Strong Stability of Elastic Control Systems with Dissipative Saturating Feedback

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Abstract

We will consider, with a focus on saturating feedback control laws, two problems associated with damping in a bounded acoustic cavity $\Omega \subset \mathbb{R}^3$. Our objective is to verify (compare [9], [11]) that these are strongly stable: for every finite-energy solution, the acoustic energy goes to zero as $t \to \infty$. We will, in each case, formulate the problem in terms of a contraction semigroup of nonlinear operators on an appropriate Hilbert space and compare this with the corresponding semigroups without saturation — following [2] in using the spectral methods of [1] to show strong stabilization for those linear semigroups.

Key words: strong stability, saturating feedback, wave equation with boundary control, structural acoustic interaction

1 Introduction

We will be considering, as indicating some relevant technical difficulties in the analysis, the question of stabilization for two paradigmatic examples of elastic systems when the feedback is dissipative but 'saturating'. Both of these examples involve the wave equation

$$z_{tt} - \Delta z = 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \tag{1.1}$$

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(e.g., modelling acoustic vibration in a bounded enclosure Ω) with interaction available only at part of the boundary. In each of our examples the uncontrolled system is conservative and linear and it will be possible to show that a corresponding linear feedback does stabilize the system. Our principal concern here is to see that the saturating control law, coinciding with the linear feedback for small enough data, still provides global stabilization: the relevant physical energy goes to 0 as $t \to \infty$. As the saturation is inherently nonlinear, our context will be the theory of contraction semigroups of nonlinear operators on Hilbert space.

For our first example the boundary conditions for (1.1) will have the form

$$z_\nu = \varphi \quad \text{on } \partial\Omega \tag{1.2}$$

where the control φ is to have support restricted to some specified $\Gamma \subset \partial\Omega$ and will be determined by some implementable feedback law with the objective of *stabilization*: we consider finite energy solutions of (1.1) so

$$\mathcal{E}(t) = \mathcal{E}_{kinetic} + \mathcal{E}_{potential} := \tfrac{1}{2}\left[\|z_t\|^2 + \|\nabla z\|^2\right] \quad < \infty \tag{1.3}$$

and wish to ensure that — always — we have $\mathcal{E}(t) \to 0$ for the feedback-controlled solutions. Formally, we may differentiate $\mathcal{E}(\cdot)$, use (1.1), and then apply the Divergence Theorem to the term $\langle z_t, \Delta z\rangle$ to get, in view of (1.2),

$$\frac{d\mathcal{E}}{dt} = \int_{\partial\Omega} z_t z_\nu = \langle z_t, \varphi\rangle_\Gamma\,. \tag{1.4}$$

For our first example we note that conceptually the simplest feedback to consider would be to determine a boundary control by velocity feedback, setting

$$\varphi := -z_t \quad \text{restricted to } \Gamma. \tag{1.5}$$

This gives $d\mathcal{E}/dt = -\|(\text{trace of } z_t \text{ on } \Gamma)\|^2$ so at least the control action would be dissipative. Apart from showing that we then would eventually have stabilization, there are three considerable difficulties with this:

- justification of the formal computation, especially noting that taking the trace on Γ is an unbounded operator from $L^2(\Omega)$ and finiteness of the energy only leads us to expect $z_t \in L^2(\Omega)$.
- implementation — for which we are first concerned with the possibility that an actuator on Γ (e.g., involving coupling with a 'smart material' under computer control) might only be capable of producing output with some fixed bound M on the realizable control φ.

- since (1.1), (1.2) with (1.5), and (1.3) are invariant under addition of a spatial constant to z, our feedback cannot be expected to eliminate such constants; the energy (1.3) does not provide a norm for the full state.

For the second of these difficulties we are led to consider the use of a 'saturating' control law: if φ_0 is the control specified by the original control law (e.g., (1.5)), then we would actually use the determination: $\varphi = F(\varphi_0)$ with

$$F(y) := \min\{1, M/\|y\|\} = \begin{cases} y & \text{if } \|y\| \leq M, \\ My/\|y\| & \text{if } \|y\| \geq M, \end{cases} \tag{1.6}$$

to ensure that $\|\varphi\|$ does not exceed the saturation threshhold M.

Our major thrust in this paper is to show — for this and correspondingly for our second example — that the linear control law (1.5) is asymptotically stable and that we can overcome the technical difficulties to compare the effects of (1.5) and its saturating version to show that the latter also provides stabilization. For showing the asymptotic stability of (1.5), we follow [2] in applying [1]. For consideration of the saturating version it had been our intention to follow [9], but it seems more convenient to present, in the next section, a new abstract comparison theorem which makes the comparison directly, without the intervening reference to the uncontrolled situation used in [9]. [It is worth noting at this point the emphasis in our Theorem 1 on *output stabilization* for 'nice' initial data.]

Our second example corresponds to damping of the acoustic vibration in Ω through interaction with a plate occupying a (2-D, flat) portion of the boundary $\partial\Omega$ — the well-known *structural acoustic model* - [8,4]. Here the feedback is through coupling of the acoustic vibration in Ω with the vibration in the plate Γ. We again use 'velocity feedback' to dampen those vibrations and so the entire system, but for the structural acoustic model this will appear as distributed control on (part of) Γ (i.e., in the system equations) rather than as boundary control on $\Gamma \subset \partial\Omega$ as in our first example.

Thus, for the second problem we have

$$\begin{array}{ll} z_{tt} = \Delta z & \text{in } \Omega \subset \mathbb{R}^3 \\ z_\nu = w_t & \text{on } \Gamma \subset \partial\Omega \\ z_\nu + cz = 0 & \text{on } \Gamma' = \partial\Omega \setminus \Gamma \\ w_{tt} + \Delta^2 w + z_t = -\varphi & \text{in } \Gamma\ (\subset \mathbb{R}^2) \\ w = 0,\ \Delta w = 0 & \text{at } \partial\Gamma \subset \mathbb{R}^2 . \end{array} \tag{1.7}$$

Remark 1 *Note that the Laplacians Δ appearing in (1.7) are different in the equations for z and for w, with the first a 3-dimensional Laplacian associated with Ω and the second a 2-dimensional Laplacian associated with Γ. The introduction of the term cz on Γ' with $0 \leq c \not\equiv 0$ will add a boundary term to the energy, which then provides a norm. This choice for the example, slightly modifying our formulation, is purely for expository purposes. The simplest possibility in this setting for dissipative feedback would be to observe $y = aw_t$ on supp $(a) \subset \Gamma$ and then determine the control φ by a saturating feedback $\varphi = aF(y)$ with F as in (1.6) so we would have linear feedback $\varphi = a^2 w_t$ when w_t is not too big but φ remains bounded with $\|\varphi\| \leq M$ always. It would not have been difficult to include this as an example, proceeding along the lines we present, but we choose, instead, to provide a somewhat 'more unbounded' example, involving a version of Kelvin–Voigt damping.*

For our second example, we will assume observation here of $y := \Delta(aw_t)$ on supp $(a) \subset \Gamma$ and then set $\varphi = -a\Delta F(y)$, again with F as in (1.6). [Note that, if we did not have saturation and took $a \equiv 1$ on Γ, then this control term would just be $-\Delta^2 w_t$, i.e., Kelvin–Voigt damping.] The main technical difficulty associated with this problem is that the domains of the corresponding generators are *not compact* with respect to finite energy norms. On the other hand, it is known that compactness of the resolvent operators is used critically for the proofs of strong stability (see [11] and references therein). While this type of difficulty can be dealt with, via spectral analysis (Tauberian type of theorems — as [1]) for linear problems, we are not aware of any prior results pertaining to nonlinear hyperbolic dynamics and strong stabilizability in the absence of compactness.

In the next section we will provide a somewhat abstract formulation of our approach and then treat the details for each of the examples in Sections 3, 4. The key to our approach is Theorem 1, a somewhat simplified version of Theorem 2 of [9], comparing the saturating and the linear versions of the feedback. For the linear version strong stability will follow from the Arendt-Batty result [1], as in [2], using a spectral computation.

The two examples in Sections 3 and 4 illustrate the strength and applicability of the abstract theory developed in the next section. Indeed, the strong stability results obtained for these examples do not follow from standard approaches. To wit, a major difficulty in Example 1 is the presence of nontrivial steady states in a classical formulation of the abstract wave equation. This of course could be eliminated (as done in the literature) by modifying the boundary conditions on $\partial\Omega$ or by creating a "hole" in the domain with zero Dirichlet boundary conditions prescribed on a portion of the boundary. However, in our tratment we do not do this, aiming for the ultimate goal of stabilizing the acoustic energy only.

In the second example the difficulty is more serious as it is related to an intrinsic lack of compactness of the resolvent operator.

2 General theory

We first give a new version of Theorem 2 of [9]. Note that all the spaces here are Hilbert spaces and all the operators are single-valued. For our applications, $\mathbf{S}(\cdot)$ will be the evolution semigroup of the controlled system of interest, i.e., with (nonlinear) saturating feedback, and $\tilde{\mathbf{S}}(\cdot)$ will be the (linear) comparison semigroup associated with the corresponding feedback without saturation. While the underlying ideas are the same as in [9], the present argument is made both simpler and a bit more general by avoiding the further comparison with the uncontrolled sytem with no feedback at all. For our examples the feedback can be expressed — in the context of sufficiently smooth solutions — through a linear operator $\mathbf{C} : \mathbf{x} \mapsto y$ giving the 'observation output' and we will be taking $\psi(\mathbf{x}) := \|\mathbf{C}\mathbf{x}\|^2$ so the critical assertion (C) in the proof below is just 'output stabilization' for the system with smooth initial data.

Theorem 1 *Assume*

(H1) $-\mathbf{A}$ *and* $-\tilde{\mathbf{A}}$ *are, respectively, generators of continuous contraction semigroups* $\mathbf{S}(\cdot)$ *and* $\tilde{\mathbf{S}}(\cdot)$ *on a Hilbert space* $\mathcal{X}$*, i.e.,* $\mathbf{A}, \tilde{\mathbf{A}}$ *are densely defined maximal monotone (single-valued) operators on* $\mathcal{X}$*.*

(H2) *There is a function* $\psi : \mathcal{X} \supset \mathcal{D}(\psi) \to \mathbb{R}_+$ *such that*

- *(1)* $\mathcal{D}(\mathbf{A}), \mathcal{D}(\tilde{\mathbf{A}}) \subset \mathcal{D}(\psi)$
- *(2) for* $\xi \in \mathcal{D}(\tilde{\mathbf{A}})$ *with* $\|\xi\| \le \mu,\ \|\mathbf{A}\xi\| \le \lambda$*:* $\langle \xi, \mathbf{A}\xi \rangle \to 0$ *implies* $\psi(\xi) \to 0$
- *(3) for* $\xi \in \mathcal{D}(\tilde{\mathbf{A}})$ *with* $\|\xi\| \le \mu,\ \|\tilde{\mathbf{A}}\xi\| \le \lambda$*:* $\psi(\cdot)$ *is uniformly continuous:* $\mathcal{X} \to \mathbb{R}_+$*.*
- *(4) when* $\psi(\xi) \le \alpha$*:* $\xi \in \mathcal{D}(\tilde{\mathbf{A}})$ *if and only if* $\xi \in \mathcal{D}(\mathbf{A})$ *— and then* $\tilde{\mathbf{A}}\xi = \mathbf{A}\xi$

(H3) $\tilde{\mathbf{S}}(\cdot)$ *is strongly stable — i.e.,* $\tilde{\mathbf{S}}(t)x_0 \to 0$ *for each* $x_0 \in \mathcal{X}$*.*

Then $\mathbf{S}(\cdot)$ *is also strongly stable.*

PROOF: Our principal task will be to prove that:

(C) If $x_0 \in \mathcal{D}(\mathbf{A})$, then $\psi(\mathbf{S}(t)x_0) \to 0$ as $t \to \infty$.

To see from (C) that $\mathbf{S}(t)x_0 \to 0$ for $x_0 \in \mathcal{D}(\mathbf{A})$, note that it gives $\psi(\mathbf{S}(t)x_0) < \alpha$ from some time s on so, by (H2-4), $\tilde{\mathbf{A}}$ then coincides with $\mathbf{A}$ and the convergence to 0 of $\mathbf{S}(t)x_0 = \tilde{\mathbf{S}}(t-s)x(s)$ follows from (H3). Now, if we are given arbitrary $x_0' \in \mathcal{X}$, then for any $\varepsilon > 0$ we can choose $x_0 \in \mathcal{D}(\mathbf{A})$ with $\|x_0' - x_0\| < \varepsilon/2$ and note, first, that $\|\mathbf{S}(t)x_0' - \mathbf{S}(t)x_0\| \le \varepsilon/2$ for all $t > 0$ and then that $x(t) := \mathbf{S}(t)x_0 \to 0$ as $t \to \infty$ so, from some time on, $\|x(t)\| \le \varepsilon/2$

whence $\|\mathbf{S}(t)x_0'\| \le \varepsilon$ — i.e., one has $\mathbf{S}(t)x_0' \to 0$ as $t \to 0$ for all $x_0' \in \mathcal{X}$.

To show **(C)** we fix $x_0 \in \mathcal{D}(\mathbf{A})$ and set $\mu := \|x_0\|$, $\lambda := \|\mathbf{A}x_0\|$, $\mathbf{x}(t) := \mathbf{S}(t)x_0$, $\tilde\psi(t) := \psi(\mathbf{x}(t))$. Note that for $t > 0$ we have $\|\mathbf{x}(t)\| \le \mu$ and, by Komura-Kato Theorem (cf., [10] Prop 3.1 p. 174), we have $D_t^+\mathbf{x} = \mathbf{A}\mathbf{x}$ pointwise the function $t \to Ax(t)$ is rightcontinuous with $||Ax(t)||$ decreasing. [1].

— $\mathbf{x}(t) \in \mathcal{D}(\mathbf{A})$ (so $\tilde\psi$ is well defined) and $\|\mathbf{A}\mathbf{x}(t)\| \le \lambda$. By (H2-2) we have

$$\begin{array}{l} \forall \delta > 0,\ \exists\, \beta = \beta(\delta) = \beta(\delta; \mu, \lambda) \text{ such that:} \\ \|\xi\| \le \mu,\ \|\mathbf{A}\xi\| \le \lambda,\ \ \langle \xi, \mathbf{A}\xi\rangle \le \beta \qquad \Rightarrow \quad \psi(\xi) \le \delta \end{array} \tag{2.8}$$

and by (H2-3) we have

$$\begin{array}{l} \forall \delta > 0,\ \exists\, \gamma = \gamma(\delta) = \gamma(\delta; \mu, \lambda) \text{ such that:} \\ \text{ if } \|\xi\|, \|\xi'| \le \mu,\ \|\tilde{\mathbf{A}}\xi\|, \|\tilde{\mathbf{A}}\xi'\| \le \lambda, \\ \text{ then:} \qquad \|\xi - \xi'\| \le \gamma \quad \Rightarrow \quad |\psi(\xi) - \psi(\xi')| \le \delta. \end{array} \tag{2.9}$$

Supposing we have $\tilde\psi(\hat s) < \alpha$ for some $\hat s > 0$ so, by (H2-4), we have $x(\hat s) \in \mathcal{D}(\tilde{\mathbf{A}})$ with $\tilde{\mathbf{A}}x(\hat s) = \mathbf{A}x(\hat s)$ and $\|\tilde{\mathbf{A}}x(\hat s)\| \le \lambda$. Then we set $\hat x(t) := \tilde{\mathbf{S}}(t-\hat s)x(\hat s)$, $\hat\psi(t) := \psi(\hat x(t))$ for $t \ge \hat s$; note that $\hat\psi(\hat s) = \tilde\psi(\hat s) < \alpha$ and, again using Komura-Kato Theorem, we have $D_t^+\hat x = \tilde{\mathbf{A}}\hat x$ pointwise — $\hat x(t) \in \mathcal{D}(\tilde{\mathbf{A}})$ (so $\hat\psi$ is well defined) and $\|\tilde{\mathbf{A}}\hat x(t)\| \le \lambda$.

If (C) were to fail, there would be some (fixed) $x_0 \in \mathcal{D}(\mathbf{A})$ for which, using the notation above, one has some $\delta > 0$ and a sequence $t_k \to \infty$ with $\tilde\psi(t_k) > 2\delta$; without loss of generality we assume $2\delta < \alpha$. Now observe that, for arbitrary $0 < s < t \to \infty$,

$$2\int_s^t \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle = \int_s^t -\frac{d\,\|\mathbf{x}\|^2}{dt}\,dt = \|\mathbf{x}(s)\|^2 - \|\mathbf{x}(t)\|^2 \le \|\mathbf{x}(s)\|^2 \tag{2.10}$$

so, as $\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \ge 0$, there is a sequence $s_k \to \infty$ at which $\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \to 0$ so, by (H2-2), we have $\tilde\psi(s_k) \to 0$.

Without loss of generality, we assume each $\tilde\psi(s_k) < \delta < \alpha$ and introduce $\hat x = \hat x_k$ by taking $\hat s = s_k$ in the notation above, i.e., $\hat x_k(t) := \tilde{\mathbf{S}}(t - s_k)\mathbf{x}(s_k)$

[1] This equality depends, of course, on the single-valuedness of $\mathbf{A}$. We also note that we are here taking $D_t^+\mathbf{x}$ to be the forward derivative ($\lim[\mathbf{x}(t+h) - \mathbf{x}(t)]/h$ for $h \searrow 0+$)

and $\hat{\psi}(t) = \hat{\psi}_k(t) := \psi(\hat{x}(t))$ for $t \geq s_k$. Note that each $\hat{\psi}_k$ is continuous in t for $t \geq s_k$, since $\hat{x}_k(\cdot)$ is continuous — indeed,

$$\|\hat{x}(t) - \hat{x}(s)\| \leq \left\| \int_s^t \hat{x}^{\cdot} \right\| \leq \int_s^t \|\mathbf{A}\hat{x}\| \leq \lambda|t-s| \tag{2.11}$$

— and we have (H2-3). There is then a maximal interval $\mathcal{I}_k = [s_k,\, t_k]$ on which $\hat{\psi} \leq \alpha$. On this interval $\hat{x}^{\cdot} = \tilde{\mathbf{A}}\hat{x} = \mathbf{A}\hat{x}$ by (H2-4) so, by uniqueness of the solution, $\hat{x}(t) = \mathbf{S}(t - s_k)\hat{x}(s_k) = \mathbf{x}(t)$ on $\mathcal{I}_k$. By continuity, there is necessarily a subinterval $\bar{\mathcal{I}}_k = [\bar{s}_k, \bar{t}_k] \subset \mathcal{I}_k$ such that, noting that $\hat{\psi}_k = \tilde{\psi}$ on $\mathcal{I}_k$,

$$\tilde{\psi}(\bar{s}_k) = \delta \leq \tilde{\psi}(t) \leq 2\delta = \tilde{\psi}(\bar{t}_k) \quad \text{on } \bar{\mathcal{I}}_k.$$

Note that, by (2.8), having $\tilde{\psi} \geq \delta$ requires that $\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle \geq \beta = \beta(\delta) > 0$ on $\bar{\mathcal{I}}_k$. Also, by (2.9), $\hat{\psi}(\bar{t}_k) - \hat{\psi}(\bar{s}_k) = \delta$ requires that $\|\hat{x}(\bar{t}_k) - \hat{x}(\bar{s}_k)\| \geq \gamma = \gamma(\delta)$ whence, by (2.11), we have $\bar{t}_k - \bar{s}_k \geq \gamma/\lambda$. Combining these shows that the integral over $\bar{\mathcal{I}}_k$ of $\langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle$ is not less than $\beta\gamma/\lambda > 0$, independent of k. Since, without loss of generality, we can assume that the intervals $\bar{\mathcal{I}}_k$ are disjoint, this would contradict the bound on $\int \langle \mathbf{x}, \mathbf{A}\mathbf{x}\rangle\, dt$ of (2.10) — i.e., we must have $\tilde{\psi} \to 0$ as in (C). ∎

For each of our systems the state space $\mathcal{X}$ will have the form $\mathcal{X} = \mathcal{U} \times \mathcal{H}$ where $\mathcal{U}$ is a Hilbert space of 'configurations,' with a norm related to the potential energy, and $\mathcal{H}$ is a space of velocities (momenta), with the L^2 norm related to the kinetic energy. We will introduce an 'evolution triple' $\mathcal{Z} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{Z}^*$ and an operator $\mathcal{A} : \mathcal{Z} \xrightarrow{\text{cont}} \mathcal{Z}^*$ for which $\mathbf{A}\mathbf{x} = \mathcal{A}\mathbf{x}$ for $\mathbf{x} \in \mathcal{D}(\mathbf{A}) := \{\mathbf{x} \in \mathcal{Z} \subset \mathcal{X} : \mathcal{A}\mathbf{x} \in \mathcal{X} \subset \mathcal{Z}^*\}$; note that writing $\mathcal{A}\mathbf{x}$ as an element of $\mathcal{Z}^*$ provides a weak form which evades our difficulties with unbounded operators.

We observe at this point that, for any Hilbert space $\mathcal{Y}$, the function F — which is just the 'nearest point projection' on $\mathcal{Y}$ to the ball of radius M — is a continuous, maximal monotone operator; indeed, F is the gradient of the C^1 convex functional given by $\{\frac{1}{2}r^2 \text{ if } r \leq M;\ Mr - \frac{1}{2}M^2 \text{ if } r \geq M\}$ with $r = \|y\|$. We will have, in each case,

$$\langle \mathbf{x}_1 - \mathbf{x}_2, \mathcal{A}\mathbf{x}_1 - \mathcal{A}\mathbf{x}_2\rangle_{[\mathcal{Z}:\mathcal{Z}^*]} = \langle y_1 - y_2, F(y_1) - F(y_2)\rangle_{\mathcal{Y}} \geq 0 \tag{2.12}$$

so $\mathcal{A}$ (hence also $\mathbf{A}$) is monotone. To show that $\mathbf{A}$ is maximal monotone, it is sufficient to show that $(\mathbf{A} + \mathbf{1})$ is surjective. Since (2.12) does not give coercivity, we proceed indirectly and consider a 'reduced' problem involving a related operator $\mathcal{M} : \mathcal{V} \xrightarrow{\text{cont}} \mathcal{V}^*$ with $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$. This $\mathcal{M}$ will be monotone and coercive, which will enable us to show that $-\mathbf{A}$ generates a contraction semigroup $\mathbf{S}(\cdot)$ on $\mathcal{X}$. A similar construction applies to the linear problem, with $F(\cdot)$ replaced by the identity on $\mathcal{Y}$. To verify the hypothesis (H3) above we will, in each case, apply a result by Arendt and Batty [1]:

Theorem 2 *Let $\tilde{\mathbf{S}}(\cdot)$ be a C_0 contraction semigroup of linear operators on a Hilbert space $\mathcal{X}$ with infinitesimal generator $-\tilde{\mathbf{A}}$. If $\sigma(\tilde{\mathbf{A}})\bigcap i\mathbb{R}$ is at most countable and contains no point spectrum, then $\tilde{\mathbf{S}}(\cdot)$ is strongly asymptotically stable, i.e., $\tilde{\mathbf{S}}(t)x_0 \to 0$ as $t \to \infty$ for every $x_0 \in \mathcal{X}$.*

Our argument is then completed by verifying the hypothesis of Theorem 2 on $\sigma(\tilde{\mathbf{A}})$ — indeed, we will show for our examples that $\sigma(\tilde{\mathbf{A}})\bigcap i\mathbb{R} = \emptyset$ — obviously to be considered in the complexifications of the relevant spaces. This verification will similarly involve reduced problems.

Our key steps in the analysis of each of these examples will thus be the following:

(1) Reformulate the problem as a first order system
  (a) Specify the relevant Hilbert spaces: $\mathcal{X} = \mathcal{U} \times \mathcal{H}$, $\mathcal{Z} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{Z}^*$, etc.
  (b) Write (explicitly) the operator $\mathcal{A} : \mathcal{Z} \to \mathcal{Z}^*$ (and note its monotonicity).
(2) Show that $(\mathbf{A} + \mathbf{1}) : \mathcal{X} \supset \mathcal{D}(\mathbf{A}) \to \mathcal{X}$ is surjective. [This shows the nonlinear operator $\mathbf{A}$ is maximal monotone; the same argument shows the linear operator $\tilde{\mathbf{A}}$ is maximal monotone.]
(3) Show that $(\tilde{\mathbf{A}} - ir\mathbf{1})$ is invertible for $0 \neq r \in \mathbb{R}$
  (a) Write the equations to be solved
  (b) Use the 'first equation' to eliminate the $\mathcal{U}$-component, obtaining a 'reduced problem' for the $\mathcal{H}$-component
  (c) Formulate this in terms of a continuous operator $\mathcal{M}_r : \mathcal{V} \to \mathcal{V}^*$
  (d) Note that $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact and that $(\mathcal{M}_r + s\mathbf{1})$ is monotone and coercive for large $s > 0$ so $\mathbf{M}_r$ has compact resolvent
  (e) Use the detectability assumption to show $\mathbf{M}_r$ is injective and so invertible
(4) Show that $\tilde{\mathbf{A}}$ is invertible.

3 Example 1: Boundary feedback for the wave equation

We are here considering the damping of acoustic energy, as in (1.3), for the wave equation (1.1) defined on a bounded domain $\Omega \subset R^3$ with saturating velocity feedback in (1.2) — i.e., we consider finite energy solutions of

$$z_{tt} = \Delta z \ \text{ in } \Omega \qquad \text{with } z_\nu = -aF(az_t) \text{ on } \partial\Omega \tag{3.13}$$

where $a(\cdot)$ is a bounded measurable function on $\partial\Omega$ with nontrivial support Γ and F, defined in (1.6) for the observation $y = az_t$, gives the saturation at threshold $M > 0$. We will follow the four steps above to show that the associated acoustic energy $\mathcal{E}(t)$ of (1.3) always decays to 0 as $t \to \infty$.

We impose a technical detectability assumption regarding Γ:

(D) There is no eigenfunction: $\Delta z = -cz$ $(z_\nu \equiv 0)$ with $c \neq 0$ for which z vanishes on Γ.

Remark 2 *We note that, on the strength of Holmgren's Uniqueness Theorem, the detectability condition holds true whenever Γ is an open set in $\partial\Omega$.*

Theorem 3 *Assume the detectability condition* **(D)**. *Then we have $\mathcal{E}(t) \to 0$ as $t \to \infty$ for every solution of* (3.13) *with initial data of finite energy, where $\mathcal{E}(t)$ is defined by* (1.3).

PROOF: As Step 1 we will reformulate (3.13) as a first order system. The usual way to do this would be to consider state components (z, z_t), but that choice would lead to difficulties with degeneracy of the 'energy norm.' Instead, we introduce the Hilbert spaces

$$\mathcal{U} = L^2_{grad}(\Omega),\ \mathcal{H} = L^2(\Omega),\ \mathcal{X} = \mathcal{U} \times \mathcal{H},\ \mathcal{V} = H^1(\Omega),\ \mathcal{Z} = \mathcal{U} \times \mathcal{V} \tag{3.14}$$

where we note that $L^2_{grad}(\Omega) = \{\nabla h : h \in \mathcal{V}\}$ is a closed subspace of $L^2(\Omega \to \mathbb{R}^3)$ and so is a Hilbert space in its own right. Indeed, this follows from the fact that $L^2_{grad}(\Omega)$ coincides with the range of ∇ considered as a map : $\frac{H^1(\Omega)}{R} \to [L_2(\Omega)]^3$. Since this map is closed (in fact continuous) with continuous inverse, its range is closed.

We write the state as $\mathbf{x} = (\xi, u)^\top \in \mathcal{X}$, with ξ and u corresponding to ∇z and z_t, respectively, so $\frac{1}{2}\|\mathbf{x}\|^2_{\mathcal{X}}$ is just the energy $\mathcal{E}$ of (1.3). The wave equation (3.13) then takes the form

$$\begin{array}{l} \xi_t = \nabla u \\ u_t = \nabla \cdot \xi \end{array} \quad \text{on } \Omega \qquad \text{with } \xi \cdot \nu = -aF(au) \text{ on } \partial\Omega. \tag{3.15}$$

Note that if we are given the usual (finite energy) initial data: $z = \overset{\circ}{z} \in \mathcal{V}$ and $z_t = \overset{\circ}{z}{}' \in \mathcal{H}$ at $t = 0$, then the data for (3.15) will just be $\mathbf{x}_0 = (\nabla \overset{\circ}{z}, \overset{\circ}{z}{}')^\top \in \mathcal{X}$. If we can solve (3.15) for $\mathbf{x} = (\xi, u)^\top$, then we have $z_t \equiv u$ and we can recover z, itself, either from $z = \overset{\circ}{z} + \int_0^t u$ or from

$$\nabla z = \xi \quad \text{with} \int_\Omega z = \int_\Omega \overset{\circ}{z} + \int_0^t \left[\int_\Omega u \right].$$

We wish to consider (3.15) as $\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = 0$. It is preferable to give a weak formulation for $\mathcal{A}$, since just writing $\mathbf{A} = \begin{pmatrix} 0 & -\nabla \\ -\nabla\cdot & 0 \end{pmatrix}$ would hide the boundary conditions, including the feedback of particular interest to us, in the domain specification. A simple (formal) calculation shows that

$$\begin{aligned} \langle \tilde{\mathbf{x}}, \mathbf{A}\mathbf{x} \rangle_{\mathcal{X}} &= \int_\Omega \tilde{\mathbf{x}} \cdot \begin{pmatrix} 0 & -\nabla \\ -\nabla\cdot & 0 \end{pmatrix} \mathbf{x} \\ &= \textstyle\int_\Omega [\nabla \tilde{u} \cdot \xi - \tilde{\xi} \cdot \nabla u] + \int_\Gamma a\tilde{u}F(au) \end{aligned}$$

and, as this last makes continuous sense whenever $\tilde{\mathbf{x}}, \mathbf{x}$ are in $\mathcal{Z}$, we define an operator $\mathcal{A} : \mathcal{Z} \to \mathcal{Z}^*$ by

$$\mathcal{A}\mathbf{x} := \left[\tilde{\mathbf{x}} \mapsto \int_\Omega [\nabla \tilde{u} \cdot \xi - \tilde{\xi} \cdot \nabla u] + \int_\Gamma a\tilde{u}F(au) \right] \tag{3.16}$$

and, now simply specifying $\mathcal{D}(\mathbf{A})$ as the pre-image of $\mathcal{X}$ under $\mathcal{A}$, use this to define $\mathbf{A}\mathbf{x} := \mathcal{A}\mathbf{x}$. We easily see from (3.16) that (2.12) holds here so $\mathcal{A}$ (whence also $\mathbf{A}$) is monotone.

For Step 2 we must show solvability of: $\mathbf{A}\mathbf{x} + \mathbf{x} = \mathbf{f}$ for arbitrary $\mathbf{f} = (f, g)^\intercal \in \mathcal{X}$, i.e., find $(\xi, u)^\intercal \in \mathcal{D}(\mathbf{A})$ [noting that $\mathcal{D}(\mathbf{A}) \subset \mathcal{Z} \subset \mathcal{X}$] such that

$$\int_\Omega [\nabla \tilde{u} \cdot \xi - \tilde{\xi} \cdot \nabla u] + \int_\Gamma a\tilde{u}F(au) + \int_\Omega [\tilde{\xi} \cdot \xi + \tilde{u}u] = \int_\Omega [\tilde{\xi} \cdot f + \tilde{u}g]$$

for all $(\tilde{\xi}, \tilde{u})^\intercal \in \mathcal{Z}$. With $\tilde{u} = 0$ this gives $\xi = \nabla u + f$ so, setting

$$\begin{aligned} \mathcal{M}u &:= [\tilde{u} \mapsto \textstyle\int_\Omega [\nabla \tilde{u} \cdot \nabla u + \tilde{u}u] + \int_\Gamma a\tilde{u}F(au)] \in \mathcal{V}^* \quad \text{for } u \in \mathcal{V} \\ \tilde{f} &:= [\tilde{u} \mapsto \textstyle\int_\Omega [\nabla \tilde{u} \cdot f - \tilde{u}g]] \in \mathcal{V}^*, \end{aligned}$$

we have the reduced problem of solving $\mathcal{M}u = \tilde{f}$. Since we easily see that $\mathcal{M} : \mathcal{V} \to \mathcal{V}^*$ is coercive on $\mathcal{V}$ as well as continuous and monotone, this can be solved for $u \in \mathcal{V} \subset \mathcal{H}$ and we then set $\xi = \nabla u + f$ to get the desired

$\mathbf{x} = (\xi, u)^\top \in \mathcal{D}(\mathbf{A}) \subset \mathcal{X}$. This proves that $(\mathbf{A}+\mathbf{1})$ is surjective to $\mathcal{X}$ so $\mathbf{A}$ is maximal monotone and $-\mathbf{A}$ generates a contraction semigroup $\mathbf{S}(\cdot)$ on $\mathcal{X}$.

The same argument shows that $-\tilde{\mathbf{A}}$ (where, we recall, $-\tilde{\mathbf{A}}$ corresponds to the linear problem) generates a contraction semigroup $\tilde{\mathbf{S}}(\cdot)$ on $\mathcal{X}$ so we have verified the hypothesis (H1) of Theorem 1. The observation operator for this example is

$$\mathbf{C} : \mathcal{Z} \longrightarrow \mathcal{Y} := L^2(\Gamma) \subset L^2(\partial\Omega) : \mathbf{x} = (\xi, u)^\top \longmapsto a[\text{trace of } u] \tag{3.17}$$

so $\psi : \mathbf{x} \mapsto \|y\|_{\mathcal{Y}}^2 = \|\mathbf{C}\mathbf{x}\|_{\mathcal{Y}}^2$ is uniformly continuous on bounded subsets of $\mathcal{Z}$ — although it is not well-defined on $\mathcal{X}$. Since $\{\mathbf{x} \in \mathcal{D}(\mathbf{A}) : \|\mathbf{x}\|_{\mathcal{X}} \leq \mu,\ \|(\mathbf{A}+\mathbf{1})\mathbf{x}\|_{\mathcal{X}} \leq \lambda+\mu\}$ is bounded in $\mathcal{Z}$ (and similarly for $\tilde{\mathbf{A}}$), we have verified (H2-1) and (H2-3). Then (H2-2) follows from (2.12) and the definition (1.6). Finally, (H2-4), with $\alpha = M^2$, follows from the definitions of $\mathbf{A}, \tilde{\mathbf{A}}$, which differ only for $\|y\| > M$.

For Steps 3,4 we are considering solvability, for arbitrary $\mathbf{f} = (f, g)^\top \in \mathcal{X}$, of: $(\tilde{\mathbf{A}} - ir\mathbf{1})\mathbf{x} = \mathbf{f}$ so we are seeking $\mathbf{x} = (\xi, u)^\top \in \mathcal{D}(\tilde{\mathbf{A}}) \subset \mathcal{Z} \subset \mathcal{X}$ such that

$$\int_\Omega [\overline{\nabla\tilde{u}} \cdot \xi - \overline{\tilde{\xi}} \cdot \nabla u] + \int_\Gamma a^2 \overline{\tilde{u}} u + \int_\Omega ir[\overline{\tilde{\xi}} \cdot \xi + \overline{\tilde{u}} u] = \int_\Omega [\overline{\tilde{\xi}} \cdot f + \overline{\tilde{u}} g]$$

for all $(\tilde{\xi}, \tilde{u})^\top \in \mathcal{Z}$ — where, of course, the overlines indicate conjugation since we must now work in the complexified spaces. As earlier, with $\tilde{u} = 0$ we will get: $-\nabla u + ir\xi - f = 0$ in $\mathcal{U}$.

When $r \neq 0$ we may then use this to eliminate ξ: with $\xi = (\nabla u + f)/(-ir)$ the equation is equivalent to seeking $u \in \mathcal{V} \subset \mathcal{H}$ such that: $\mathcal{M}_r u = \tilde{f}$ where

$$\begin{aligned} \mathcal{M}_r u &:= \left[\tilde{u} \mapsto \textstyle\int_\Omega [\overline{\nabla\tilde{u}} \cdot \nabla u - r^2 \overline{\tilde{u}} u] - ir \int_\Gamma a^2 \overline{\tilde{u}} u\right] \in \mathcal{V}^* \quad \text{for } u \in \mathcal{V} \\ \tilde{f} &:= \left[\tilde{u} \mapsto \textstyle\int_\Omega [\overline{\nabla\tilde{u}} \cdot f - \overline{\tilde{u}} g]\right] \in \mathcal{V}^*. \end{aligned}$$

We easily verify that $(\mathcal{M}_r + s\mathbf{1}) : \mathcal{V} \to \mathcal{V}^*$ is coercive as well as continuous and monotone when $s > r^2$, so there exists $(\mathcal{M}_r + s\mathbf{1})^{-1} : \mathcal{V}^* \to \mathcal{V} \hookrightarrow \mathcal{H}$. Since the embedding $\mathcal{V} := H^1(\Omega) \hookrightarrow \mathcal{H} := L^2(\Omega)$ is compact, this shows $\mathbf{M}_r$ has compact resolvent and so is continuously invertible on $\mathcal{H}$, whence $(\tilde{\mathbf{A}} - ir\mathbf{1})$ is continuously invertible on $\mathcal{X}$, if $\mathcal{M}_r$ is injective. Thus we wish to show $u = 0$ when $\mathcal{M}_r u = 0$. The imaginary part of $\langle u, \mathcal{M}_r u\rangle_{[\mathcal{V}:\mathcal{V}^*]}$ is $\int_\Gamma a^2|u|^2$ so $\mathcal{M}_r u = 0$ gives $y = au = 0 \in \mathcal{Y}$ (i.e., u vanishes on $\operatorname{supp}(a)$) and the homogeneous equation becomes simply the weak form of the eigenfunction equation: $-\Delta u = r^2 u$ with $u_\nu = 0$. Since we are here considering $r \neq 0$ so $u \neq$const., the assumption **(D')** above then gives $u = 0$. This shows that for $0 \neq r \in \mathbb{R}$ we have $ir \notin \sigma(\tilde{\mathbf{A}})$.

When $r = 0$ above, we must proceed differently: we have $\nabla u + f = 0$ and will now use this to eliminate u. Note that $f \in \mathcal{U}$ so there is some $h \in \mathcal{V}$ with $f = \nabla h$; thus $\nabla(u + h) = 0$ gives $u = -h + c$ where the constant c is to be determined. We also note that we are seeking $\xi \in \mathcal{U}$ in the form $\xi = \nabla v$ by the definition of $L^2_{grad}(\Omega)$. Then our equation, with $\tilde{\xi} = \nabla \tilde{u}$, just becomes the weak form of

$$-\Delta v = g \qquad \text{with } v_\nu = a^2(h - c).$$

It is well-known that this has a solution if and only if $\int_\Omega g + \int_\Gamma v_\nu = 0$, so we have the unique determination that $c = [\int_\Omega g + \int_\Gamma a^2 h] \,/\, [\int_\Gamma a^2]$. The resulting v is not unique, but this indeterminacy is nugatory: $\xi = \nabla v$ is unique in $\mathcal{U}$. This establishes a map $\tilde{\mathbf{A}}^{-1} : \mathcal{X} \to \mathcal{D}(\tilde{\mathbf{A}}) \subset \mathcal{X}$ and, since $\tilde{\mathbf{A}}$ is clearly a closed linear operator, the map $\tilde{\mathbf{A}}^{-1}$ is continuous, showing that $0 \notin \sigma(\tilde{\mathbf{A}})$ and so completing Step 4.

This verifies the spectral hypothesis of Theorem 2 which in turn, verifies (H3), completing the verification of the hypotheses of Theorem 1. It now follows from that Theorem that we have the desired asymptotic strong stability: $\mathbf{S}(t)\mathbf{x}_0 \to 0$ in $\mathcal{X}$ as $t \to \infty$. ■

Remark 3 *Note that we are* not *asserting that $z \to 0$. [Indeed, it is not even clear from our results that we can be certain that z remains bounded as $t \to \infty$ since we do not know that $\int_\Omega z$ remains bounded: we know convergence of $\int_\Omega z_t$ to 0 as $t \to \infty$, but not its integrability on $\mathbb{R}_+$, which would give the boundedness of z.] We* have, *however, shown — as promised — that the acoustic energy $\mathcal{E}(t)$ will always go to 0 as $t \to \infty$.*

4 Example 2: The structural acoustic model

We recall the coupled feedback system (1.7) from Section 1:

$$\begin{aligned} &z_{tt} - \Delta z = 0 \text{ on } \Omega, \qquad && z_\nu = \begin{cases} w_t \text{ on } \Gamma \subset \partial\Omega \\ -cz \text{ on } \Gamma' = \partial\Omega \setminus \Gamma \end{cases} \\ &w_{tt} + \Delta^2 w + a\Delta F(\Delta[a w_t]) = -z_t \text{ on } \Gamma, && \\ & && w,\ \Delta w = 0 \text{ at } \partial\Gamma \end{aligned} \tag{4.18}$$

where a is a nontrivial smooth function on $\Gamma \subset \partial\Omega$ with support in the interior of Γ and the function c is assumed positive.

We consider finite energy solutions of (4.18) with

$$\mathcal{E}(t) = \mathcal{E}_{kinetic} + \mathcal{E}_{potential} \tag{4.19}$$

where, for this example,

$$\begin{aligned} \mathcal{E}_{kinetic} &\equiv \tfrac{1}{2}\left[\int_\Omega |z_t|^2 + \int_\Gamma |w_t|^2\right] \\ \mathcal{E}_{potential} &\equiv \tfrac{1}{2}\left[\int_\Omega |\nabla z|^2 + \int_{\Gamma'} cz^2 + \int_\Gamma |\Delta w|^2\right] \end{aligned}$$

As before, if we (formally) apply the Divergence Theorem we obtain $d\mathcal{E}/dt = -\int_\Gamma yF(y) \leq 0$ (now with $y := \Delta[aw_t]$), so our feedback is again dissipative: the energy is nonincreasing.

Our main aim is to ensure that we have $\mathcal{E}(t) \to 0$ as $t \to 0$ for all finite energy initial data. This property was shown in [2] for the *linear model* (i.e., when $F \leftrightarrow \mathbf{1}$) with Kelvin-Voigt damping on the wall Γ (i.e., $a(x) > 0$). The theorem stated below extends the result of [2] to the *nonlinear* case of a saturated feedback with partial damping active on a subset of Γ.

Theorem 4 *Assume that the support of $a(\cdot)$ contains a nonempty open set $\Gamma_0 \subset \Gamma$. Then the system* (4.18) *is strongly stable: $\mathcal{E}(t) \to 0$ as $t \to \infty$* (4.19) *for every solution with initial data of finite energy.*

Remark 4 *As mentioned before, this example illustrates a situation when strong stability is obtained in an absence of compactness for the resolvent operator. We also note that the same result holds for other choices of well-posed boundary conditions (free, clamped) associated with the plate equation in (4.18).*

PROOF: We proceed with the four steps at the end of Section 2, but present these in somewhat less detail than in the last section.

As Step 1 we begin by reformulating (4.18) as a first-order system on an appropriate state space. We will here take

$$\begin{aligned} \mathcal{U} = \mathcal{V} := H^1(\Omega) \times H^2_*(\Gamma) \quad &\text{with } H^2_*(\Gamma) := \{w \in H^2(\Gamma) : w|_{\partial\Gamma} = 0\} \\ \mathcal{H} := L^2(\Omega) \times L^2(\Gamma) \qquad &\mathcal{X} := \mathcal{U} \times \mathcal{H} \qquad \mathcal{Z} := \mathcal{U} \times \mathcal{U}. \end{aligned} \tag{4.20}$$

The inner product of the Hilbert space $\mathcal{X}$ will be

$$\langle \tilde{\mathbf{x}}, \mathbf{x} \rangle := \int_{\Omega} [\nabla \tilde{z} \cdot \nabla z + \tilde{u} u] + \int_{\Gamma'} c \tilde{z} z + \int_{\Gamma} [(\Delta \tilde{w})(\Delta w) + v \tilde{v}] \,, \tag{4.21}$$

noting that with nontrivial $c > 0$ on Γ' this induces a norm on $\mathcal{X}$.

We will write $\xi = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{U}$ and $\omega = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H}$ so, as is usual for a second order system, we are taking the state to be $\mathbf{x} = (\xi, \omega)^{\mathsf{T}} = (z, w, u, v)^{\mathsf{T}} \in \mathcal{X}$ with u, v to be identified with z_t, w_t, giving $\omega = \xi_t$. Then (4.18) becomes

$$\dot{\mathbf{x}} + \mathbf{A}\mathbf{x} = 0 \quad \text{where } \mathbf{A} : \begin{pmatrix} z \\ w \\ u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} -u \\ -v \\ -\Delta z \\ \Delta^2 w + u|_{\Gamma} + a \Delta F(\Delta[av]) \end{pmatrix} \tag{4.22}$$

with the domain $\mathcal{D}(\mathbf{A})$ given by:

$$\mathcal{D}(\mathbf{A}) := \left\{ \mathbf{x} \in \mathcal{Z} : \begin{array}{cc} \Delta z \in L^2(\Omega), & \Delta^2 w + a \Delta F(\Delta[av]) \in L^2(\Gamma), \\ z_\nu = \begin{cases} -cz \text{ on } \Gamma' \\ v \text{ on } \Gamma, \end{cases} & \Delta w = 0 \text{ at } \partial\Gamma \end{array} \right\}.$$

Remark 5 *Note that the domain $D(\mathbf{A})$ is not compactly imbedded in $\mathcal{X}$. To see this it suffices to consider the following set*

$$\begin{aligned} \tilde{D} \equiv \{(z, w, u, v) \in B_{\mathcal{X}}(0, M); supp\ w \in \Gamma_0, v + w = 0, \\ \Delta z = 0, z_\nu + cz = 0, \text{ on } \Gamma', z_\nu = v; \text{ on } \Gamma\} \end{aligned} \tag{4.23}$$

Clearly $\tilde{D}\mathcal{D}(\mathbf{A})$ is a bounded (in $\mathcal{X}$) subset of $\mathcal{D}(\mathbf{A})$, yet not compact in $\mathcal{X}$.

The weak formulation of this is

$$\begin{aligned} \langle \tilde{\mathbf{x}}, \mathbf{A}\mathbf{x} \rangle_{\mathcal{X}} = \textstyle\int_{\Omega} [\nabla \tilde{u} \cdot \nabla z - \nabla \tilde{z} \cdot \nabla u] + \int_{\Gamma'} c[\tilde{u} z - \tilde{z} u] + \int_{\Gamma} [\tilde{v} u - \tilde{u} v] \\ + \textstyle\int_{\Gamma} [(\Delta \tilde{v})(\Delta w) - (\Delta \tilde{w})(\Delta v)] + \int_{\Gamma} \Delta[a\tilde{v}]\, F(\Delta[av]) \end{aligned}$$

and, as this last makes continuous sense whenever $\tilde{\mathbf{x}}, \mathbf{x}$ are in $\mathcal{Z} := \mathcal{U} \times \mathcal{U}$, we define, as a realization of $\mathbf{A}$, the operator $\mathcal{A} : \mathcal{Z} \to \mathcal{Z}^*$ for this example by

$$\mathcal{A}\mathbf{x} := \left[\tilde{\mathbf{x}} \mapsto \begin{array}{c} \int_\Omega [\nabla \tilde{u} \cdot \nabla z - \nabla \tilde{z} \cdot \nabla u] + \int_{\Gamma'} c[\tilde{u} z - \tilde{z} u] + \int_\Gamma [\tilde{v} u - \tilde{u} v] \\ + \int_\Gamma [(\Delta \tilde{v})(\Delta w) - (\Delta \tilde{w})(\Delta v)] + \int_\Gamma \Delta[a\tilde{v}]\, F(\Delta[av]) \end{array} \right] \tag{4.24}$$

and note that we again have (2.12) with $y = \Delta[av] \in \mathcal{Y} = L^2(\Gamma)$ well-defined for $v \in H^2_*(\Gamma)$; we define $\tilde{\mathbf{A}}$ much as in (4.24), with $F \hookleftarrow \mathbf{1}$. Then $\mathbf{A} : \mathcal{X} \supset \mathcal{D}(\mathbf{A}) \to \mathcal{X}$ and similarly $\tilde{\mathbf{A}} : \mathcal{X} \supset \mathcal{D}(\tilde{\mathbf{A}}) \to \mathcal{X}$ are well-defined and monotone on $\mathcal{X}$.

For Step 2 we again wish to solve $\mathcal{A}\mathbf{x} + \mathbf{x} = \mathbf{f}_*$ with $\mathbf{f}_*$ arbitrary in $\mathcal{X} \subset \mathcal{Z}^*$. While, as in the previous section, we could do all our computations using the weak formulation, we will here write things in operator form. Thus, the equation: $\mathcal{A}\mathbf{x} + \mathbf{x} = \mathbf{f}_*$ becomes the system

$$\begin{array}{rr} -u + z = z_* \in H^1(\Omega) & -v + w = w_* \in H^2_*(\Gamma) \\ -\Delta z + u = u_* \in L^2(\Omega) & z_\nu = \begin{cases} -cz & \text{on } \Gamma' \\ v & \text{on } \Gamma \end{cases} \\ \Delta^2 w + u|_\Gamma + a\Delta\, F(\Delta[av]) + v = v_* \in L^2(\Gamma) & w = 0 = \Delta w \text{ at } \partial\Gamma \end{array}$$

The first equations give $u = z - z_*$, $v = w - w_*$ which we use to eliminate u, v in the last equations, obtaining the reduced system for $\xi = (z, w)^\intercal$, which we can now write as: $\mathcal{M}\xi = \tilde{f} \in \mathcal{V}^*$ where

$$\mathcal{M}\xi : \begin{pmatrix} \tilde{u} \\ tv \end{pmatrix} \longmapsto \begin{cases} \int_\Omega [\nabla \tilde{u} \cdot \nabla z + \tilde{u} z] + \int_{\Gamma'} c\tilde{u} z \\ + \int_\Gamma [(\Delta \tilde{v})(\Delta w) + \tilde{v} w] + \int_\Gamma (\Delta[a\tilde{v}])\, F(\Delta[a(w - w_*)]) \end{cases}$$

$$\tilde{f} : \begin{pmatrix} \tilde{u} \\ tv \end{pmatrix} \longmapsto \int_\Omega \tilde{u}(u_* + z_*) + \int_{\Gamma'} c\tilde{u} w_* + \int_\Gamma \tilde{v}(v_* + z_*|_\Gamma + w_*)$$

As in the previous section, $\mathcal{M} : \mathcal{V} \to \mathcal{V}^*$ is coercive, continuous, and monotone on $\mathcal{V}$ — hence surjective [3] — and we use this to show that $\mathbf{A}$ is maximal monotone, hence $-\mathbf{A}$ generates a contraction semigroup $\mathbf{S}(\cdot)$ on $\mathcal{X}$; again the same argument also shows that $-\tilde{\mathbf{A}}$ generates a contraction semigroup $\tilde{\mathbf{S}}(\cdot)$ on $\mathcal{X}$. We have verified the hypothesis (H1) of Theorem 1. The observation operator for this second example is

$$\mathbf{C} : \mathcal{Z} \longrightarrow \mathcal{Y} := L^2(\Gamma) : \mathbf{x} = (z, w, u, v)^\intercal \longmapsto y := \Delta[av] \tag{4.25}$$

and $\psi : \mathbf{x} \mapsto \|y\|_{\mathcal{Y}}^2 = \|\mathbf{C}\mathbf{x}\|_{\mathcal{Y}}^2$ is uniformly continuous on bounded subsets of $\mathcal{Z}$ since that bounds v in $H_*^2(\Gamma)$ and we have assumed the coefficient function $a(\cdot)$ is smooth enough. The remainder of the verification for the hypothesis (H2) of Theorem 1 is essentially as for Example 1.

For Steps 3,4 we are again considering solvability, for arbitrary $\mathbf{f}_* \in \mathcal{X}$, of: $(\tilde{\mathbf{A}} - ir\mathbf{1})\mathbf{x} = \mathbf{f}_*$ in the context of the complexified spaces. As above, we use the first equations of the corresponding system to get: $u = irz - z_*$, $v = irw - w_*$ and use these to eliminate $u.v$ in the last equations to obtain the reduced system:

$$
\begin{aligned}
-\Delta z - r^2 z = u_* + irz_* \qquad & z_\nu = \begin{cases} -cz & \text{on } \Gamma' \\ irw - w_* & \text{on } \Gamma \end{cases} \\
\Delta^2 w - r^2 w + ir\left[a\Delta^2 a w + z|_\Gamma\right] = a\Delta^2 a w_* + irw_* & \\
& w = 0 = \Delta w \text{ at } \partial\Gamma.
\end{aligned}
\tag{4.26}
$$

Again, this may be expressed in terms of a continuouslinear operator $\mathcal{M}_r : \mathcal{V} \to \mathcal{V}^*$. Without explicitly writing the full form of $\mathcal{M}_r$, we note that, for $\xi = (z, w)^\intercal \in \mathcal{V}$, we have

$$
\begin{aligned}
\langle \xi, (\mathcal{M}_r + s\mathbf{1})\xi \rangle_{\mathcal{H}} = & \int_\Omega [-\overline{z}(\Delta z) + (s - r^2)|z|^2] \\
& + \int_\Gamma [\overline{w}(\Delta^2 w) + (s - r^2)|w|^2] + ir \int_\Gamma \overline{w}[a\Delta^2 w + z] \\
= & \int_\Omega [|\nabla z|^2 + (s - r^2)|z|^2] + \int_{\Gamma'} c|z|^2 \\
& + r \int_\Gamma c \, Im\{\overline{w} z|_\Gamma\} + \int_\Gamma [|\Delta w|^2 + (s - r^2)|w|^2] \\
& + ir \int_\Gamma |\Delta(aw)|^2.
\end{aligned}
\tag{4.27}
$$

Since trace: $H^1(\Omega) \to L^2(\partial\Omega)$ is compact, we have an estimate $\|z\|_\Gamma \le \varepsilon \|z\|_{H^1(\Omega)}$ $+ C_\varepsilon \|z\|_{L^2(\Omega)}$ and one easily sees that $(\mathcal{M}_r + s\mathbf{1})$ is monotone and coercive for large enough $s \in \mathbb{R}$. As for the previous example, this shows $\mathbf{M}_r$ has compact resolvent (for arbitrary $r \in \mathbb{R}$) so invertibility will folow from injectivity. Setting $r = 0$ in (4.27), we see that this is immediate in that case — $\mathcal{M}_0 \xi = 0$ gives $0 = \langle \xi, \mathbf{M}_0 \xi \rangle = \|\xi\|_{\mathcal{V}}^2$ so $\xi = 0$, completing Step 4.

For Step 3 we wish to show that the hypothesis of Theorem 4 regarding the support of $a(\cdot)$ is sufficient to ensure the requisite detectability. This will now involve two separate applications of the classical Holmgren Uniqueness Theorem, once for each component of the coupled system. Taking the imaginary part of (4.27) when $s = 0$ and $\mathcal{M}_r \xi = 0$, we see that this gives $\Delta(aw) \equiv 0$ on Γ and this, with the boundary condition: $w = 0 = \Delta w$ at $\partial\Gamma$, gives $aw \equiv 0$ on Γ so $w \equiv 0$ on Γ_0, whence also $\Delta^2 w \equiv 0$ on Γ_0. Using this in the second equation of (4.26), now taken as homogeneous and interpreted pointwise, then gives $z \equiv 0$ on Γ_0. We also have $z_\nu \equiv 0$ on Γ_0 from the boundary condition of the first equation of (4.26) and so can apply the Holmgren Theorem to assert that z must vanish on all of Ω. With this, the second equation of (4.26) becomes: $\Delta^2 w - r^2 w = 0$ and, with $w \equiv 0$ on the open set Γ_0, we may apply Holmgren's Unique Continuation Theorem to this fourth order elliptic equation to see that $w \equiv 0$ on all of Γ. This shows that $ir \notin \sigma(\tilde{\mathbf{A}})$ for arbitrary $r \in \Re$, verifying (H3) of Theorem 1 by use of the Arendt-Batty Theorem 2, and so completing our proof. ∎

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