &0 < \|h\|_{L^\infty(0,T)} = \|\eta\|_{L^\infty(0,1)} \quad \text{with } h|_{0,T} = 0, \quad \int_0^T h(t)\, \mathrm{d}t = T \\ &\text{and} \quad \begin{aligned} \|\dot{h}^2/h\|_{L^\infty(0,T)} &= T^{-2}\|(\eta')^2/\eta\|_{L^\infty(0,1)}, \\ \|\ddot{h}^2/h\|_{L^\infty(0,T)} &= T^{-4}\|(\eta'')^2/\eta\|_{L^\infty(0,1)}. \end{aligned} \end{aligned} \tag{2.9}$$

We also note that this gives $\|f\|_h^2 \leqslant \|\eta\|_{L^\infty(0,1)}\|f\|_{\mathcal{Q}}^2$.

Lemma 2.3. *Let $U = (\theta, u, v) = (\theta, \mathbf{A}w, \mathbf{M}^{1/2} w_t) \in C([0, T] \to \mathcal{H})$ be a solution of* (1.11)*, which we consider in the form*

$$\mathbf{A}^{-1}\dot{\theta} = -\theta + \alpha\mathbf{M}^{-1/2} v, \qquad \mathbf{A}^{-1}\dot{u} = \mathbf{M}^{-1/2} v, \qquad \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v} = -\alpha\theta - u, \tag{2.10}$$

$$\textit{so} \quad \begin{aligned} \theta &= -\alpha^{-1}\bigl(u + \mathbf{A}^{-1}\mathbf{M}^{1/2}\dot{v}\bigr) = \alpha\mathbf{M}^{-1/2} v - \mathbf{A}^{-1}\dot{\theta}, \\ u &= -\alpha\theta - \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v}, \\ v &= \mathbf{A}^{-1}\mathbf{M}^{1/2}\dot{u} = \alpha^{-1}\mathbf{M}^{1/2}\bigl(\theta + \mathbf{A}^{-1}\dot{\theta}\bigr). \end{aligned} \tag{2.11}$$

Then one has the identities:

$$\begin{aligned} \|u\|_h^2 &= \|v\|_h^2 - \alpha\langle \theta, u\rangle_h + \int_0^T \dot{h}\langle \mathbf{M}^{1/2}\mathbf{A}^{-1} u, v\rangle\, \mathrm{d}t, \\ \|v\|_h^2 &= \|\theta\|_h^2 + \alpha^{-1}\left[\langle \theta, u\rangle_h + \langle \mathbf{M}^{1/2}\theta, v\rangle_h - \int_0^T \dot{h}\langle \mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle\, \mathrm{d}t\right], \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
\|\theta\|_h^2 + \|v\|_h^2 = {} & (\alpha + \alpha^{-1})\langle\theta, \mathbf{M}^{-1/2}v\rangle_h + \alpha^{-1}\langle\mathbf{M}^{1/2}v, \theta\rangle_h \\
& + \frac{1}{2}\int_0^T \dot{h}\,\langle v, \mathbf{A}^{-1}v\rangle\,\mathrm{d}t - 2\alpha^{-1}\int_0^T \dot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle\,\mathrm{d}t \\
& + \alpha^{-1}\int_0^T \ddot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, \mathbf{A}^{-1}v\rangle\,\mathrm{d}t,
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\int_0^T \dot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle\,\mathrm{d}t &= \alpha^{-1}\biggl[\int_0^T \dot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}u, \theta\rangle\,\mathrm{d}t - \int_0^T \dot{h}\,\langle v, \mathbf{M}^{1/2}\mathbf{A}^{-1}\theta\rangle\,\mathrm{d}t \\
&\qquad - \int_0^T \ddot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}u, \mathbf{A}^{-1}\theta\rangle\,\mathrm{d}t\biggr], \\
\int_0^T \dot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle\,\mathrm{d}t &= \int_0^T \dot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta, u\rangle\,\mathrm{d}t - \alpha\int_0^T \dot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}v, u\rangle\,\mathrm{d}t \\
&\qquad - \int_0^T \ddot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta, \mathbf{A}^{-1}u\rangle\,\mathrm{d}t, \\
\langle\mathbf{M}^{1/2}\theta, v\rangle_h &= \langle\mathbf{M}\theta, u\rangle_h - \alpha\langle\mathbf{M}^{1/2}u, v\rangle_h - \int_0^T \dot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta, u\rangle\,\mathrm{d}t.
\end{aligned}
\tag{2.14}
$$

Proof. First we notice that finite energy solutions $U(t) = (\theta(t), u(t), v(t)) \in \mathcal{H} = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ lead to equations in (1.11) that are defined with values in $L^2(\Omega)$. Note also that when our computations are performed with $\gamma > 0$ we have $\mathcal{D}(\mathbf{M}_\gamma) = \mathcal{D}(\mathbf{A})$ and both operators are self-adjoint on $L^2(\Omega)$; one sees immediately[4] that

$\mathbf{M}\mathbf{A}^{-1}$ and $\mathbf{M}^{1/2}\mathbf{A}^{-1/2}$ are bounded on $L^2(\Omega)$

and, indeed, this boundedness is uniform in γ for bounded $\gamma \geqslant 0$.

As a consequence, the following regularity of higher time derivatives of weak solutions θ, w, v_t follows directly from (1.4), interpreted as in (2.10):

$$
\begin{aligned}
\mathbf{A}^{-1}\dot{\theta} &\in C\bigl([0,T] \to L^2(\Omega)\bigr), \\
\mathbf{A}^{-1}\dot{u} &\in C\bigl([0,T] \to \mathcal{D}\bigl(\mathbf{M}^{1/2}\bigr)\bigr), \\
\mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v} &\in C\bigl([0,T] \to L^2(\Omega)\bigr), \\
\mathbf{A}^{-2}\mathbf{M}^{1/2}\ddot{v} &\in C\bigl([0,T] \to L^2(\Omega)\bigr).
\end{aligned}
\tag{2.15}
$$

Thus, all the calculations performed below involve $L^2(\Omega)$ inner products of $L^2(\Omega)$ functions and so are well defined and justified.

To see (2.12), first consider the weighted scalar product of v with $v = \mathbf{A}^{-1}\mathbf{M}^{1/2}\dot{u}$ from (2.11), apply (2.8), and substitute $-\mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v} = \alpha\theta + u$ from (2.10); with a bit of re-arrangement, this is the first

[4] We owe this to the fact that we are considering simply supported boundary conditions so the operators **M** and **A** (along with their fractional powers) commute. This need not be the case if one were to consider other (clamped or free) mechanical boundary conditions for the plate.

identity of (2.12). Similarly, taking the weighted product of v with $v = \mathbf{M}^{1/2}\alpha^{-1}(\theta + \mathbf{A}^{-1}\dot{\theta})$ leads to the second identity.

For (2.13), we first differentiate $\mathbf{A}^{-1}\mathbf{M}^{1/2}\dot{v} = -u - \alpha\theta$, apply $\mathbf{A}^{-1}$ and use (2.10) to get the identity $\theta = (\alpha + \alpha^{-1})\mathbf{M}^{1/2}v + \alpha^{-1}\mathbf{M}^{1/2}\mathbf{A}^{-2}\ddot{v}$ so, taking the weighted product of θ with this,

$$\|\theta\|_h^2 = (\alpha + \alpha^{-1})\langle \mathbf{M}^{1/2}v, \theta\rangle_h + \alpha^{-1}\langle \mathbf{M}^{1/2}\mathbf{A}^{-2}\ddot{v}, \theta\rangle_h. \tag{2.16}$$

The contribution of the second term on the right of (2.16) gives

$$\alpha^{-1}\langle \mathbf{M}^{1/2}\mathbf{A}^{-2}\ddot{v}, \theta\rangle_h = -\alpha^{-1}\langle \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v}, -\mathbf{A}^{-1}\dot{\theta}\rangle_h - \alpha^{-1}\int_0^T \dot{h}\langle \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{v}, \mathbf{A}^{-1}\theta\rangle \,\mathrm{d}t.$$

Substitute $[-\theta + \alpha\mathbf{M}^{-1/2}v]$ for $\mathbf{A}^{-1}\dot{\theta}$ and use the identity

$$\langle v, \mathbf{A}^{-1}\dot{v}\rangle_h = -\frac{1}{2}\int_0^T \dot{h}\,\langle v, \mathbf{A}^{-1}v\rangle \,\mathrm{d}t,$$

obtained by integration by parts, to get

$$\begin{aligned}
\alpha^{-1}\langle \mathbf{M}^{1/2}\mathbf{A}^{-2}\ddot{v}, \theta\rangle_h = {} & -\frac{1}{2}\int_0^T \dot{h}\langle \mathbf{A}^{-1}v, v\rangle \,\mathrm{d}t - \alpha^{-1}\langle v, \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{\theta}\rangle_h \\
& - \alpha^{-1}\int_0^T \dot{h}\langle v, \mathbf{M}^{1/2}\mathbf{A}^{-1}\theta - \mathbf{M}^{1/2}\mathbf{A}^{-2}\dot{\theta}\rangle \,\mathrm{d}t \\
& + \alpha^{-1}\int_0^T \ddot{h}\langle v, \mathbf{M}^{1/2}\mathbf{A}^{-2}\theta\rangle \,\mathrm{d}t \\
& \quad \text{again substitute } [-\theta + \alpha\mathbf{M}^{-1/2}v] \text{ for } \mathbf{A}^{-1}\dot{\theta} \\
= {} & -\frac{1}{2}\int_0^T \dot{h}\langle \mathbf{A}^{-1}v, v\rangle \,\mathrm{d}t + \alpha^{-1}\langle \mathbf{M}^{1/2}v, \theta\rangle_h - \|v\|_h^2 \\
& - \alpha^{-1}\int_0^T \dot{h}\langle v, 2\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta - \alpha\mathbf{A}^{-1}v\rangle \,\mathrm{d}t \\
& + \alpha^{-1}\int_0^T \ddot{h}\langle v, \mathbf{M}^{1/2}\mathbf{A}^{-2}\theta\rangle \,\mathrm{d}t.
\end{aligned}$$

Collecting terms and going back to (2.16), we now obtain (2.13).

Finally, for the first identity of (2.14), first substitute $\alpha^{-1}(\theta + \mathbf{A}^{-1}\dot{\theta})$ for $\mathbf{M}^{-1/2}v$ on the left, apply (2.8) to the second part of this, and then use $\mathbf{A}^{-1}\dot{u} = \mathbf{M}^{-1/2}v$. For the second, substitute $\mathbf{A}^{-1}\dot{u}$ for $\mathbf{M}^{-1/2}v$ on the left, again apply (2.8), and then use $\mathbf{A}^{-1}\dot{\theta} = -\theta + \alpha\mathbf{M}^{-1/2}v$. Last, by taking the weighted product of θ with $v = \mathbf{M}^{1/2}\mathbf{A}^{-1}\dot{u}$, we obtain the third of these identities. $\square$

2.2. *Proof of Theorem 2.1*

With $\gamma \geqslant 0$, the energy identity (2.3) gives $\|U(T)\| \leqslant \|U(t)\|$ for $0 \leqslant t \leqslant T$, where we recall

$$\|U(t)\|^2 \equiv \|u(t)\|^2 + \|\mathbf{M}^{1/2}w_t(t)\|^2 + \|\theta(t)\|^2.$$

Hence, taking h as above,

$$T\|U(T)\|^2 = \int_0^T h(t)\|U(T)\|^2\,\mathrm{d}t \leqslant \int_0^T h(t)\,\|U(t)\|^2\,\mathrm{d}t \equiv \|U(\cdot)\|_h^2$$

so:

$$\text{If}\quad \|U(\cdot)\|_h^2 \leqslant \psi(T,\alpha)\,\|\varphi\|_{\mathcal{Q}}^2, \quad \text{then}\quad \mathfrak{C}(T;\alpha) \leqslant \sqrt{\psi/T}. \tag{2.17}$$

In each of our cases we will be able to use Lemma 2.3 to obtain a form for $\|U\|_h^2$ which can be suitably estimated in terms of $\|\varphi\|_{\mathcal{Q}}^2$ so (2.17) will give (2.4).

Thermal observation: $\varphi = \mathbf{M}^{1/2}\theta$

We begin by using (2.12) and then substituting the first identity of (2.14) to get the combined identity:

$$\begin{aligned}
\|U\|_h^2 &= \|\theta\|_h^2 + \|u\|_h^2 + \|v\|_h^2 \\
&= 3\|\theta\|_h^2 + \left[2\alpha^{-1} - \alpha\right]\langle\theta,u\rangle_h + 2\alpha^{-1}\langle\mathbf{M}^{1/2}\theta, v\rangle_h \\
&\quad - \alpha^{-1}\biggl[-\int_0^T \dot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta,u\rangle\,\mathrm{d}t + 3\int_0^T \dot{h}\,\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta,v\rangle\,\mathrm{d}t \\
&\quad + \int_0^T \ddot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta,\mathbf{A}^{-1}u\rangle\,\mathrm{d}t\biggr]
\end{aligned} \tag{2.18}$$

in which we have ensured that all terms on the right involve θ – note that we have freely used the symmetry of the product and the self-adjointness of $\mathbf{A}^{-1}, \mathbf{M}$ along with commutativity of $\mathbf{A}$ and $\mathbf{M}$ in obtaining (2.18). We can now estimate each of these six terms in turn, using the Cauchy inequality, (2.7), and the fact that $\mathbf{M}\mathbf{A}^{-1}$ is bounded for each γ, etc. Noting (2.9) – and then absorbing in $c = c_\varepsilon$, below, such assorted constants as $\|\mathbf{A}^{-1}\| = 1/\zeta_0$ and $\|\mathbf{M}\mathbf{A}^{-1}\|$ – we obtain, term by term,

$$\begin{aligned}
\|U\|_h^2 &\leqslant c\left[1 + \left(\alpha^{-2}+\alpha^2\right) + \alpha^{-2} + \alpha^{-2}T^{-2} + \alpha^{-2}T^{-4}\right]\|\theta\|_{\mathcal{Q}}^2 \\
&\quad + c\alpha^{-2}\|\mathbf{M}^{1/2}\theta\|_h^2 + \varepsilon\left[\|u\|_h^2 + \|v\|_h^2\right].
\end{aligned} \tag{2.19}$$

Observe that $\|\theta\| \leqslant \|\mathbf{M}^{1/2}\theta\|$.

We can then choose $\varepsilon = 1/2$ here to absorb those terms into the left and obtain (2.17) for $\varphi = \mathbf{M}^{1/2}\theta$ – with (re-adjusting c slightly)

$$\psi = \psi_\theta(T,\alpha) \leqslant c\left[\left(\alpha^2 + 1 + \alpha^{-2}\right) + \alpha^{-2}T^{-2} + \alpha^{-2}T^{-4}\right].$$

Note that this estimate is uniform in γ for $\gamma \geqslant 0$ bounded.

With a further re-adjustment of the constant, we may omit dominated terms before taking the square root of ψ/T term by term. Of the five terms here, ‘1’ is uniformly dominated by $[\alpha^2 + \alpha^{-2}]$ so that term can be dropped; the terms α^{-2} and $\alpha^{-2}T^{-2}$ are each dominated by $\alpha^{-2}T^{-4}$ for $0 < T \leqslant T_0$, so those can also be dropped. Thus (2.17) gives the estimate asserted for $\mathfrak{C}_\theta$ in (2.4).

Remark 2.4. In the case of thermal control and positive γ, we note that the estimate just obtained for global observation of θ yields *exact controllability* rather than only nullcontrollability (valid with $\gamma = 0$). Indeed, from the energy identity (2.3) we have

$$\|U(0)\|^2 = \|U(T)\|^2 + 2\int_0^T \|\nabla\theta\|^2 \, dt \leqslant T^{-1}\left[\psi_\theta(T,\alpha) + 2\right] \|\varphi\|_Q.$$

This is consistent with the basic parabolic theory where internal controls in $L_2([0,T] \to H^{-1}(\Omega))$ yield exact controllability for the heat equation: the dual formulation of this corresponds to observing $\|\mathbf{A}^{1/2}\theta\|$.

Displacement observation: $\varphi = \mathbf{M}u = \mathbf{MA}w$

Here we want an identity for $\|U\|_h$ in which each term on the right now involves u:

$$\begin{aligned}\|U\|_h^2 = {} & 3\|u\|_h^2 + \left[2\alpha - \alpha^{-1}\right]\langle\theta, u\rangle_h + \langle\mathbf{M}^{1/2}v, u\rangle_h \\ & + 2\alpha^{-1}\int_0^T \dot{h}\,\langle\mathbf{MA}^{-1}\theta, u\rangle \, dt - 3\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle \, dt \\ & - \alpha^{-1}\int_0^T \ddot{h}\langle\mathbf{MA}^{-1}\theta, \mathbf{A}^{-1}u\rangle \, dt. \end{aligned} \tag{2.20}$$

We get this by starting with the identities (2.12)

$$\begin{aligned}\|\theta\|_h^2 + \|v\|_h^2 &= 2\|v\|_h^2 - \alpha^{-1}\left[\langle\theta, u\rangle_h + \langle\mathbf{M}^{1/2}\theta, v\rangle_h - \int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle \, dt\right] \\ &= 2\|u\|_h^2 + 2\alpha\langle\theta, u\rangle_h - 2\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle \, dt \\ &\quad - \alpha^{-1}\left[\langle\theta, u\rangle_h + \langle\mathbf{M}^{1/2}\theta, v\rangle_h - \int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle \, dt\right] \\ &= 2\|u\|_h^2 + (2\alpha - \alpha^{-1})\langle\theta, u\rangle_h - \alpha^{-1}\langle\mathbf{M}^{1/2}\theta, v\rangle_h \\ &\quad - 2\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle \, dt + \alpha^{-1}\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle \, dt \end{aligned}$$

from which we obtain

$$\begin{aligned}\|U\|_h^2 = {} & 3\|u\|_h^2 + (2\alpha - \alpha^{-1})\langle\theta, u\rangle_h + \alpha^{-1}\langle\mathbf{M}^{1/2}\theta, v\rangle_h \\ & - 2\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle \, dt + \alpha^{-1}\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}\theta, v\rangle \, dt. \end{aligned}$$

The terms $\langle\mathbf{M}^{1/2}\theta, v\rangle_h$ and $\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle \, dt$ are replaced by the quantities on the right-hand sides of the last two identities in (2.14) to give

$$
\begin{aligned}
\|U\|_h^2 = 3\|u\|_h^2 &+ (2\alpha - \alpha^{-1})\langle\theta, u\rangle_h \\
&- \alpha^{-1}\left[\langle\theta, \mathbf{M}u\rangle_h - \alpha\langle u, \mathbf{M}^{1/2}v\rangle_h - \int_0^T \dot{h}\,\langle\mathbf{M}\mathbf{A}^{-1}\theta, u\rangle\,\mathrm{d}t\right] \\
&- 2\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}u, v\rangle\,\mathrm{d}t \\
&+ \alpha^{-1}\left[\int_0^T \dot{h}\langle\mathbf{M}\mathbf{A}^{-1}\theta, u\rangle\,\mathrm{d}t - \alpha\int_0^T \dot{h}\langle\mathbf{M}^{1/2}\mathbf{A}^{-1}v, u\rangle\,\mathrm{d}t - \int_0^T \ddot{h}\langle\mathbf{M}\mathbf{A}^{-1}\theta, \mathbf{A}^{-1}u\rangle\,\mathrm{d}t\right]
\end{aligned}
$$

and we then get (2.20) on collecting terms.

As before, we now estimate each of the 6 terms of (2.20) in turn to get

$$
\begin{aligned}
\|U\|_h^2 \leqslant c\big[1 + (\alpha - \alpha^{-1})^2 + (\alpha^{-2} + 1)T^{-2} + \alpha^{-2}T^{-4}\big]\|u\|_{\mathcal{Q}}^2 \\
+ c\alpha^{-2}\|\mathbf{M}u\|^2 + c\|\mathbf{M}^{1/2}u\|^2 + \varepsilon\big[\|\theta\|_h^2 + \|v\|_h^2\big]
\end{aligned}
\tag{2.21}
$$

which, after dropping dominated terms, gives (2.17) for $\varphi = \mathbf{M}u = \mathbf{M}\mathbf{A}w$ with

$$
\psi(T, \alpha) \leqslant c\big[\alpha^2 + \alpha^{-2}T^{-4}\big]
$$

uniformly in $0 \leqslant \gamma \leqslant \bar{\gamma}$. Exactly as for $\mathfrak{C}_\theta$, we now get the estimate asserted for $\mathfrak{C}_u$ in (2.4).

[We remark that, in the case of displacement observation, we have required a much stronger observation operator, necessary in order to reconstruct the thermal variable θ. If, however, one were just interested in partial observability (without a need to reconstruct θ), the same techniques give the desired estimate with observation only of $\|u\|$.]

Velocity observation: $\varphi = \mathbf{M}w_t$

For this case it is convenient to proceed in two steps. We already have (2.13) and now estimate that term by term on the right to obtain

$$
\|\theta\|_h^2 + \|v\|_h^2 \leqslant c\big[(\alpha^2 + \alpha^{-2}) + T^{-2} + \alpha^{-2}T^{-2} + \alpha^{-2}T^{-4}\big]\,\|v\|_h^2 + c\alpha^{-2}\|\mathbf{M}^{1/2}v\|_h^2 + \varepsilon\|\theta\|_h^2
$$

so after setting $\varepsilon = 1/2$ and dropping dominated terms we get

$$
\|\theta\|_h^2 \leqslant c\big[\alpha^2 + T^{-2} + \alpha^{-2}T^{-4}\big]\|v\|_h^2 + c\alpha^{-2}\|\mathbf{M}^{1/2}v\|_h^2. \tag{2.22}
$$

We can then use the first identity of (2.12) to get

$$
\begin{aligned}
\|u\|_h^2 &\leqslant c\big(\big[1 + T^{-2}\big]\|v\|_h^2 + \alpha^2\|\theta\|_h^2\big) + \varepsilon\|u\|_h^2 + c\alpha^{-2}\|\mathbf{M}^{1/2}v\|_h^2 \\
&\leqslant c\big[1 + T^{-2} + \alpha^2(\alpha^2 + T^{-2} + \alpha^{-2}T^{-4})\big]\|\mathbf{M}^{1/2}v\|_h^2 + c\alpha^{-2}\|\mathbf{M}w_t\|_h^2 + \varepsilon\|\theta\|_h^2
\end{aligned}
\tag{2.23}
$$

using (2.22). Adding (2.22) and (2.23), then dropping dominated terms, we get (2.17) for $\varphi = \mathbf{M}w_t$ with

$$
\psi(T, \alpha) \leqslant c\big[\alpha^4 + (1 + \alpha^{-2})T^{-4}\big]
$$

uniformly in $0 \leqslant \gamma \leqslant \bar{\gamma}$. As for the previous cases, we now compute $\sqrt{\psi/T}$ to get the estimate for $\mathfrak{C}_v$ asserted in (2.4).

[Again we remark that when velocity is observed and $\gamma > 0$, the observation requires two additional derivatives (or one additional as compared with the finite energy space). However, if one seeks partial observation only, then this approach gives the same estimate when only observing $\|\mathbf{M}^{1/2}w_t\|$.] □

We may remark here that in the Euler–Bernoulli case ($\gamma = 0$) all three of the observed quantities $\theta, \mathbf{A}w, w_t$ are measured in the $L_2(\Omega)$ topology while in the Kirkhoff case the situation is not symmetric: the observed quantities are then $\theta \in H^1(\Omega)$, but $\mathbf{A}w, w_t \in H^2(\Omega)$. This difference of topology between the mechanical and thermal observed variables disappears if one is willing to compromise the full reconstruction of the state and accept partial observability, i.e., observability of only the mechanical part. In that case observation of each of the three variables can be considered in the $H^1(\Omega)$-norm. It is the reconstruction of the parabolic component from the hyperbolic part of the system which costs an additional derivative.

3. Global observation: spectral expansion

In this section we continue to consider (1.4) – the same system (1.2), (2.1) as in Section 2 – and will obtain results quite similar to those of Section 2 by the use of a rather different approach: spectral expansion to reduce each problem to the consideration of a family of finite-dimensional problems to which we can apply the general finite-dimensional results taken as Theorem 3.3 from [21] in Subsection 3.1 below. We are assuming here that $\mathbf{A}$ is self-adjoint and positive definite with $\mathcal{D}(\mathbf{A}) \sim H^2(\Omega)$, but it is otherwise irrelevant to have $\mathbf{A}$ as in (1.3). In this section we again seek a unified theory, necessarily accommodating the possibility that $\gamma > 0$, and defer some special consideration of the nonrotational case $\gamma = 0$ to later. A major point here is that being able to cite general results both simplifies the arguments considerably and clarifies the structure of the problem.

The principal result we obtain in this section is:

Theorem 3.1. *We consider the linear thermoelastic system* (1.4) *– in particular,* (1.2), (2.1). *For every $\alpha > 0$ and every $T > 0$ it is possible to observe any single state component on $\mathcal{Q}_T$ and determine the full state (w, w_t, θ) on Ω at the final time T, equivalently $U(T) = (\theta, u, v)$. The linear operator: $\varphi \mapsto U(T)$ is continuous from $L^2(\mathcal{Q}_T)$ to $\mathcal{H} = (L^2(\Omega))^3$ with norm $\mathfrak{C} = \mathfrak{C}(T;\alpha)$ as in* (3.3) *for each of the cases*

- Thermal observation ($\varphi = \mathbf{M}^{1/2}\theta$ *so* $\mathbf{b}_0 = [1\ 0\ 0]$ *in* (3.13)) *or*
- Displacement observation ($\varphi = \mathbf{M}u = \mathbf{MA}w$*, corresponding to observation with w topologized in $L^2([0,T] \to H^4(\Omega))$ if $\gamma > 0$*) *or*
- Velocity observation ($\varphi = \mathbf{M}^{1/2}v = \mathbf{M}w_t$*),*

and this operator norm has the asymptotics of (1.5) *– more precisely,*

$$\begin{aligned}
&\|U(T)\|_{\mathcal{H}} = \|U(T)\| \leqslant K\alpha^{-1}T^{-5/2}\|\mathbf{M}^{1/2}\theta\|_{L^2(\mathcal{Q})},\\
&\|U(T)\|_{\mathcal{H}} = \|U(T)\| \leqslant K\alpha^{-1}T^{-5/2}\|\mathbf{MA}w\|_{L^2(\mathcal{Q})},\\
&\|U(T)\|_{\mathcal{H}} = \|U(T)\| \leqslant K\frac{\sqrt{1+\alpha^2}}{\alpha}T^{-5/2}\|\mathbf{M}w_t\|_{L^2(\mathcal{Q})}
\end{aligned} \tag{3.1}$$

for $0 < T < \tau_*$. Here K is independent of T, α, γ – and, of course, is independent of the particular $U(\cdot) \equiv (\theta(\cdot), \mathbf{A}w(\cdot), \mathbf{M}^{1/2}w_t(\cdot))$ satisfying (1.4) *– but $\tau_* > 0$ may depend on α if $\alpha \to 0, \infty$, although it is independent of γ.*

Remark 3.2. The estimated asymptotics (3.1) are, of course, entirely consistent with those of (2.4), but must be interpreted somewhat differently: we emphasize that, because of its reliance on [21], our result here can only be the existence of an iterated limit – e.g., existence of

$$\lim_{\alpha\to\infty} \alpha\Big[\lim_{T\to 0} T^{5/2}\mathfrak{C}(T;\alpha)\Big] \tag{3.2}$$

for thermal observation – and, in view of the α-dependence of τ_* for (3.1), asserts nothing about situations in which simultaneously $T \to 0$ and $\alpha \to \infty$ in some joint fashion. This seems the best one might expect based on the results given in [21]. While more detailed extension of those results might permit us to obtain by this method the consideration of Theorem 2.1 for limits joint in α, T, we confine ourselves to following the available general results, hence looking at the asymptotics as $T \to 0$ and then obtaining the asymptotics

$$\mathfrak{C}(T;\alpha) = \mathcal{O}(\alpha^{-1})\mathcal{O}(T^{-5/2}) \quad \text{as } T \to 0 \tag{3.3}$$

as $\alpha \to 0$, etc., interpreting (3.1) in the sense of (3.2).

[An interesting variant here would be to consider the simultaneous observation of *two* of these variables, e.g., determining the velocity $v = w_t$ from observation of *both* temperature and displacement so $\varphi = (\theta, u) = (\theta, \mathbf{A}w)$. This vectorial observation could be treated using more general results in [21], but not through the version presented here as Theorem 3.3.]

3.1. *A general finite-dimensional result*

In this subsection we note a general result for abstract finite-dimensional systems which will then be applied to the thermoelastic system. This result is adapted from the principal result of [21], specialized to consider scalar observation as the form needed for our present purposes. For details and more general results of this nature, see [21,24].

Theorem 3.3. *Let $\mathbf{b}_0$ be an arbitrary (row) vector in $\mathbb{R}^N$ and let L be an $N \times N$ matrix such that $\{\mathbf{b}_0, \mathbf{b}_0L, \ldots, \mathbf{b}_0L^{N-1}\}$ is linearly independent. Then, for every solution of*

$$\dot{\mathbf{x}} = L\mathbf{x} \text{ with observation of } \varphi = \mathbf{b}_0 \cdot \mathbf{x}(\cdot) \tag{3.4}$$

we have

$$\big|\mathbf{x}(T)\big| \leqslant \mathfrak{C}\|\varphi\|_{L^2(0,T)} = \mathfrak{C}\left[\int_0^T |\mathbf{b}_0 \cdot \mathbf{x}|^2 \,\mathrm{d}t\right]^{1/2} \tag{3.5}$$

with the asymptotics for $\mathfrak{C} = \mathfrak{C}(T) = \mathfrak{C}(T; L, \mathbf{b}_0)$ as $T \to 0$ given by

$$\lim_{T\to 0} T^{N-1/2}\mathfrak{C}(T; L, \mathbf{b}_0) = \psi = \frac{c_N}{\beta_*} \quad \text{with } \beta_* = |\mathbf{b}_*|, \tag{3.6}$$

where

$$\mathbf{b}_* = \text{projection of } \mathbf{b}_0L^{N-1} \text{ onto } \big[\operatorname{span}\{\mathbf{b}_0, \ldots, \mathbf{b}_0L^{N-2}\}\big]^{\perp}. \tag{3.7}$$

Clearly, given $r > 1$ *there is some* $\tau = \tau(r) > 0$ *such that*

$$\mathfrak{C}(T; L, \mathbf{b}_0) \leqslant r\psi T^{-N+1/2} \quad \text{for } 0 < T \leqslant \tau \tag{3.8}$$

with $\psi = c_N/\beta_*$ *as in* (3.6). *We note here that one can take* τ *uniform as* L *varies over any compact set* $\mathcal{M}$ *of admissible matrices, i.e., such that* $\{\mathbf{b}_0, \ldots, \mathbf{b}_0 L^{N-1}\}$ *remains linearly independent.*

[We remark that the constant c_N in (3.6) depends only on the dimension N; thus, having $\beta_* = |\mathbf{b}_*|$ as in (3.7) provides the *only* dependence of ψ in (3.6), (3.8) on the matrix L or on the observation vector $\mathbf{b}_0$. Note that for the thermoelastic example (1.4) we have $N = 3$ so for each of the finite-dimensional problems of (3.15) the blowup rate is $T^{-N+1/2} = T^{-5/2}$.]

Proof. The discussions in [21] and [24] actually obtain the first term in an asymptotic expansion for the minimum norm of a control from 0 to a target state ξ and this is here adapted to the nullcontrol problem ($\xi_0 \mapsto 0$) dual to the present observation problem by setting $\xi = -\mathrm{e}^{TL}\xi_0$. Analysis shows that one is essentially estimating $\|Q_T^{-1}\|^{1/2}$ where $Q_T = \int_0^T [\mathbf{y}(t) \otimes \mathbf{y}^*(t)]\,\mathrm{d}t$ is the nullcontrollability operator for $\dot{\mathbf{y}} = -L^*\mathbf{y}$, $\mathbf{y}(0) = \mathbf{b}_0{}^*$. With this ‘translation’, (3.5) and (3.6) follow directly from [21]. The uniformity of τ with respect to $L \in \mathcal{M}$ for the inequality (3.8) is immediate from (3.6) once one looks carefully at the argument in [21] to observe that all steps in estimating $T^{N-1/2}|\mathbf{x}(T)|$ are continuous in T, L, subject to the independence assumption on $\{\mathbf{b}_0, \ldots, \mathbf{b}_0 L^{N-1}\}$. $\square$

We note here some related facts, but will only prove what is actually needed as it is needed.

Lemma 3.4. *Assume the hypotheses of Theorem* 3.3. *Then one has*

(1) $\mathfrak{C}(T; L, \lambda\mathbf{b}_0) = \mathfrak{C}(T; L, \mathbf{b}_0)/\lambda$,

(2) $\mathfrak{C}(T; \lambda L, \mathbf{b}_0) = \sqrt{\lambda}\,\mathfrak{C}(\lambda T; L, \mathbf{b}_0)$,

(3) $\mathfrak{C}(T; L, \mathbf{b}_0) \leqslant \|A^{-1}\|\,\mathfrak{C}(T; ALA^{-1}, \mathbf{b}_0 A^{-1})$,

(4) *if* $\|\mathrm{e}^{tL}\| \leqslant b$ *for all* $t > 0$, *then* $\mathfrak{C}(nT; L, \mathbf{b}_0) \leqslant b\,\mathfrak{C}(T; L, \mathbf{b}_0)/\sqrt{n}$;

(5) *if* $\|\mathrm{e}^{tL}\| \leqslant b\,\mathrm{e}^{-\omega t}$ *with* $\omega > 0$, *for all* $t > 0$, *then* $\mathfrak{C}(2nT; L, \mathbf{b}_0) \leqslant b\,\mathrm{e}^{-\omega nT}\mathfrak{C}(T; L, \mathbf{b}_0)/\sqrt{n}$.

3.2. The family of finite-dimensional problems

Our problem is to estimate $U(T)$ in terms of observation of a single component – suitably topologized by the introduction of an observation operator $\mathbf{S} = \sigma(\mathbf{A})$ so $\varphi = \mathbf{b}_0 \cdot \mathbf{S}U(\cdot)$ – for $0 < t < T$. To reduce this to a family of finite-dimensional problems, we begin by considering the set $\{(\zeta_m, g_m)\colon\ m = 0, 1, \ldots\}$ of eigenpairs of $\mathbf{A}$, taking the eigenvalues as ordered so $0 < \zeta_0 \leqslant \zeta_1 \leqslant \cdots \to \infty$ and taking the eigenfunctions $\{g_m\}$ to be orthonormal in $L^2(\Omega)$. We then expand the solution to get

$$U(t; x) = \sum_m U_m(t) g_m(x) \tag{3.9}$$

with ($\mathbb{R}^3$-valued) coefficients: $U_m = \langle U, g_m\rangle = \int_\Omega U(\cdot, x) g_m(x)\,\mathrm{d}x$; since $\mathbf{A}g_m = \zeta_m g_m$, it follows from (1.4) that each U_m satisfies

$$\mathrm{d}U_m/\mathrm{d}t = \zeta_m L\, U_m, \quad \text{where } Lg_m = \mathbf{L}g_m. \tag{3.10}$$

An important observation is that (1.10) and our choice of state vector $U = (\theta,\ \mathbf{A}w,\ \mathbf{M}^{1/2}w_t)$ give

$$\|U(t)\| \leqslant \|U(s)\|, \qquad |U_m(t)| \leqslant |U_m(s)| \quad \text{for } s \leqslant t. \tag{3.11}$$

Noting that $\mu = \mu_m = 1 + \gamma\zeta_m$ is the eigenvalue of $\mathbf{M}$ corresponding to g_m, (3.10) gives (compare [5])

$$L = L_\mu = \begin{pmatrix} -1 & 0 & \alpha/\sqrt{\mu} \\ 0 & 0 & 1/\sqrt{\mu} \\ -\alpha/\sqrt{\mu} & -1/\sqrt{\mu} & 0 \end{pmatrix}. \tag{3.12}$$

In each case, the observation φ will be restricted to one component (thermal, mechanical displacement, or velocity):

$$\varphi = \mathbf{b}_0 \cdot \mathbf{S}U(t) \quad \begin{array}{l} \text{with } \mathbf{b}_0 = [1\ 0\ 0] \text{ or } [0\ 1\ 0] \text{ or } [0\ 0\ 1] \\ \text{to select } \theta \text{ or } u = \mathbf{A}w \text{ or } v = \mathbf{M}^{1/2}w_t. \end{array} \tag{3.13}$$

Expanding the observation similarly gives

$$\varphi(t,x) = \mathbf{b}_0 \cdot \mathbf{S}\, U(t,x) = \sum_m \sigma_m \varphi_m(t)\, g_m(x) \quad \text{for } x \in \Omega \tag{3.14}$$

with scalar coefficients: $\varphi_m(t) = \mathbf{b}_0 \cdot U_m(t)$; here $\sigma_m = \sigma(\zeta_m)$ is the eigenvalue of the observation operator $\mathbf{S}$ for the eigenvector g_m. The orthonormality of $\{g_m\}$ then gives

$$\|U(T)\|_{\mathcal{H}}^2 = \|U(T)\|^2 = \sum_m \|U_m(T)\|^2, \qquad \|\varphi(\cdot)\|^2 = \sum_m \sigma_m^2 \|\varphi_m\|_{L^2(0,T)}^2, \tag{3.15}$$

where we are using, e.g., the $\mathcal{H}$-norm for $U(t)$ but the Euclidean norm on $\mathbb{R}^3$ for each $U_m(t)$. Thus, we have reduced the problem to a decoupled family of 3-dimensional problems, each as considered in Theorem 3.3: our task now is to obtain observation inequalities for each U_m in terms of the corresponding φ_m, proving for each of these problems that

$$|U_m(T)| \leqslant \mathfrak{C}_* \, \sigma_m \|\varphi_m\|_{L^2(0,T)} \tag{3.16}$$

with $\mathfrak{C}_*$ independent of m to get $\|U(T)\| \leqslant \mathfrak{C}_* \|\varphi\|$, as desired.

3.3. Proof of Theorem 3.1

Proof. We wish to apply (3.8) of Theorem 3.3 to get the observation estimate (3.16) for each m, i.e., for solutions U_m of (3.10), so $\mathfrak{C}_* = T^{-N+1/2}\psi$ ($\psi = \psi(\alpha)$ independent of m). This uniform estimate then gives $U(T) \leqslant T^{-N+1/2}\psi(\alpha)\|\varphi\|$ from (3.15) as desired. The difficulty is the uniformity: unfortunately, (except for the special case $\gamma = 0$) we have $1 + \gamma\zeta_m \to \infty$ so the closure $\mathcal{M} = \overline{\{L_\mu\colon \mu = \mu_m\}}$ is compact, but fails the independence requirement as $\mu_m \to \infty$ to give $1/\sqrt{\mu} = 0$.

To obtain the desired uniformity we re-scale the components, e.g., keeping θ but making replacements $u \hookleftarrow \kappa u$ and $v \hookleftarrow \nu v$. Thus we set

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad \text{and} \quad \hat{L} = \begin{pmatrix} -1 & 0 & \alpha/\nu\sqrt{\mu} \\ 0 & 0 & \kappa/\nu\sqrt{\mu} \\ -\alpha\nu/\sqrt{\mu} & -\nu/\kappa\sqrt{\mu} & 0 \end{pmatrix}, \tag{3.17}$$

i.e., $\hat{L} = DLD^{-1}$. We then let $\mathbf{x}(s) = \mathbf{x}_m(s) = DU_m(t)$ with $s = \zeta t$ so

$$\mathrm{d}\mathbf{x}/\mathrm{d}s = \hat{L}\mathbf{x}, \qquad \mathbf{b}_0 \cdot U_m(t) = \left[\mathbf{b}_0 D^{-1}\right] \cdot \mathbf{x}(s) = \rho\, \mathbf{b}_0 \cdot \mathbf{x}(s),$$

noting that $\mathbf{b}_0 D^{-1} = \rho \mathbf{b}_0$ where ρ is 1 or $1/\kappa$ or $1/\nu$, depending on our selection of $\mathbf{b}_0$. [Note that $\zeta, \mu, \hat{L}, \kappa, \nu, D, \rho, \sigma$ all have (suppressed) subscripts$_m$ throughout this proof as we proceed with the decoupled sequence of finite-dimensional problems.]

Now, for any $n = 1, 2, \ldots$ and $k = 1, \ldots, n$, we recall (3.11) and have

$$\begin{aligned} \left|U_m(T)\right| &\leqslant \left|U_m\left(\frac{k}{n}T\right)\right| = \left|D^{-1}\mathbf{x}\left(k\frac{\zeta T}{n}\right)\right| \leqslant \|D^{-1}\|\left|\mathbf{x}\left(k\frac{\zeta T}{n}\right)\right| \\ &\leqslant \|D^{-1}\|\mathfrak{C}\left(\frac{\zeta T}{n}; \hat{L}, \mathbf{b}_0\right)\left[\int_{(k-1)\zeta T/n}^{k\zeta T/n} \left[\mathbf{b}_0 \cdot \mathbf{x}(s)\right]^2 \mathrm{d}s\right]^{1/2} \end{aligned} \tag{3.18}$$

so, summing over $k = 1, \ldots, n$ and changing variables,

$$\begin{aligned} n\left|U_m(T)\right|^2 &\leqslant \|D^{-1}\|^2 \mathfrak{C}^2\left(\frac{\zeta T}{n}; \hat{L}, \mathbf{b}_0\right) \int_0^{\zeta T} \left[\mathbf{b}_0 \cdot \mathbf{x}(s)\right]^2 \mathrm{d}s \\ &= \|D^{-1}\|^2 \mathfrak{C}^2\left(\frac{\zeta T}{n}; \hat{L}, \mathbf{b}_0\right) \frac{\zeta}{\rho^2} \int_0^T \left[\mathbf{b}_0 \cdot U_m(t)\right]^2 \mathrm{d}t. \end{aligned} \tag{3.19}$$

We always have $\zeta = \zeta_m \geqslant \zeta_0$ so, once we will have a suitable $\mathcal{M}$ to apply (3.8), we can then choose n so $n \leqslant \zeta/\zeta_0 \leqslant 2n$ and will have $\zeta T/n \leqslant \tau$ whenever $T < \tau_*$ on setting $\tau_* = \tau/2\zeta_0$ (independent of m). Using (3.8) for (3.19), we then have

$$\begin{aligned} \left|U_m(T)\right| &\leqslant \frac{\|D^{-1}\|}{\rho}\sqrt{\frac{\zeta}{n}}\mathfrak{C}\left(\frac{\zeta T}{n}; \hat{L}, \mathbf{b}_0\right)\|\varphi_m\|_{L^2(0,T)} \\ &\leqslant \frac{\|D^{-1}\|}{\rho}\sqrt{\frac{\zeta}{n}} r \frac{c_3}{\beta_*}\left(\frac{\zeta T}{n}\right)^{-5/2}\|\varphi_m\| \end{aligned}$$

$$\leqslant \frac{rc_3}{\zeta_0^2}\left(\frac{\|D^{-1}\|}{\sigma\rho\beta_*}\right)T^{-5/2}\sigma\|\varphi_m\| \quad \text{for } 0 < T \leqslant \tau_*, \tag{3.20}$$

computing β_* for $\hat{L}, \mathbf{b}_0$ and using the choice of n to have $\zeta/n \geqslant \zeta_0$ for the final step. In comparison with (3.16) we will eventually finish this computation by selecting the 'observation operator' $\mathbf{S}$ to give $\sigma = \sigma_m$ such that the expression $\|D^{-1}\|/\sigma\rho\beta_*$ in (3.20) will be bounded uniformly in m.

To complete the proof we now wish to find – case by case, depending on $\mu = \mu_m$ and on our selection of $\mathbf{b}_0$ – an appropriate choice of $D = D_m$ which will make $\mathcal{M} = \overline{\{\hat{L} = DLD^{-1}\colon \mu \in [1,\infty)\}}$ admissible for the application of (3.8) in Theorem 3.3 to justify uniformity in our derivation of (3.20). From (3.17) we immediately note that, in each case, we must have

$$1/\nu, \quad \nu, \quad \kappa/\nu, \quad \nu/\kappa \quad = \mathcal{O}(\sqrt{\mu}) \tag{3.21}$$

for $\mathcal{M}$ to be bounded. This assumes that α is bounded; similarly, we will necessarily assume that α is bounded away from 0 in showing linear independence.

For the case of *thermal observation*, so $\mathbf{b}_0 = [1\ 0\ 0]$ and

$$\mathbf{b}_0\hat{L} = \begin{bmatrix} -1 & 0 & \frac{\alpha}{\nu\sqrt{\mu}} \end{bmatrix}, \qquad \mathbf{b}_0\hat{L}^2 = \begin{bmatrix} \left(1 - \frac{\alpha^2}{\mu}\right) & \frac{\alpha}{\kappa\mu} & \frac{-\alpha}{\nu\sqrt{\mu}} \end{bmatrix},$$

we see that for the independence requirement, in addition to (3.21), we need $\kappa = \mathcal{O}(1/\mu)$. With a little manipulation, it is easily seen that the only usable scaling in this case is to take $\kappa \sim 1/\mu$ and $\nu \sim 1/\sqrt{\mu}$; to within constant factors, this makes

$$\|D^{-1}\| = \sqrt{\mu}, \qquad \mathbf{b}_* = [0 \quad a \quad 0] \quad \text{so} \quad \beta_* = \alpha, \qquad \rho = 1. \tag{3.22}$$

Thus, to get $\sigma \geqslant \sqrt{\mu}$ we choose $\mathbf{S} = \mathbf{M}^{1/2}$ so, from (3.20) and (3.15), we get

$$\|U(T)\| \leqslant K\alpha^{-1}T^{-5/2}\|\mathbf{M}^{1/2}\theta\|_{L^2(\mathcal{Q})} \quad \text{for } 0 < T \leqslant \tau_* = \tau/2\zeta_0 \tag{3.23}$$

with $K = rc_3/\zeta_0^2$ for some $r > 1$ and with $\tau_* = \tau/2\zeta_0$ where τ is as in (3.8) for this r and the above choice of $\mathcal{M}$ – again noting that we have assumed α bounded and bounded away from 0 in constructing $\mathcal{M}$.

For *displacement observation*, so $\mathbf{b}_0 = [0\ 1\ 0]$ and

$$\mathbf{b}_0\hat{L} = \begin{bmatrix} 0 & 0 & \frac{\kappa}{\nu\mu} \end{bmatrix}, \qquad \mathbf{b}_0\hat{L}^2 = \begin{bmatrix} \frac{-\alpha\kappa}{\mu} & \frac{-1}{\mu} & 0 \end{bmatrix},$$

we must supplement (3.21) by requiring, e.g., that $\nu/\kappa = \mathcal{O}(1/\sqrt{\mu})$ and $1/\kappa = \mathcal{O}(1/\mu)$. This is possible if and only if $\kappa \sim \mu$ and $\nu \sim \sqrt{\mu}$ in this case, whence

$$\|D^{-1}\| = 1, \qquad \mathbf{b}_* = [\alpha \quad 0 \quad 0] \quad \text{so} \quad \beta_* = \alpha, \qquad \rho = 1/\kappa = 1/\mu. \tag{3.24}$$

Thus we need $\sigma \sim \mu$ and must choose $\mathbf{S} = \mathbf{M}$ to get, as with (3.23),

$$\|U(T)\| \leqslant K\alpha^{-1}T^{-5/2}\|\mathbf{M}\mathbf{A}\theta\|_{L^2(\mathcal{Q})} \quad \text{for } 0 < T \leqslant \tau_* = \tau/2\zeta_0. \tag{3.25}$$

Finally, for *velocity observation*, so $\mathbf{b}_0 = [0\ 0\ 1]$ and

$$\mathbf{b}_0\hat{L} = \left[\ \tfrac{-\alpha\nu}{\sqrt{\mu}} \quad \tfrac{-\nu}{\kappa\sqrt{\mu}} \quad 0\ \right], \qquad \mathbf{b}_0\hat{L}^2 = \left[\ \tfrac{\alpha\nu}{\sqrt{\mu}} \quad 0 \quad -\tfrac{1+\alpha^2}{\mu}\ \right],$$

we must supplement (3.21) by requiring, e.g., that $\nu/\sqrt{\mu}$ and $\nu/\kappa\sqrt{\mu}$ each be bounded away from 0, which necessitates $\kappa \sim 1$ and $\nu \sim \sqrt{\mu}$ in this case. The computation of $\mathbf{b}_*$ from (3.7) is here a bit messier, but we get

$$\|D^{-1}\| = 1, \qquad \beta_* = \frac{\alpha}{\sqrt{1+\alpha^2}}, \qquad \rho = 1/\nu = 1/\sqrt{\mu}. \tag{3.26}$$

This time, since $\|D^{-1}\|/\rho = \sqrt{\mu}$, we are led to choose $\mathbf{S} = \mathbf{M}^{1/2}$ so the observation is $\varphi = \mathbf{S}v = \mathbf{M}w_t$. Thus, in this case, we get

$$\|U(T)\| \leqslant K\,\frac{\sqrt{1+\alpha^2}}{\alpha}T^{-5/2}\,\|\mathbf{M}w_t\|_{L^2(\mathcal{Q})} \quad \text{for } 0 < T \leqslant \tau_*. \tag{3.27}$$

Collecting (3.23), (3.25), (3.27) is just the desired result (3.1). □

3.4. A variant when $\gamma = 0$

The computations of κ, ν in the proof above show why the observational topologies in (3.1) are sharp in each case when $\gamma > 0$. However, when $\gamma = 0$ so $\mathbf{M} \equiv \mathbf{1}$ it is possible to determine the state in $\mathcal{H}$ from internal observation of the deflection w taken with a weaker observational topology than was used above in Theorem 3.1, i.e., now taking w in $L^2(\mathcal{Q}_T)$ for the observation. The proof is quite similar to that above but now depends on the fact that when $\gamma = 0$ we have the exponential decay estimate (4.10) of Lemma 4.3 below. However, the reduction in regularity from $H^2(\Omega)$ to $L^2(\Omega)$ does result in an increase in the blowup rate from $\mathcal{O}(T^{-5/2})$ to $\mathcal{O}(T^{-7/2})$ as $T \to 0$, as well as increases in the blowup rates as $\alpha \to 0, \infty$.

Theorem 3.5. *For every $\alpha > 0$ and every $T > 0$ it is possible to observe the displacement w, topologized in $L^2(\mathcal{Q}_T)$ and determine the full state (w, w_t, θ) at the final time T, topologized in $\mathcal{H}_* = H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, for the linear thermoelastic system* (4.1), (2.1). *The map*: $\varphi = w \mapsto (w, w_t, \theta)|_{t=T}$ *is then continuous with norm $\mathfrak{C} = \mathfrak{C}(T;\alpha)$, as in* (1.12), *with the asymptotics*

$$\begin{aligned} &\mathfrak{C}(T;\alpha) \leqslant \psi T^{-7/2} \quad \textit{as } T \to 0 \textit{ for fixed } \alpha \\ &\quad \textit{with } \psi = \mathcal{O}(1/\omega\beta_*) = \begin{cases} \mathcal{O}(\alpha^{-3}) & \textit{as } \alpha \to 0, \\ \mathcal{O}(\alpha) & \textit{as } \alpha \to \infty. \end{cases} \end{aligned} \tag{3.28}$$

Proof. We only sketch the proof as it is essentially similar to that of Theorem 3.1. We now use Lemma 4.3 (4.10) to replace the first inequality in (3.18) by

$$\left|U_m(T)\right| = \left|\mathrm{e}^{\zeta L}U_m\left(\frac{k}{n}T\right)\right| \leqslant \kappa^2\,\mathrm{e}^{-\omega(1-k/n)\zeta T}\left|U_m\left(\frac{k}{n}T\right)\right|$$

and later, taking n even, sum over $k = 1, \ldots, n/2$. We are taking $\mathbf{S} = \mathbf{A}^{-1}$ (so $\sigma_m = 1/\zeta_m$) to have $\varphi = w = \mathbf{S}u$. Thus, with $D = 1$ so $\rho = 1$ and then taking $n \approx 2\,\zeta_m/\zeta_0$, (3.20) will now become

$$\begin{aligned}\left|U_m(T)\right| &\leqslant \kappa^2\,\mathrm{e}^{-\omega\zeta T/2}\sqrt{\frac{\zeta}{n/2}}\,\mathfrak{C}\left(\frac{\zeta T}{n/2}; L, \mathbf{b}_0\right)\|\varphi_m\|_{L^2(0,T/2)} \\ &\leqslant \frac{rc_3\kappa^2}{\beta_*\zeta_0^2}T^{-5/2}\,\mathrm{e}^{-\omega\zeta T/2}\zeta_m\sigma_m\|\varphi_m\|\end{aligned}$$

and (3.28) follows on bounding $\mathrm{e}^{-\omega\zeta T/2}\zeta$ by $c/\omega T$ and then combining the asymptotics for β_* used earlier for Theorem 3.1 with the asymptotics for $\omega(\alpha)$ of Lemma 4.3. $\square$

4. Boundary observation ($\gamma = 0$)

In this section we consider the same system (1.2), (2.1) as in Sections 2 and 3 but our concern in this section is with geometrically localized observation – restricted to (part of) the boundary $\partial\Omega$ – for a single component: $z = \theta$ or u or v. Since the boundary conditions (2.1) specify that $z \equiv 0$ on $\partial\Omega$ in each case, we will consider observation of the normal derivative so $\varphi = z_\nu$; this is particularly natural for $z = \theta$ where the observation is just the resultant heat flux. Again, a principal point of this subsection is that being able to cite general results – here Theorem 4.2, taken from [23] – simplifies the arguments considerably when that analysis is applicable. We might again comment along the lines of Remark 3.2.

For boundary observation we cannot have as complete a reduction as for the case of observation on all of Ω considered earlier: we will obtain a family of infinite-dimensional problems for each of which we are concerned with the 'window problem' of nonharmonic analysis. The technical restrictions of the available result here [23] sharply restrict the applicability of this approach (cf. [22]). In particular, we must restrict our attention to the case $\gamma = 0$ so $\mathbf{M} = \mathbf{1}$ and the system becomes

$$\begin{aligned} w_{tt} + \Delta^2 w - \alpha\Delta\theta &= 0, \\ \theta_t - \Delta\theta + \alpha\Delta w_t &= 0, \end{aligned} \qquad L = L(\alpha) = \begin{pmatrix} -1 & 0 & \alpha \\ 0 & 0 & 1 \\ -\alpha & -1 & 0 \end{pmatrix}. \tag{4.1}$$

Further, since the sparsity hypothesis (4.5) of Theorem 4.2 effectively restricts the applicability to quadratically spaced exponent sequences and so to spatially 1-dimensional problems, our approach in this section is to use spectral expansion to reduce our problem to a family of such spatially 1-dimensional problems and (compare [13]) note that we are taking Ω to be a cylindrical product region $\Omega = (0,1) \times \Omega_*$ with boundary observation at the base $\Gamma_0 = \{0\} \times \Omega_*$ in order to make such a reduction possible.

In the case $\gamma = 0$, boundary nullcontrollability in an arbitrary short time of thermoelastic beams ($\dim\Omega = 1$) was established (without any asymptotics) in [11]. For thermoelastic plates with rotational forces ($\gamma > 0$), there is a large body of literature yielding null or partial controllability results for T sufficiently large (in line with finite speed of propagation exhibited by singular support of the underlying PDE), but this type of result is not relevant to the topic we study.

To our best knowledge, as already noted in the Introduction, the result stated in Theorem 4.1 is the first to provide asymptotics as $T \to 0$ in the case of boundary control in one component only – for thermoelastic plates or, more generally, for any nonscalar system.

The principal result we obtain in this section is:

Theorem 4.1. *We consider the linear thermoelastic system* (4.1), (2.1) *on a product region* $\Omega = (0,1) \times \Omega_*$. *Then for every* $\alpha > 0$ *and every* $T > 0$ *it is possible to observe* $\varphi = z_\nu$ *on* $\Gamma_0 = \{0\} \times \Omega_*$, *taken in* $L^2([0,T] \times \Gamma_0)$, *for any of the components* z *of a solution and determine from this the full state* (θ, w, w_t) *at the final time* T, *taken in* $L^2(\Omega) \times H^2(\Omega) \times L^2(\Omega)$, *equivalently, to determine* $U(T)$ *in* $L^2(\Omega \to \mathbb{R}^3)$. *For each of the cases*

- Thermal observation ($\varphi = \theta_\nu$ *so* $\mathbf{b}_0 = [1\ 0\ 0]$) *or*
- Displacement observation ($\varphi = u_\nu = [\mathbf{A}w]_\nu$, *corresponding to observation with* w *topologized in* $L^2([0,T] \to H^2(\Omega))$ *so* $\mathbf{b}_0 = [0\ 1\ 0]$) *or*
- Velocity observation ($\varphi = v_\nu = [w_t]_\nu$ *so* $\mathbf{b}_0 = [0\ 0\ 1]$),

the map: $\varphi \mapsto U(T)$ *is continuous from* $L^2([0,T] \times \Gamma_0)$ *to* $\mathcal{H} = L^2(\Omega \to \mathbb{R}^3)$ *with norm* $\mathfrak{C} = \mathfrak{C}(T;\alpha)$ *uniformly bounded for* T, α *bounded and bounded away from* 0, *and with asymptotics as in* (1.6): *more particularly,*

$$\begin{aligned} &\mathfrak{C}(T;\alpha) \leqslant \psi(\alpha)\,\mathrm{e}^{B/T} \quad \text{as } T \to 0 \text{ for fixed } \alpha \\ &\quad \text{with } \psi(\alpha) = \mathcal{O}\big(\alpha^{-1}\big) \text{ as } \alpha \to 0 \end{aligned} \tag{4.2}$$

for each of the three cases.

4.1. A general result of nonharmonic analysis

In this subsection we note a general result for the 'window problem' of nonharmonic analysis which will then be applied to the thermoelastic system above. Noting that it was originally motivated by precisely the concerns at issue here, we cite the principal result of [23], specialized to treat eigenvalue sequences with quadratic separation – i.e., restricting $\nu(\cdot)$ in (4.5) below to the form $\delta\sqrt{s}$, leading to the estimate (4.7). For details and more general results of this nature, see [23,22], etc.

Theorem 4.2. *Let* $\Lambda = \{\lambda_m = \tau_m + \mathrm{i}\sigma_m\colon\ m = 1, 2, \ldots\}$ *be a complex sequence satisfying*:

$$\sigma_m \geqslant 0, \tag{4.3}$$

uniform separation: *for some* $s_0 > 0$,

$$|\lambda_{m'} - \lambda_m| \geqslant s_0 \quad (m' \neq m), \tag{4.4}$$

uniform sparsity: *for some* $\delta > 0$,

$$\#\big\{\lambda \in \Lambda\colon\ 0 < |\lambda - \lambda_*| \leqslant s\big\} \leqslant \nu(s) = \delta\sqrt{s} \quad \text{uniformly for } \lambda_* \in \Lambda. \tag{4.5}$$

Then for each $T > 0$ *there is a constant* $\mathfrak{C} = \mathfrak{C}(T;\Lambda)$ *such that*

$$\sum_m \big|c_m\,\mathrm{e}^{\mathrm{i}\lambda_m T}\big|^2 \leqslant \mathfrak{C}^2 \int_0^T \Big|\sum_m c_m\,\mathrm{e}^{\mathrm{i}\lambda_m t}\Big|^2\,\mathrm{d}t \tag{4.6}$$

with

$$\mathfrak{C} = \mathfrak{C}(T;\Lambda) \leqslant A\,\mathrm{e}^{B/T}, \tag{4.7}$$

where A, B *depend only on the values of* s_0, δ *in* (4.4), (4.5)*, but not otherwise on the sequence* Λ.

Thus, we necessarily have uniformity of the estimate (4.6), (4.7) over families $\{\Lambda\}$ of such sequences for which we can use fixed $s_0, \delta, T > 0$ and we have blowup in the estimate as $T \to 0$, exponential to the order of $1/T$ as in (1.6).

4.2. Reduction to a family of problems

We consider spectral expansion for solutions of (4.1), (2.1). Before proceeding with the expansion we note some spectral properties of the particular matrix $L = L(\alpha)$ of (4.1).

Lemma 4.3. *Let* $L = L(\alpha)$ *be as in* (4.1) *for* $0 < \alpha < \infty$. *Then* L *has distinct eigenvalues* $\{\xi_0, \xi_1, \xi_3\}$ *and so a basis* $\mathcal{B} = \mathcal{B}(\alpha) = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ *of corresponding eigenvectors. The eigenvalues have negative real parts with*

$$0 < \omega = \omega(\alpha) = -\max_j\{\Re(\xi_j)\} \sim \begin{cases} \alpha^2/4 & \text{as } \alpha \to 0, \\ \alpha^{-2} & \text{as } \alpha \to \infty. \end{cases} \tag{4.8}$$

There is a constant κ (*uniform in* α) *such that*

$$\frac{1}{\kappa}\left\|\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}\right\| \leqslant \|a_0\mathbf{v}_0 + a_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2\| \leqslant \kappa\left\|\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}\right\| \tag{4.9}$$

for all complex triples a_0, a_1, a_2 *and we then have*

$$\|\mathrm{e}^{tL(\alpha)}\| \leqslant \kappa^2\,\mathrm{e}^{-\omega(\alpha)\,t} \tag{4.10}$$

for all $t, \alpha > 0$.

Proof. The characteristic polynomial of L is

$$p = p(\xi) = p(\xi;\alpha) = \xi^3 + \xi^2 + (\alpha^2 + 1)\xi + 1.$$

Since $p' = 2\xi^2 + (\xi + 1)^2 + \alpha^2 > 0$, $p(0) = 1 > 0$, and $p(-1) = -\alpha^2 < 0$, there is a unique real root ξ_0 which lies strictly between -1 and 0. There is also a conjugate pair of roots $\xi_1, \xi_2 = -a \pm \mathrm{i}b$ with $a > 0$ since the sum of the roots is $\xi_0 - 2a = -1$; as the product of the roots is $\xi_0(a^2 + b^2) \equiv -1$, we have $b \geqslant \sqrt{26}\,a$ uniformly in α.

Since the roots are distinct, they depend analytically on the parameter α^2 of p. At $\alpha = 0$ we have $\xi_0(0) = -1$ and $\xi_{1,2} = \pm\mathrm{i}$; thus differentiating $p(\xi) \equiv 0$ with respect to α^2 shows

$$\begin{aligned} &\xi_0 = -1 + (1/2)\alpha^2 + [\text{higher-order terms}], \\ &\xi_{1,2} = \pm\mathrm{i} + \big([-1 \pm \mathrm{i}]/4\big)\alpha^2 + [\text{h.o.t.}] \end{aligned}$$

giving the asymptotics of ω as $\alpha \to 0$. Setting $\xi = \alpha\eta$ and $\hat{\alpha} = 1/\alpha$, we note that $0 = \hat{\alpha}^2 p(\alpha\eta) = \eta^3 + \hat{\alpha}\eta^2 + (1+\hat{\alpha}^2)\eta + \hat{\alpha}^3$ giving $\eta_j \to 0, \pm\mathrm{i}$ as $\hat{\alpha} \to 0$ ($\alpha \to \infty$). This gives $\xi_{1,2} \sim \pm\mathrm{i}\alpha$ so, since $\Pi_j \xi_j \equiv -1$, we get $\xi_0 \sim -\alpha^{-2}$ as $\alpha \to \infty$, completing the argument for (4.8).

Since the eigenvalues are always distinct, the set of eigenvectors remains linearly independent as α varies in $(0,\infty)$. We denote this ordered basis of eigenvectors by $\mathcal{B} = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$, consistently taking $\mathbf{v}_0$ to correspond to the real eigenvalue ξ_0 and taking $\mathbf{v}_1, \mathbf{v}_2$ to correspond, respectively, to the conjugate eigenvalues ξ_1, ξ_2 with positive/negative imaginary part, respectively. Since $L(\alpha)$ is real, we can ask that $\mathbf{v}_0$ have real entries and that $\mathbf{v}_1, \mathbf{v}_2$ be conjugate; for convenience, we keep the norm of each eigenvector constant as α varies.

By standard perturbation results (cf., e.g., [12]) $\mathcal{B} = \mathcal{B}(\alpha)$ can be taken to vary smoothly with respect to the parameter α of $L(\alpha)$. Since $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ are also eigenvectors of $\alpha^{-1}L$, which is analytic in $\hat{\alpha} = \alpha^{-1}$, we can connect this α-parametrized trajectory to a trajectory in $\hat{\alpha}$ with $\hat{\alpha} \to 0$ as $\alpha \to \infty$. We fix

$$\begin{aligned} \mathcal{B}(0) &= \{(1 \quad 0 \quad 0), (0 \quad 1 \quad \mathrm{i}), (0 \quad 1 \quad -\mathrm{i})\}, \\ \mathcal{B}(\infty) &= \{(0 \quad 1 \quad 0), (1 \quad 0 \quad -\mathrm{i}), (1 \quad 0 \quad \mathrm{i})\} \end{aligned} \tag{4.11}$$

and note, since the limits exist, that the trajectory: $\alpha \mapsto \mathcal{B}(\alpha)$ maps $[0,\infty]$ to a compact set in $[\mathbb{C}^3]^3$. Since each $\mathcal{B}(\alpha) = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ is a basis, we have (4.9) – with κ uniform in α by the compactness. Finally, using (4.9) for the spectral expansion $\mathrm{e}^{tL}[\sum_j a_j \mathbf{v}_j] = \sum_j (a_j\, \mathrm{e}^{t\xi_j})\mathbf{v}_j$ gives (4.10). $\square$

Lemma 4.4. *Set* $\beta_j = \mathbf{b}_0 \cdot \mathbf{v}_j(\alpha)$ *and* $B = B(\alpha; \mathbf{b}_0) = \min_j |\beta_j|$. *Then*

$$B(\alpha) \sim \begin{cases} \alpha/\sqrt{2} \quad \alpha/2 \quad \alpha/2 & \text{as } \alpha \to 0+, \\ \alpha^{-1} \quad \alpha^{-1} \quad \alpha^{-2} & \text{as } \alpha \to \infty, \end{cases} \tag{4.12}$$

respectively, for the three cases $\mathbf{b}_0 = [1\ 0\ 0]$, $[0\ 1\ 0]$, $[0\ 0\ 1]$.

Proof. Since $\{\mathbf{v}_j\}$ are eigenvectors, $\mathbf{b}_0 \cdot \mathbf{v} = 0$ would also give $\mathbf{b}_0 L \cdot \mathbf{v} = 0$ and $\mathbf{b}_0 L^2 \cdot \mathbf{v} = 0$, which is impossible because $\mathcal{B}(\alpha)$ is a basis.

To consider the asymptotics of $\mathcal{B}(\alpha)$ as $\alpha \to 0+$, we differentiate the eigenvector equation with respect to α to get $[L - \xi]\mathbf{w} = -[L' - \xi']\mathbf{v}$ with $\mathbf{w} = \mathbf{v}'$. Since $[L - \xi]$ is necessarily singular (rank 2), we have supplemented this by $\mathbf{v} \perp \mathbf{w}$ in keeping $|\mathbf{v}|$ constant to determine $\mathbf{w}$. We then obtain

$$\begin{aligned} \mathbf{v}_0 &= (1 \quad \alpha/2 \quad -\alpha/2) + [\text{h.o.t.}], \\ \mathbf{v}_{1,2} &= ([1 \pm \mathrm{i}]\alpha/2 \quad 1 \pm \mathrm{i}) + [\text{h.o.t.}] \end{aligned}$$

as $\alpha \to 0$ which gives the first line of (4.12). Similarly, although with a bit more work, one obtains the given asymptotics as $\hat{\alpha} \to 0, \alpha \to \infty$. $\square$

We now obtain the desired reduction by a spectral expansion.

Note that $\Omega = (0,1) \times \Omega_*$ is a separable geometry for $\mathbf{A} = -\Delta$ given by (1.3) – i.e., that the eigenfunctions form an orthonormal basis for $L^2(\Omega)$ having the product form $\{e_k(x) f_\ell(y)\}$. For (1.3) with this Ω we have

$$e_k(x) = (1/\sqrt{2}) \sin k\pi x \quad (k = 1, 2, \ldots) \ \text{ for } x \in (0,1); \tag{4.13}$$

similarly, $\{f_\ell(y)$ for $y \in \Omega_*$: $\ell = 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\Omega_*)$ consisting of the eigenfunctions of the cross-sectional Laplacian $-\Delta_*$ on Ω_* with homogeneous Dirichlet conditions at $\partial\Omega_*$. The corresponding eigenvalues of $-\Delta_*$ are $\{\mu_\ell\}$ (taken with appropriate multiplicities; we note that each $\mu_\ell > 0$). One verifies immediately that the eigenvalues of $\mathbf{A}$ are then $(\pi^2 k^2 + \mu_\ell)$ with

$$\mathbf{A}[e_k f_\ell] = \left(\pi^2 k^2 + \mu_\ell\right) e_k f_\ell \tag{4.14}$$

for each k, ℓ.

If we expand the dependence on $y \in \Omega_*$ with respect to the orthonormal basis $\{f_\ell\}$, writing U in the form

$$U(t, x, y) = \sum_\ell U_\ell(t, x)\, f_\ell(y) \tag{4.15}$$

with time-dependent vectorial coefficients $U_\ell(t, x) = \langle U(t, x, \cdot),\ f_\ell\rangle$, then by (1.11) and (4.14) each of these coefficients satisfies the vectorial partial differential equation (system) in one space dimension:

$$[U_\ell]_t = -\mu_\ell L[U_\ell]_{xx} \quad \text{on } (0, T) \times (0, 1) \text{ with } U_\ell = 0 \text{ at } x = 0, 1. \tag{4.16}$$

This is the anticipated reduction to a decoupled family of 1-dimensional problems and we will be applying Theorem 4.2 to each of these separately.

That application, however, will require further expansion and we now expand U_ℓ with respect to the orthonormal basis $\{e_k\}$, now writing U in the form

$$U(t, x, y) = \sum_{k,\ell} U_{k,\ell}(t)\, e_k(x) f_\ell(y) \tag{4.17}$$

with vectorial coefficients $U_{k,\ell}(t) = \langle U(t),\ e_k f_\ell\rangle = \int_0^1 U_\ell e_k \,\mathrm{d}x$ satisfying the ordinary differential equations

$$\dot U_{k,\ell} = \left(\pi^2 k^2 + \mu_\ell\right) L U_{k,\ell}. \tag{4.18}$$

Thus we have

$$U_{k,\ell}(t) = \sum_{j=0,1,2} a_{j,k,\ell}\ \mathrm{e}^{\xi_j(\pi^2 k^2 + \mu_\ell)t}\, \mathbf{v}_j, \tag{4.19}$$

where $\{(\xi_j, \mathbf{v}_j)\}$ are the eigenpairs of L, as given in (4.1). The eigenpairs of the system operator $\mathbf{L}_* = L\mathbf{A}$ appearing in (1.11) are here

$$\left\{\xi_j\left(\pi^2 k^2 + \mu_\ell\right),\ e_k(x) f_\ell(y)\mathbf{v}_j\right\}. \tag{4.20}$$

To consider observation of the normal derivative, $\varphi = \partial z/\partial\nu$ on the base $\Gamma = \{0\} \times \Omega_*$ – which is here $-\partial z/\partial x$ at $x = 0$ – one first chooses $\mathbf{b}_0$ to select the appropriate component for observation: $z = \theta$

or $z = u = -\Delta w$ or $z = v = w_t$. Then using (4.19) and noting that (4.13) gives $-\partial e_k/\partial x = -\pi k/\sqrt{2}$ at $x = 0$, one gets

$$\begin{aligned} \varphi(t,y) &= -z_x(t,0,y) = -(\mathbf{b}_0 \cdot U)_x(t,0,y) \\ &= \sum_{k,\ell} \frac{-\pi k}{\sqrt{2}}\, \mathbf{b}_0 \cdot U_{k,\ell}(t) f_\ell(y) \\ &= \sum_{\ell} \varphi_\ell(t) f_\ell(y), \quad \text{where} \\ \beta_j = \mathbf{b}_0 \cdot \mathbf{v}_j \ &\text{ and } \ \varphi_\ell(t) = \sum_{j,k} \frac{-\pi k}{\sqrt{2}} \beta_j a_{j,k,\ell}\, \mathrm{e}^{\xi_j(\pi^2 k^2 + \mu_\ell)t}. \end{aligned} \tag{4.21}$$

4.3. Proof of Theorem 4.1

Proof. We will use (4.17), (4.19), (4.21), together with Theorem 4.2 to obtain the desired estimate (4.2). We begin by choosing $\mathbf{b}_0$ to select the relevant component of U for observation and then considering (4.17), noting the orthonormality of $\{e_k f_\ell\}$ in $L^2(\Omega)$ and of $\{f_\ell\}$ in $L^2(\Omega_*)$ to get

$$\begin{aligned} \|U(T)\|^2 &= \sum_{k,\ell} |U_{k,\ell}(T)|^2 = \sum_{k,\ell} \left| \sum_{j=0,1,2} a_{j,k,\ell}\, \mathrm{e}^{\xi_j(\pi^2k^2+\mu_\ell)T} \mathbf{v}_j \right|^2 \\ &\leqslant \kappa^2 \sum_{\ell} \sum_{j,k} \left| a_{j,k,\ell}\, \mathrm{e}^{\lambda_{j,k}^{[\ell]} T} \right|^2 \\ &\leqslant \frac{2\kappa^2}{\pi^2 B^2} \sum_{\ell} \left[\sum_{j,k} \left| a_{j,k,\ell}\, \beta_j \frac{\pi k}{\sqrt{2}}\, \mathrm{e}^{\lambda_{j,k}^{[\ell]} T} \right|^2 \right], \end{aligned} \tag{4.22}$$

where we have used (4.9) for the first inequality and then, for the second, multiplying and dividing by $(\beta_j \pi k/\sqrt{2})^2 \geqslant \pi^2 B^2/2$ with B as in Lemma 4.4. Using the orthonormality of $\{f_\ell\}$, we then have

$$\|\varphi(t,\cdot)\|^2 = \sum_{\ell} \|\varphi_\ell(t)\|^2 = \sum_{\ell} \left| \sum_m c_m^{[\ell]}\, \mathrm{e}^{\mathrm{i}\lambda_m^{[\ell]} t} \right|^2. \tag{4.23}$$

Letting $\{m = 1, 2, \ldots\}$ represent a re-indexing of the pairs $\{j, k\}$, we have

$$c_m = c_m^{[\ell]} = a_{j,k,\ell} \beta_j \frac{\pi k}{\sqrt{2}} \tag{4.24}$$

for each ℓ and consider the family of sequences

$$\begin{aligned} \Lambda^{[\ell]} &= \{\lambda_m^{[\ell]}\} \quad \text{with} \\ \lambda_m^{[\ell]} = \lambda_{j,k}^{[\ell]} &= -\mathrm{i}\xi_j(\pi^2 k^2 + \mu_\ell) \end{aligned} \tag{4.25}$$

for $j = 0, 1, 2$; $k = 1, 2, \ldots$ as in (4.20) – each corresponding to the partial differential equation $(4.16)_\ell$.

To show that Theorem 4.2 will be applicable for each φ_ℓ and will give estimates uniform in ℓ, our next major task is to show that the sequences $\Lambda^{[\ell]} = \{\lambda_m^{[\ell]}\}$ each satisfy the separation and sparsity conditions (4.4), (4.5) with parameters s_0, δ which are independent of ℓ. [These s_0, δ may also be taken uniform in α for α bounded and bounded away from 0.]

For each ℓ think of the sequence $\Lambda = \Lambda^{[\ell]}$ in (4.20) as consisting of branches on the three rays $\mathcal{R}_j = \{t\xi_j\colon\ t > 0\}$ for $j = 0, 1, 2$. In verifying (4.4), we note that the separation would be minimized by reducing μ to 0 and is then bounded below by the smaller of the distances from $\lambda_{0,1}$ either to $\lambda_{0,2} = -3\pi^2 r$ or to the rays $\mathcal{R}_1, \mathcal{R}_2$. Since $\min\{|\xi_j|\} = r = r(\alpha)$ is bounded away from 0 for bounded α and the angular separation of the rays is bounded below, there is thus a uniformly applicable value of s_0.

We next obtain a bound (dependent on s, α, but uniform in ℓ) for the sparsity parameter δ. Note that

$$\nu_*(s) = \#\{\lambda \in \Lambda\colon\ 0 < |\lambda - \lambda_*| \leqslant s\} = \#[\mathcal{D}_s \cap \Lambda] - 1$$

with $\mathcal{D}_s = \{z \in \mathbb{C}\colon\ |z - \lambda_*| \leqslant s\}$. Clearly, each such disk $\mathcal{D}_s$ is contained in an annulus $\mathcal{A} = \mathcal{A}_{S,s} = \{z \in \mathbb{C}\colon\ S \leqslant |z| \leqslant S + 2s\}$ so $1 + \nu_*$ is bounded by the sum (over the branches $j = 0, 1, 2$) of

$$\#\{k\colon\ S \leqslant (\pi^2 k^2 + \mu)|\xi_j(\alpha)| \leqslant S + 2s\}.$$

This is maximized by again reducing μ to 0 and then taking S as small as possible (i.e., $S = \pi^2|\xi_j(\alpha)|$) so

$$\begin{aligned}\nu_*(s;\alpha) &\leqslant 3\,\#\{k\colon\ 1 \leqslant k^2 \leqslant 1 + 2s/(\pi^2 r)\}\\ &\leqslant 3\sqrt{1 + \frac{2s}{\pi^2 r(\alpha)}}.\end{aligned}$$

Since we are only concerned with $s > s_0$, this gives (4.5) with the desired uniformity of δ for bounded α.

Applying Theorem 4.2 now gives (4.6), specifically

$$\begin{aligned}\sum_{j,k}\left|a_{j,k,\ell}\beta_j \frac{\pi k}{\sqrt{2}}\,\mathrm{e}^{\lambda_{j,k}^{[\ell]}T}\right|^2 &\leqslant \mathfrak{C}(T;\Lambda^{[\ell]})^2 \int_0^T |\varphi_\ell(t)|^2\,\mathrm{d}t\\ &\leqslant \mathfrak{C}_*(T;\alpha)^2 \int_0^T |\varphi_\ell(t)|^2\,\mathrm{d}t,\end{aligned}\tag{4.26}$$

noting that $\mathfrak{C}(T;\Lambda^{[\ell]})$ is bounded uniformly in ℓ since we have shown similar uniformity of the parameters s_0, γ whence we have $\mathfrak{C}_* = \mathfrak{C}_*(T)$ uniformly in α for α bounded and bounded away from 0. We then use (4.26) to continue (4.22):

$$\begin{aligned}\|U(T)\|^2 &\leqslant \frac{2\kappa^2}{\pi^2 B^2}\sum_\ell\left[\sum_{j,k}\left|a_{j,k,\ell}\,\beta_j \frac{\pi k}{\sqrt{2}}\ \mathrm{e}^{\lambda_{j,k}^{[\ell]}T}\right|^2\right]\\ &\leqslant \frac{2\kappa^2}{\pi^2 B^2}\mathfrak{C}_*^2 \int_0^T \|\varphi(t,\cdot)\|^2\,\mathrm{d}t,\end{aligned}\tag{4.27}$$

using (4.23). Of course, (4.27) is just (1.12) – and the asymptotic estimate (4.2) follows immediately from (4.7) and (4.12). □

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