A note on stabilization with saturating feedback

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Abstract. We assume, for a distributed parameter control system, that a linear stabilizing feedback is available. We then seek a stabilizing feedback, necessarily nonlinear, subject to an *a priori* bound on the control.

1. Introduction

We are concerned with feedback stabilization of a linear system

$$\dot{x} = \mathbf{A}x \tag{1}$$

for which it is known that the solution semigroup $\mathbf{S}(\cdot)$ generated by the (unbounded) operator $\mathbf{A}$ is a contraction semigroup on the Hilbert space $\mathcal{H}$. We suppose that control actuation is available

$$\dot{x} = \mathbf{A}x + \mathbf{B}z \tag{2}$$

where $\mathbf{B}$ is a linear operator from some auxiliary Hilbert space $\mathcal{U}$ to $\mathcal{H}$. If the observation $y = \mathbf{B}^*x$ is also available, then the feedback

$$z = -y = -\mathbf{B}^*x \tag{3}$$

is clearly dissipative ($\langle x, \mathbf{B}z\rangle = -\|y\|^2 \le 0$) and we assume *a priori* that this linear feedback system

$$\dot{x} = \mathbf{A}x - \mathbf{B}\mathbf{B}^*x = (\mathbf{A}+\mathbf{\Gamma})x \qquad \text{with } \mathbf{\Gamma} := -\mathbf{B}\mathbf{B}^* \tag{4}$$

is asymptotically stable:

$$\text{For every solution of (4) one has } x(t) \to 0 \text{ as } t \to \infty. \tag{5}$$

Our principal concern here is that the control actuation (2) might be available only subject to a control constraint: $\|z(t)\| \le 1$ (perhaps after re-scaling). In such a situation we wish to replace (3) by the 'saturating control law'

$$z = F(y) := \begin{cases} -y & (\|y\| \le 1) \\ -y/\|y\| & (\|y\| \ge 1) \end{cases} \qquad (y = \mathbf{B}^*x), \tag{6}$$

i.e., we would try as hard as possible, subject to the constraint, to employ (3). The resulting feedback system is now nonlinear:

$$\dot{x} = \mathbf{A}x + G(x) \qquad \text{with } G : \xi \mapsto \mathbf{B}F(\mathbf{B}^*\xi) \tag{7}$$

and in some sense lies 'between' (1) and (4) — letting $\mathbf{\Gamma}$ be the dissipative linear operator: $\xi \mapsto -\mathbf{B}\mathbf{B}^*\xi$ of (4), the control term $G(\cdot)$ of (7) has the form (with $\rho := \min\{1, 1/\|y\|\}$):

$$G(\xi) := \mathbf{B}F(\mathbf{B}^*\xi) = \rho\mathbf{\Gamma}\xi \quad \text{with } 0 < \rho \le 1 \tag{8}$$

where $\rho = 0$ would reduce the dynamics to (1) while $\rho = 1$, on the other hand, would correspond to (4). Our concern here is whether we retain asymptotic stability for this saturating control — i.e., we ask: Is it true that

$$\text{For every solution of (7) one has } x(t) \to 0 \text{ as } t \to \infty? \tag{9}$$

This form of control was treated for the distributed parameter case by Slemrod [7] and that paper has been a primary stimulus to this. We note that Slemrod, in [7], was particularly interested in demonstrating (9) for a specific beam model; he imposed hypotheses which were satisfied in that model and which then permitted application of some available general results, especially for verification of (5) for that model. Here we wish to take (5), the asymptotic stability of the linear feedback system, as given *a priori* so, under weaker supplementary assumptions than in [7] on the operators $\mathbf{A}, \mathbf{B}$, we will be able to show the asymptotic stability of the system (7) using the saturating feedback (6) as a simple consequence of its relation to (1) and (4). In particular, for this implication we will not need to impose compactness assumptions either on the resolvent $(\lambda\mathbf{1} - \mathbf{A})^{-1}$ or on $\mathbf{B}$ — indeed, it will be possible (in Section 4 below) even to relax the continuity requirement for $\mathbf{B}$. For further treatment of some more detailed examples, note [5], which will concentrate on the cases of boundary control and non-compact resolvents. Our analysis here will rely primarily on the theory of semigroups of nonlinear operators, for which we will take [1] and [8] as principal references.

What we have in mind, primarily, is a system governed by a partial differential equation (as a wave equation or a non-dissipative beam or plate equation) for which energy is conserved and can be used to furnish a 'natural' Hilbert space norm. (Note that in this case $\mathbf{S}(\cdot)$ would be a unitary group.) Of course, some damping might already be included in (1), but we view the stabilization as principally coming from the use of feedback control as a mechanism for energy dissipation. Quite different in spirit, although also covered by this hypothesis, would be a diffusion problem in which, in a context of no-flux boundary conditions, $\mathbf{A}$ has a nontrivial nullspace so some additional dissipation would be needed for asymptotic stability.

As an example, consider for (2) the controlled damped wave equation

$$w_{tt} - a^2\Delta w_t = \Delta w + bz \quad \text{on } \Omega \quad \text{with } w = 0 \text{ at } \partial\Omega \tag{10}$$

To formulate this as above, we take $z \in \mathcal{U} := L^2(\Omega')$ where $\Omega \supset \Omega' \supset \operatorname{supp} b(\cdot)$, introduce $u = w_t$ and $v = \nabla w$, set $x := \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H} := L^2(\Omega) \times L^2_{\text{grad}}(\Omega)$, and set

$$\mathbf{A} := \mathbf{A}_0 + \mathbf{A}_1 = \begin{pmatrix} 0 & \nabla\cdot \\ \nabla & 0 \end{pmatrix} + \begin{pmatrix} a^2\Delta & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{B}z := \begin{pmatrix} bz \\ 0 \end{pmatrix}$$

with suitable specification of the domain $\mathcal{D}(\mathbf{A})$. Then (2) becomes (10) and (4) becomes

$$w_{tt} - a^2\Delta w_t = \Delta w - b^2 w_t$$

This is clearly dissipative and, even with $a \equiv 0$, one expects (5) for nontrivial b by results of scattering theory — although a uniform stabilization rate would necessarily depend on the support of $b(\cdot)$. Note also that this $\mathbf{A}$ will not have compact resolvent and if the support of a is small this term will not give stability without the feedback.

2. Asymptotic stability

In this section we present the major result of this note: an affirmative answer to the question raised about (9). We have explicitly assumed that $\mathbf{A}$ is the infinitesimal generator of a contraction semigroup $\mathbf{S}(\cdot)$ on the Hilbert space $\mathcal{H}$ and implicitly assumed, in that discusssion, the boundedness of $\mathbf{B} : \mathcal{U} \to \mathcal{H}$. For the moment (until Section 4) we continue to assume continuity of $\mathbf{B}$ so $G : \xi \mapsto \mathbf{B}F(\mathbf{B}^*\xi)$ is uniformly Lipschitzian: $\mathcal{H} \to \mathcal{H}$. We will now wish to consider the feedback system

$$\begin{aligned} &\dot{x} = \mathbf{A}x + \mathbf{B}z \ \text{ with } z = G[x], \text{ i.e.,} \\ &z = \min\{1,\ 1/\|\mathbf{B}^*x\|\}\,\mathbf{\Gamma}x \quad (\mathbf{\Gamma} := -\mathbf{B}\mathbf{B}^*) \end{aligned} \tag{11}$$

Theorem 1. *Let $\mathbf{A}$ be the infinitesimal generator of a C_0 contraction semigroup on $\mathcal{H}$ and let $\mathbf{B} : \mathcal{U} \to \mathcal{H}$ be continuous. Then* (11) *will be* asymptotically output stable *— i.e.,*

$$\text{For every solution of (11) one has } y(t) \to 0 \text{ as } t \to \infty. \tag{12}$$

where $y(t) := \mathbf{B}^*x(t)$ is the output.

Proof. There is no question about solvability of (11) — one need only consider (compare [3]) the usual integral equation for the mild solution

$$x(t) = \mathbf{S}(t)x_0 + \int_0^t \mathbf{S}(t-s)G[x(s)]\,ds, \tag{13}$$

noting that $G[\cdot] = \mathbf{B}F[\mathbf{B}^*\cdot]$ is uniformly Lipschitzian. As an alternative argument, one notes first that, as $\mathbf{A}$ is the infinitesimal generator of a contraction semigroup, it must be m-dissipative (In the Hilbert space context this is equivalent to $-\mathbf{A}$ being maximal monotone) and then that $-G(\cdot)$ is the gradient of the convex continuous functional

$$\begin{aligned} &\varphi : \xi \mapsto \varphi_1(\|\mathbf{B}^*\xi\|) \\ &\qquad\qquad \text{with} \quad \varphi_1(r) := \begin{cases} (1/2)r^2 & (0 \le r \le 1) \\ r - 1/2 & (r \ge 1) \end{cases} \end{aligned} \tag{14}$$

whence G is also m-dissipative. Since $\mathcal{D}(G) = \mathcal{H}$, it follows (cf., [1] or [8]) that $\hat{\mathbf{T}} := (\mathbf{A} + G)$ is m-dissipative so $\hat{\mathbf{T}}$ generates a contraction semigroup $\mathbf{S}_G(\cdot)$ on $\overline{\mathcal{D}(\mathbf{A})} = \mathcal{H}$, giving the solution of (7) — i.e., the solution of (11) with initial data $x(0) = x_0$ is given by $x(t) = \mathbf{S}_G(t)[x_0]$ with $\|\mathbf{S}_G(t)[\xi] - \mathbf{S}_G(t)[\xi']\| \le \|\xi - \xi'\|$ for all $t \ge 0$. More precisely, we note that, for any solution x of (11), we have

$$\left(\tfrac{1}{2}\|x\|^2\right)^{\cdot} = \langle x, \dot{x}\rangle = \langle x, \mathbf{A}x\rangle + \langle x, G(x)\rangle \le \langle x, G(x)\rangle$$

whence

$$\|x(t)\|^2 - \|x(s)\|^2 \le 2\int_s^t \langle x, G(x)\rangle\,dr = -2\int_s^t \rho\|y\|^2\,dr \tag{15}$$

with $\rho = \min\{1, 1/\|y\|\}$ as in (8).

As given, this argument is purely formal, since the treatment above of $\left(\|x\|^2\right)^{\cdot} = 2\langle x, \dot{x}\rangle$ is legitimate pointwise only if $x \in \mathcal{D}(\mathbf{A})$. However, a standard argument enables us to get the desired consequence (15) in any case: we let $x_\mu(\cdot)$ be the solution of $(11)_\mu$ — i.e., of (11) with $\mathbf{A}$ replaced by its Yosida approximation $\mathbf{A}_\mu := \mu\mathbf{A}(\mu\mathbf{1} - \mathbf{A})^{-1}$ — and note first that $(15)_\mu$ is valid (as $\mathbf{A}_\mu$ is continuous) and then that as $\mu \to \infty$ one has $x_\mu(t) \to x(t)$ uniformly on any $[0, T]$ so, in the limit, we obtain (15).

Two immediate consequences of (15) are that:

- $\|x(\cdot)\|$ is nonincreasing so x (hence y) is bounded and
- since $\rho(\cdot)$ in (15) must be bounded away from 0 for bounded $\|y\|$, i.e., on $\mathbb{R}_+$, the output $y(\cdot)$ must be in $L^2(\mathbb{R}_+ \to \mathcal{U})$.

Now consider any particular solution $x(\cdot) = \mathbf{S}_G(t)[x_0]$ of (11); clearly

$$\|x(t)\| = \|\mathbf{S}_G(t-s)[x(s)]\| \le \|x(s)\| \le \|x_0\| \qquad (0 \le s \le t).$$

We next note that $x(\cdot)$ (and so $y(\cdot)$ also, by the assumed boundedness of $\mathbf{B}^*$) is uniformly continuous since

$$\|x(t) - x(s)\| = \|\mathbf{S}_G(s)[\mathbf{S}_G(t-s)[x_0]] - \mathbf{S}_G(s)[x_0]\| \le \|\mathbf{S}_G(t-s)[x_0] - x_0\|$$

and $\mathbf{S}_G(\cdot)$ is continuous at 0 in the strong (pointwise) topology.

We now prove (12) by contradiction: supposing it were false, there would be a solution x as above and an increasing sequence of times $t_k \to \infty$ such that $\|y(t_k)\| > 2\varepsilon$ for some fixed $\varepsilon > 0$. By the uniform continuity of $y(\cdot)$ shown above, we have an interval $\mathcal{I}_k$ centered at t_k and of width at least $\delta = \delta(\varepsilon)$ on which $\|y\| > \varepsilon$ so, since the boundedness of $\|y\|$ ensures existence of some $\rho_* > 0$ with $\rho \geq \rho_*$ uniformly, the contribution of each $\mathcal{I}_k$ to the integral of $-\langle x, G(x)\rangle$ must be at least $\rho_* \varepsilon^2 \delta > 0$ and, as we may assume without loss of generality that these intervals are disjoint, the total contribution up to $t_k + \delta$ must then be at least $k\rho_* \varepsilon^2 \delta$. Since (15) implies $-\int_0^\infty \langle x, G(x)\rangle \leq \|x_0\|^2$ this gives a contradiction. ■ ■

Corollary 1. *If, in addition to the hypotheses of Theorem 1, we also have* (5)*, then we have* (9)*.*

Proof. We need only note that, by the Theorem, there is some time τ such that $\|y(t)\| < 1$ for $t > \tau$ so $G(x) \equiv \mathbf{\Gamma} x = -\mathbf{B}\mathbf{B}^* x$ from τ on. The remaining evolution of (7) coincides thenceforth with that of (4), starting at $x(\tau)$ — for which convergence to 0 was assumed. ■ ■

3. Some remarks

Remark 3: It is known [6] that there are control systems (2) such that, for certain observation operators $\mathbf{C}$, there is no possible causal feedback (even of much more general form than here) which would give asymptotic output stabilization of $y = \mathbf{C}x$. In these examples one may have $\mathbf{A}$ generating a contraction semigroup and $\mathbf{B}, \mathbf{C}$ continuous, as in Theorem 1 and, further, can assume exact nullcontrollability in fixed finite time (so exponential state stabilization would be possible with full state observation). Here, however, we have seen that those possibilities cannot occur with $\mathbf{C} = \mathbf{B}^*$.

At this point we also note the possibility of a gain operator $\mathbf{K}$ for which $z = -\mathbf{K}y = -\mathbf{K}\mathbf{C}x$ is stabilizing. If this feedback were purely dissipative ($\mathbf{\Gamma} = -\mathbf{B}\mathbf{K}\mathbf{C}$ selfadjoint and negative), we could redefine so $\mathbf{B} \hookleftarrow [-\mathbf{\Gamma}]^{1/2}$ and $\mathcal{U} \hookleftarrow \mathcal{H}$. The entire argument then would proceed as before.

Remark 3: It is clear from the proof given that Theorem 1 applies to feedback operators G as in (8) with more general $\rho(\cdot)$ provided the dependence of ρ on $r = \|y\| = \|\mathbf{B}x\|$ is moderately smooth and satisfies:

$$\rho : (0, \infty) \to (0, \infty) \text{ is continuous with } r\rho(r) \text{ nondecreasing.}$$

[To keep the control bounded, e.g., to impose a constraint that $\|u\| \leq M$, we would also have to require that $\rho(r) \leq M/r$; this constraint is, however, irrelevant to the Theorem.] Note that the resulting operator G is the gradient of the convex functional

$$\xi \mapsto \varphi(\|\mathbf{B}^*\xi\|) \qquad \text{with } \varphi(r) := \int_0^r \hat{r}\rho(\hat{r})\, d\hat{r} \tag{16}$$

To enable the proof above for Corollary 1, we need only add the requirement that $\rho \equiv 1$ on some nontrivial interval $0 < r < r_0$.

Another generalization would be to consider J continuous control operators $\mathbf{B}_j : \mathcal{U}_j \to \mathcal{H}$ with observation $y = (y_1, \ldots, y_J)$, taking $y_j = \mathbf{B}_j^* x$ with values in $\mathcal{U}_j$. The linear feedback would be $u_j = -y_j$ so (4) would become

$$\dot{x} = \mathbf{A}x + \mathbf{B}_1 u_1 + \cdots + \mathbf{B}_J u_J = \mathbf{A}x + \mathbf{\Gamma}x \tag{17}$$

with $\mathbf{\Gamma} = \mathbf{\Gamma}_1 + \cdots + \mathbf{\Gamma}_J$ where $\mathbf{\Gamma}_j = -\mathbf{B}_j \mathbf{B}_j^*$. Correspondingly, we set

$$G_j(\xi) := \rho_j(\|\mathbf{B}_j \xi\|)\mathbf{\Gamma}_j \xi \qquad G(\xi) = \sum_j G_j(\xi)$$

with each ρ_j as just above so (11) would become

$$\dot{x} = \mathbf{A}x - \sum_j \rho_j(\|\mathbf{B}_j x\|)\mathbf{B}_j \mathbf{B}_j^* x = \mathbf{A}x + G(x). \tag{18}$$

Replacing (11) by (18) as an equation and in (12), the proof provided above for the Theorem continues to hold with the obvious minor modifications. The same applies to Corollary 1.

[It might be of some interest to extend this further to an integral form:

$$G(\xi) = -\int_{\mathcal{A}} \rho_\alpha(\|\mathbf{A}_\alpha^*\xi\|)\,\mathbf{B}_\alpha\mathbf{B}_\alpha^*\xi\,\mu(d\alpha)$$

with $\mathbf{B}_\alpha : \mathcal{U}_\alpha \to \mathcal{H}$ so $\mathbf{B}^* : \mathcal{H} \to \bigoplus_\alpha \mathcal{U}_\alpha =: \mathcal{U}$. With reasonable hypotheses (e.g.: $\|\mathbf{B}_\alpha\| \leq \beta$ and each ρ_α as above), a version of our argument shows the convergence $y(t) := \mathbf{B}x(t) \to 0$ as $t \to \infty$ in the sense of $\|y\| = \|\alpha \mapsto y_\alpha\| := \left[\int \|y_\alpha\|^2\,\mu(d\alpha)\right]^{1/2}$. Unfortunately, there seems no easy way to obtain the Corollary in this setting.]

We also note that the linearity of $\mathbf{A}$ has not been used — e.g., we did not rely on (13) but rather on the theory of nonlinear semigroups — so our analysis would apply also when (1) is nonlinear. We will not pursue this.

Remark 3: While we have shown in Corollary 1 the convergence $x(t) \to 0$ as $t \to \infty$, our argument provides no *rate* of convergence — and, indeed, this could not be expected if no rate is specified for the linearly controlled system (4). With such a specification, necessarily exponential, our argument still does not give *uniformity* for this exponential decay: for each solution the asymptotic rate is always the same as for (4) but for initially large solutions the decay can be relatively minuscule for an initial period of indeterminate length. We will, however, provide a suggestive computation indicating why such a uniform rate for (7) might be plausible in fairly general circumstances:

One can always find a spectral variable ω for the selfadjoint linear operator $\mathbf{\Gamma}$ — i.e., one has an isomorphism of $\mathcal{H}$ with some $L^2_\mu(\Omega)$ and in this representation $\mathbf{\Gamma}$ acts by simple multiplication: there is a μ-measurable function $\gamma : \Omega \to [0, \|\mathbf{\Gamma}\|\,]$ such that

$$[\mathbf{\Gamma}\xi](\omega) = -\gamma(\omega)\xi(\omega) \qquad \text{for } \xi(\cdot) \in L^2_\mu(\Omega).$$

Note that $\|\mathbf{B}\|^2 = \|\mathbf{\Gamma}\| = \text{ess sup}\{\gamma\}$. We now consider a conservative setting so $\mathbf{S}(\cdot)$ is actually a unitary group, for which it is plausible to expect a representation

$$[\mathbf{S}(t)\xi](\omega) = \xi(\pi_t(\omega)) \qquad \text{for } t \in \mathbb{R},\ \omega \in \Omega,\ \xi(\cdot) \in L^2_\mu(\Omega)$$

where $t \mapsto \pi(t)$ is a one-parameter group of measure preserving transformations on Ω. We easily verify that the solution of (7) is given by

$$x(t;\omega) = \exp\left[-\int_0^t \hat\rho(s)\gamma\left(\pi_{t-s}(\omega)\right)\,ds\right] x_0\left(\pi_{-t}(\omega)\right)$$

where $\hat\rho(s) := \rho(\|\mathbf{B}^*x(s)\|) \geq \rho(\|\mathbf{B}\|\|x(s)\|)$. The same formula with $\hat\rho$ replaced by 1 gives the solution of (4), so a convergence rate for (5) — i.e., $\|x(\tau)\| \leq \alpha\|x_0\|$ for some $\tau > 0$ and some $\alpha < 1$ — would require that

$$\text{ess}\ \inf_\omega \left\{\int_0^\tau \gamma\left(\pi_s(\omega)\right)\,ds\right\} \geq \gamma_* := -\ln\alpha > 0.$$

We then get for (9) the rate:

$$\|x(t)\| \leq \exp\left[\,-\rho(\,\|\mathbf{B}\|\|x_0\|\,)\gamma_* t/\tau\,\right]\,\|x_0\|.$$

Remark 3: A principal reason for using feedback for stabilization is to allow for the possibility of continued excitation, i.e., the inclusion of a forcing term f in (2) so we would get

$$\dot{x} = \mathbf{A}x + G(x) + f \tag{19}$$

For an exponentially decaying linear system it is standard to estimate the stability of the response: if, e.g., f is bounded (uniformly in time), then the solution will be bounded with a bound dependent both on the decay rate and the bound on f and related comments apply to excitations which are locally L^p. For the saturating system (19) one notes that one has a bound on the dissipation provided by the feedback so even a bounded excitation (but exceeding this) might produce an unbounded trajectory: we then do not, in general, get BIBO stabilization.

4. Unbounded control/observation

In this section we wish to relax the requirement that the linear control/observation operators $\mathbf{B}, \mathbf{B}^*$ be continuous with respect to the $\mathcal{H}, \mathcal{U}$ topologies as above. This is potentially important especially for the possible consideration of stabilization through boundary control. Note the related importance of avoiding imposition of a coercivity assumption for $-\mathbf{A}$ since, while useful, this would unfortunately preclude consideration of precisely those problems in which we are most interested.

We continue to assume that $\mathbf{A}$ generates a contraction semigroup on $\mathcal{H}$, although all we will use of this is that it is dissipative and that its domain $\mathcal{D}(\mathbf{A}) := \{\xi : \mathbf{A}\xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Since $\mathbf{\Gamma} = -\mathbf{B}\mathbf{B}^*$ is now to be unbounded, it is no longer automatic that the dissipative operator $(\mathbf{A} + \mathbf{\Gamma})$ should be maximal and, with a view to Theorem 3, we will therefore assume explicitly that

(20)
i) $(\mathbf{A} + \mathbf{\Gamma})$ is the infinitesimal generator of a C_0 contraction semigroup $\mathbf{S}_{\mathbf{\Gamma}}(\cdot)$ on $\mathcal{H}$;
ii) $\mathcal{D}_{\#} := \mathcal{D}(\mathbf{A}) \cap \mathcal{D}(\mathbf{A} + \mathbf{\Gamma})$ is dense in $\mathcal{H}$.

As before, $-G(\xi) := -\rho(\|\mathbf{B}^*\xi\|)\mathbf{\Gamma}\xi = \partial\varphi(\|\mathbf{B}^*\xi\|)$ with φ given on $\mathcal{H}$ as in (14) for $\xi \in \mathcal{D}(\mathbf{B}^*) := \{\xi \in \mathcal{H} : \|\mathbf{B}^*\xi\| < \infty\}$ and $\varphi(\xi) := +\infty$ for $\xi \in [\mathcal{H} \setminus \mathcal{D}(\mathbf{B}^*)]$; one easily sees that this φ is lower semicontinuous on $\mathcal{H}$, so $-G$ is maximal monotone. It is now no longer automatic that the sum $(\mathbf{A} + G)$ is maximal and, as with (20-*i*), we assume explicitly that

(21) $(\mathbf{A} + G)$ is the infinitesimal generator of a C_0 contraction semigroup $\mathbf{S}_G(\cdot)$ on $\mathcal{H}$.

Theorem 2. *Let $\mathbf{A}, \mathbf{B}, G$ be as above. Then, for the controlled differential equation:*

$$\dot{x} = \mathbf{A}x + G(x) \qquad x(0) = x_0 \tag{22}$$

giving $x(t) = \mathbf{S}_G(t)[x_0]$, one has

- *If $x_0 \in \mathcal{D}_{\#}$, then the solution is asymptotically output stable, i.e., one has: $y(t) := \mathbf{B}^*x(t) \to 0$ as $t \to \infty$.*
- *If one assumes asymptotic stabilization on $\mathcal{D}_{\#}$ for the linear semigroup $\mathbf{S}_{\mathbf{\Gamma}}(\cdot)$ — i.e., if (5) holds for arbitrary initial data $x_0 \in \mathcal{D}_{\#}$ — then one has asymptotic stability for the saturating control system (22), now for arbitrary initial data $x_0 \in \mathcal{H}$.*

Proof. We proceed by a sequence of steps, beginning with an observation about the linear semigroup $\mathbf{S}_{\mathbf{\Gamma}}$:

Step 1: Let $\tilde{x}(t) := \mathbf{S}_{\mathbf{\Gamma}}(t)[x_0]$ be the solution of (4) with initial data in the domain of the infinitesimal generator: $x_0 \in \mathcal{D}(\mathbf{A} + \mathbf{\Gamma})$. By a theorem of Komura [4], in this case $\tilde{x}^{\,\cdot}$ as well as $\tilde{x}$ will be nonincreasing, i.e.,

$$\|\tilde{x}^{\,\cdot}(t)\| \le \|\tilde{x}^{\,\cdot}(0)\| = \|(\mathbf{A} + \mathbf{\Gamma})x_0\| =: C$$

as well as $\|\tilde{x}(t)\| \le \|x_0\|$.

Strictly speaking, we have $\tilde{x}^{\,\cdot}(t)$ only a.e. and otherwise have the 'forward derivative' $\tilde{x}^+$, but this distinction does not affect our argument.

We now show that, setting $\tilde{y}(\cdot) := \mathbf{B}^*\tilde{x}(\cdot)$, we have

- $\tilde{y}(\cdot)$ is well-defined and bounded uniformly on $\mathbb{R}_+$;
- $\tilde{y}(\cdot)$ is continuous, uniformly on $\mathbb{R}_+$.

For the first, we note that

$$\begin{aligned}\|\tilde{y}(t)\|^2 &= \langle \mathbf{B}^*\tilde{x}, \mathbf{B}^*\tilde{x}\rangle = \langle \tilde{x}, -\mathbf{\Gamma}\tilde{x}\rangle \\ &\le \langle \tilde{x}, -(\mathbf{A}+\mathbf{\Gamma})\tilde{x}\rangle = \langle \tilde{x}, -\tilde{x}^{\cdot}\rangle \le C\|x_0\|\end{aligned}$$

so $\tilde{x}(t) \in \mathcal{D}(\mathbf{B}^*)$ and $\tilde{y}(t)$ is defined and bounded on $\mathbb{R}_+$. For the second, we note similarly that (for $t > s$) we have

$$\begin{aligned}\|\tilde{y}(t) - \tilde{y}(s)\|^2 &= \langle \tilde{x}(t) - \tilde{x}(s), -\mathbf{\Gamma}[\tilde{x}(t) - \tilde{x}(s)]\rangle \\ &\le \langle \tilde{x}(t) - \tilde{x}(s), \tilde{x}^{\cdot}(t) - \tilde{x}^{\cdot}(s)\rangle \\ &\le \left\| \int_s^t \tilde{x}^{\cdot}(r)\, dr \right\| [\|\tilde{x}^{\cdot}(t)\| + \|\tilde{x}^{\cdot}(s)\|] \\ &\le 2C^2(t-s)\end{aligned}$$

giving uniform Hölder continuity

$$\|\tilde{y}(t) - \tilde{y}(s)\| \le \sqrt{2}C\,(t-s)^{1/2}. \tag{23}$$

Step 2: Let $x := \mathbf{S}_G(\cdot)[x_0]$ be the solution of (22) with x_0 in the domain of this infinitesimal generator: $x_0 \in \mathcal{D}(\mathbf{A} + G)$; as before, $\|x(t)\| \le \|x_0\|$ and $\|\dot{x}(t)\| \le \|(\mathbf{A} + G)[x_0]\| =: C$. Setting $y := \mathbf{B}^*x$, we now have

- $y(\cdot)$ is well-defined and bounded uniformly on $\mathbb{R}_+$; $\rho := \min\{1, 1/\|y\|\} \ge \rho_\# > 0$;
- $y(\cdot) \in L^2(\mathbb{R}_+ \to \mathcal{U})$ with $\left[\int_0^\infty \|y\|^2\right]^{1/2} \le (1/\sqrt{2\rho_\#})\|x_0\|$

As in Step 1, we have from [4] that $\mathcal{D}(\mathbf{A} + G)$ is invariant and that $\|x(\cdot)\|$, $\|\dot{x}(\cdot)\|$ are nonincreasing. For the first assertion we note that

$$\begin{aligned}\|y(t)\|^2 &= \langle x, -\mathbf{\Gamma}x\rangle = (1/\rho)\,\langle x, -G[x]\rangle, \\ \min\{\|y\|^2, \|y\|\} &\le \langle x, -G[x]\rangle \le \langle x, -(\mathbf{A}+G)[x]\rangle \le C\|x_0\|, \\ \|y(t)\| &\le \max\{1, C\|x_0\|\} \qquad \text{so } \rho \ge \rho_\# := 1/C\|x_0\|\end{aligned}$$

while for the second we note that $\left(\|x\|^2\right)^{\cdot} = 2\langle x, \dot{x}\rangle$ so

$$\begin{aligned}\rho_\# \int_0^t \|y\| &\le \int_0^t \rho\|y\| = \int_0^t \langle x, -G[x]\rangle \\ &\le \int_0^t \langle x, -(\mathbf{A}+G)[x]\rangle = \int_0^t \langle x, -\dot{x}\rangle \le \tfrac{1}{2}\|x_0\|^2.\end{aligned}$$

Step 3: Let $x(\cdot)$, $y(\cdot)$ be as above and suppose, for some τ, that $\|y(\tau)\| < 1$. Then $x(\tau) \in \mathcal{D}(\mathbf{A} + \mathbf{\Gamma})$ and, on some interval $\mathcal{I} = [\tau, \tau_1)$, one has $x(t) = \tilde{x}(t) := \mathbf{S}_{\mathbf{\Gamma}}(t - \tau)[x(\tau)]$.

We note that when $\|y\| \le 1$ we have $\rho = 1$ so $G[x] = \mathbf{\Gamma}x$; noting that (20-*ii*) permits us to consider the sums directly, we know that then $(\mathbf{A} + \mathbf{\Gamma})x(\tau) = (\mathbf{A} + G)[x(\tau)]$, which is in $\mathcal{H}$. Thus we have $x(\tau) \in \mathcal{D}(\mathbf{A} + \mathbf{\Gamma})$ and, setting $\tilde{x}(t) := \mathbf{S}_{\mathbf{\Gamma}}(t - \tau)[x(\tau)]$, can proceed as in Step 1; note that $\|\tilde{x}^{\cdot}(\tau)\| = \|(\mathbf{A} + \mathbf{\Gamma})x(\tau)\| = \|\dot{x}(\tau)\| \le \|\dot{x}(0)\| = C$. We then have $\tilde{y}(\cdot)$ continuous so, as $\tilde{y}(\tau) = y(\tau)$ with $\|y(\tau)\| < 1$, there is an interval $\mathcal{I}$ on which $\|\tilde{y}\| \le 1$. On that interval, $\tilde{x}^{\cdot} = (\mathbf{A} + \mathbf{\Gamma})\tilde{x} = (\mathbf{A} + G)[\tilde{x}]$ so $\tilde{x}(\cdot)$ is also a solution of (22) with the same (initial) data at $t = \tau$ as $x(\cdot)$. By uniqueness for this differential equation, we then have $x = \tilde{x}$ so long as $\|y\| = \|\tilde{y}\|$ remains below 1. By (23), a lower bound for the length of $\mathcal{I}$ is: $\tau_1 - \tau \ge [1 - \|y(\tau)\|]^2/2C$.

Step 4: With $x_0 \in \mathcal{D}(\mathbf{A} + G)$, we may now conclude, much as in the proof of Theorem 1, that the output $y(t)$ approaches 0 in $\mathcal{U}$ as $t \to \infty$.

Suppose this would be false. Then, for some $0 < \varepsilon < 1/3$ there would be arbitrarily large values of t for which $\|y(t)\| > 2\varepsilon$. Since, as in Step 2, we have $y(\cdot) \in L^2(\mathbb{R}_+ \to \mathcal{U}$, there must also be arbitrarily large values of t for which $\|y(t)\| < 2\varepsilon < 1$; since $y(\cdot)$ is continuous there,

there must be $\tau_k \to \infty$ with $\|y(\tau_k)\| = 2\varepsilon$ and with $\|y\| \geq \varepsilon$ on an interval $\mathcal{I}_k$ of length at least $\delta := \varepsilon^2/2C$; without loss of generality we may assume these intervals are disjoint. Since there are infinitely many such intervals, it then follows that $\int_0^\infty \|y\|^2 > N\varepsilon^4/2C$ for arbitrarily large N, contradicting the L^2 bound for $\|y\|$.

Step 5: If we now assume (5) for initial data in $\mathcal{D}(\mathbf{A}+\mathbf{\Gamma})$, then — at least for such initial data x_0 — the asymptotic stability $x(t) := \mathbf{S}_G(t)[x_0] \to 0$ follows from the output stabilization (just obtained in Step 4) and (5), exactly as in the proof of Corollary 1.

Step 6: We conclude by showing the asymptotic stability for general initial data:

$$x(t) := \mathbf{S}_G(t)[x_0] \to 0 \quad \text{as } t \to \infty \qquad \text{for arbitrary } x_0 \in \mathcal{H}. \tag{24}$$

To see this, we first note that $\mathcal{D}(\mathbf{A}+G)$ contains $\mathcal{D}_\#$ which is dense in $\mathcal{H}$. Thus, for any $\varepsilon > 0$ we can choose $x_0' \in \mathcal{D}(\mathbf{A}+G)$ with $\|x_0' - x_0\| \leq \varepsilon$ and let $x' = \mathbf{S}_G(\cdot)[x_0']$ be the corresponding solution. We know that there will be some τ for which $\|x'(\tau)\| \leq \varepsilon$ and we then have

$$\|x(\tau)\| \leq \|x'(\tau)\| + \|\mathbf{S}_G(\tau)[x_0'] - \mathbf{S}_G(\tau)[x_0]\| \leq 2\varepsilon.$$

By the contractivity of $\mathbf{S}_G$, we will have $\|x(t)\| \leq 2\varepsilon$ thenceforth, i.e., $x(t) \to 0$ as $t \to \infty$. ■
■

We are next concerned with verification of the hypotheses for Theorem 2 and show that a sufficient condition for this is that there be some $\vartheta < 1$ and some K such that

$$\|\mathbf{\Gamma}\xi\| \leq \vartheta\|\mathbf{A}\xi\| + K\|\xi\| \qquad \text{for all } \xi \in \mathcal{D}(\mathbf{A}) \tag{25}$$

[It is worth noting that a sufficient condition for (25) — with $\vartheta > 0$ arbitrary — is that the unbounded operator $\mathbf{\Gamma}$ be compact relative to $\mathbf{A}$, i.e., compact when viewed as a linear operator from $\mathcal{D}(\mathbf{A})$, taken with graph norm.]

Theorem 3. *Assume $-\mathbf{A}$ is a maximal monotone (unbounded) linear operator on the Hilbert space $\mathcal{H}$ with $\mathbf{B}$ such that $\mathbf{\Gamma} := -\mathbf{B}\mathbf{B}^*$ satisfies (25) with $\vartheta < 1$. Then* (20) *holds and, with G defined as in Theorem 2,* (21) *holds.*

Proof. We begin by showing (21), for which it is sufficient to show that the sum $-(\mathbf{A}+G)$ is maximal monotone, for which it is sufficient to show that $\mathbf{1} - (\mathbf{A}+G)$ is surjective. Since the Yosida approximation $\mathbf{A}_\mu$ is continuous, we know that $-(\mathbf{A}_\mu + G)$ is maximal monotone so, for arbitrary $\beta \in \mathcal{H}$, there is a unique solution $\xi = \xi_\mu$ of

$$\xi - \mathbf{A}_\mu\xi - G[\xi] = \beta \tag{26}$$

and we set $\eta = \eta_\mu := (\mathbf{1} - \mu\mathbf{A})^{-1}\xi_\mu$ and $\omega = \omega_\mu := \mathbf{A}_\mu\xi_\mu$. Note that $\langle \omega, \xi \rangle \leq 0$ as $\mathbf{A}_\mu$ is dissipative, that $\xi = \eta - \mu\omega$, and that multiplying (26) by ξ and using the monotonicity gives $\|\xi\|^2 \leq \langle \xi, \beta \rangle$ so $\|\xi_\mu\| \leq \|\beta\|$, uniformly in μ, whence also $\|\eta\| \leq \|\beta\|$.

Theorem 8 of [2] asserts that β is in the range of $(\mathbf{1} - \mathbf{A} - G)$ if (and only if) ω_μ remains bounded as $\mu \to 0+$ and we proceed to show this, using (25). Multiplying (26) by ω gives

$$\begin{aligned} \|\omega\|^2 &= \langle \omega, \omega \rangle = \langle \omega, \beta - \xi - G[\xi] \rangle \\ &\leq \|\omega\|\,\|\beta\| - \langle \omega, G[\xi] \rangle = \|\omega\|\,\|\beta\| - \rho\langle \omega, \mathbf{\Gamma}\xi \rangle \\ &= \|\omega\|\,\|\beta\| - \rho\left[\langle \omega, \mathbf{\Gamma}\eta \rangle - \mu\langle \omega, \mathbf{\Gamma}\omega \rangle\right] \\ &\leq \|\omega\|\left[\|\beta\| + \|\mathbf{\Gamma}\eta\|\right] \leq \|\omega\|\left[\|\beta\| + \vartheta\|\omega\| + K\|\eta\|\right]. \end{aligned}$$

This gives, uniformly,

$$\|\omega_\mu\| \leq \frac{1+K}{1-\vartheta}\|\beta\|$$

whence, by the Brezis result, we see that this arbitrary β is in the range of $(\mathbf{1} - \mathbf{A} - G)$, showing (21).

The argument for (20) is identical after replacing ρ by 1 above. ■ ■

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