On positive reachability for diffusion equations*

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Abstract

Counterexamples are constructed for some plausible conjectures. Typical of these: as the Maximum Principle ensures that positive boundary data give a positive state at time T from 0 initial data, one might (plausibly, but falsely) conjecture that all positive terminal states should be approximately reachable in this way, i.e., subject to the requirement that the boundary data stays nonnegative.

Key Words: .

1. Introduction

For a given terminal time $T > 0$ and a connected bounded open region Ω, we consider a diffusion equation with Dirichlet boundary data:

$$\begin{aligned} u_t = Lu &\quad \text{in } \mathcal{Q} = (0,T] \times \Omega \\ u = \varphi &\quad \text{on } \Sigma = [0,T] \times \partial\Omega \\ u = u_0 &\quad \text{on } \Omega \text{ at } t = 0 \end{aligned} \tag{1.1}$$

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where

$$L : u \mapsto \nabla \cdot A\nabla u - qu \tag{1.2}$$

with A symmetric and uniformly elliptic. Without specifying precisely, we assume A, q are 'smooth enough' for our purposes; for simplicity we also assume that A, q are independent of t. Paradigmatically, we think of L as the Laplace operator $\triangle$ (with $A \equiv I, q \equiv 0$).

It is well-known that (1.1) has a unique classical solution $u = u(\cdot, \cdot; u_0, \varphi)$ for each u_0 in $\mathcal{X} = L^2(\Omega)$ and each φ in $\Phi = L^2(\Sigma)$. This solution is smooth in $\mathcal{Q}$, but may lose regularity as $t \to 0+$ (unless u_0 is smooth) and as one approaches Σ (unless φ is smooth). Since L is the infinitesimal generator of an analytic semigroup of operators $\mathbf{S}(\cdot)$ on $\mathcal{X}$, we have $u(\cdot, \cdot; u_0, 0)$ in $C([0, T] \to \mathcal{X})$ for each $u_0 \in \mathcal{X}$ and write $\mathbf{S} = \mathbf{S}(T) : \mathcal{X} \to \mathcal{X}$ for the continuous linear operator: $u_0 \mapsto u(T, \cdot; u_0, 0)$. We similarly define a linear operator $\mathbf{B} : \varphi \mapsto u(T, \cdot; 0, \varphi)$, but this is a bit more difficult to treat. It would be convenient if $\mathbf{B}$ would be continuous from Φ to $\mathcal{X}$, but this is false. Somewhat ambiguously, we use $\mathbf{B}$ both for the operator defined for all $\varphi \in \Phi$ (without an explicit specification of codomain) and also for the unbounded closed operator: $\Phi \to \mathcal{X}$ without actually examining its domain. Thus we have

$$u(T, \cdot; u_0, \varphi) = \mathbf{S}u_0 + \mathbf{B}\varphi \tag{1.3}$$

which may be taken as defining the operators $\mathbf{S}, \mathbf{B}$.

We now consider the ranges, letting

$$\mathcal{S} = \mathbf{S}(\mathcal{X}) = \{\mathbf{S}w : w \in \mathcal{X}\}, \qquad \mathcal{B} = \mathbf{B}(\Phi) = \{\mathbf{B}\psi \in \mathcal{X} : \psi \in \Phi\}$$

be the relevant subsets of $\mathcal{X}$. Noting that $\mathcal{S}$ contains the eigenfunctions of L, as a self-adjoint operator on $\mathcal{X}$ with compact resolvent, we see that

$$\mathcal{S} \text{ is dense in } \mathcal{X} \qquad (\mathcal{X} = \overline{\mathcal{S}}). \tag{1.4}$$

Interpreting (1.1) from a system-theoretic viewpoint, we can consider the boundary data φ as a control. It is now well known (cf., e.g., [7], [8]) for the heat equation ($L = \triangle$) and, e.g., [4] for more general coefficients) that (1.1) is *exactly nullcontrollable*: for each $u_0 \in \mathcal{X}$, there is a control $\varphi \in \Phi$ making $u(T, \cdot) = \mathbf{S}u_0 + \mathbf{B}\varphi \equiv 0$. This gives $\mathbf{S}u_0 = -\mathbf{B}\varphi$ so

$$\mathcal{S} \subset \mathcal{B}. \tag{1.5}$$

Of course, it follows from (1.4) and (1.5) that (1.1) is *approximately controllable*,

$$\mathcal{B} \text{ is dense in } \mathcal{X} \qquad\qquad (\mathcal{X} = \overline{\mathcal{B}}), \tag{1.6}$$

i.e., every target state can be (approximately) reached using some control. On the other hand, since the solutions of such diffusion equations are quite smooth in the interior of Ω for $t > 0$, it is clear that $\mathcal{B}$ cannot be all of $\mathcal{X}$: neither $\mathcal{S}$ nor $\mathcal{B}$ can be L^2-closed.

Besides regularity, probably the best known property of such diffusion equations is the *Maximum Principle*, one form of which asserts that every solution of (1.1) remains nonnegative if u_0, φ are each nonnegative. If we introduce the order cones

$$\mathcal{X}_+ = \{w \in \mathcal{X} : w \geq 0 \,\text{ae on}\, \Omega\}, \qquad \Phi_+ = \{\psi \in \Phi : \psi \geq 0 \,\text{ae on}\, \Sigma\}, \tag{1.7}$$

then the Maximum Principle just asserts that their images under $\mathbf{S}, \mathbf{B}$, respectively, are in $\mathcal{X}_+$ —

$$\mathcal{S}_+ := \mathbf{S}(\mathcal{X}_+) \subset \mathcal{X}_+ \quad \text{and} \quad \mathcal{B}_+ := \mathbf{B}(\Phi_+) \subset \mathcal{X}_+ \tag{1.8}$$

This suggests asking whether the analogues of (1.6), (1.4), and (1.5) will also be valid:

Q1 Is $\mathcal{B}_+$ dense in $\mathcal{X}_+$? (Is $\mathcal{X}_+ = \overline{\mathcal{B}_+}$?)

Q2 Is $\mathcal{S}_+$ dense in $\mathcal{X}_+$? (Is $\mathcal{X}_+ = \overline{\mathcal{S}_+}$?)

Q3 Is $\mathcal{S}_+ \subset \mathcal{B}_+$? (or even: Is $\mathcal{S}_+ \subset \overline{\mathcal{B}_+}$?)

Less formally, the first of these questions may be rephrased in control-theoretic terms as asking

> *Can nonnegative targets always be (approximately) reached by using nonnegative controls?*

while the third question may similarly be rephrased as

> *Can non-positive initial states always be steered (approximately) to 0 by using nonnegative controls?*

Our object in this paper is to give negative answers to each of these questions. This is somewhat of a continuation of the concerns of [9], but we are also indebted to S.A. Avdonin for raising the first of these questions and for noting the recent paper [2] by M.I. Belishev, which already provides a negative answer to that question in the one-dimensional case for the heat equation.

2. An adjoint computation and some lemmas

Given a function $\omega \in \mathcal{X}$, we consider our diffusion equation in reversed time:

$$\begin{aligned} -v_t = Lv &\quad \text{in } \mathcal{Q} \\ v = 0 &\quad \text{on } \Sigma \\ v = \omega &\quad \text{on } \Omega \text{ at } t = T. \end{aligned} \tag{2.1}$$

Note that setting $z(t) = v(T - t)$ gives a solution z of (1.1) with $\varphi = 0$ and $u_0 = z(0) = v(T) = \omega$ so $v(0) = z(T) = \mathbf{S}\omega$. We use this for an adjoint calculation.

Let u be the solution of (1.1) so $u(T, \cdot) = \mathbf{S}u_0 + \mathbf{B}\varphi$ for some $u_0 \in \mathcal{X}$ and $\varphi \in \Phi$; we assume φ is such that $\mathbf{B}\varphi \in \mathcal{X}$ so $u(T, \cdot) \in \mathcal{X}$. Using the Divergence Theorem and L^2 inner products for Ω and Σ, we then have

$$\begin{aligned} \langle u(T), \omega\rangle_{\mathcal{X}} &= \langle \mathbf{S}u_0 + \mathbf{B}\varphi, \omega\rangle = \langle u_0, v(0)\rangle + \int_0^T [d\langle u, vdt]\, dt \\ &= \langle u_0, v(0)\rangle + \int_0^T [\langle Lu, v\rangle + \langle u, -Lv\rangle]\, dt \\ &= \langle u_0, v(0)\rangle_{\mathcal{X}} - \langle \varphi, v_\nu\rangle_\Phi \end{aligned} \tag{2.2}$$

where v_ν denotes the conormal derivative at $\partial\Omega$ — i.e., $v_\nu = A\nabla v \cdot \mathbf{n}$ with $\mathbf{n}$ the unit exterior normal; of course, this requires for legitimacy that v_ν be in Φ. The computation (2.2) confirms the expected self-adjointness of $\mathbf{S} : \mathcal{X} \to \mathcal{X}$ and shows that the adjoint $\mathbf{B}^* : \mathcal{X} \to \Phi$ of the unbounded operator $\mathbf{B} : \Phi \to \mathcal{X}$ is given by

$$\mathbf{B}^*\omega := -v_\nu \qquad \text{subject to (2.1)} \tag{2.3}$$

with the domain of $\mathbf{B}^*$ given by the requirement on ω that the solution v of (2.1) is such that v_ν is in Φ.

We have already commented on the extraordinary interior regularity of solutions of (1.1) with smooth coefficients and now comment more specifically on (2.1), particularly noting a localization of irregularity.

LEMMA 1 *For $\omega \in \mathcal{X} = L^2(\Omega)$ the solution v of (2.1) satisfies*

$$\begin{array}{l} v(t,\cdot) \text{ is smooth for } t < T\text{; in particular,} \\ \mathbf{S}\omega = v(0,\cdot) \text{ is in } C^1(\overline{\Omega}) \end{array} \tag{2.4}$$

and if the support of ω is interior to Ω (i.e., if ω vanishes in a neighborhood of $\partial\Omega$) one also has

$$\mathbf{B}^*\omega = -v_\nu \text{ is continuous on } \Sigma, \tag{2.5}$$

which shows that such ω are always in the domain of $\mathbf{B}^*$.

PROOF: Allowing for the time reversal, we note, e.g., from [3] or [5] the integral representation

$$v(t,x) = \int_\Omega G(x,y,T-t)\,\omega(y)\,dy \tag{2.6}$$

whose kernel G is the *Green's function* for the operator L on Ω. The conclusions here now follow immediately from estimating v and $D_x v$ by using in (2.6) the estimate

$$|D_s^j D_x^k G(x,y,s)| \le C s^{-[n+2j+k]/2} e^{-c|x-y|/s} \tag{2.7}$$

given, with slightly different notation, in Theorem 16.3 (Chapter IV) in [5]. Note that for (2.5) we are concerned with $x \in \partial\Omega$ so the assumption on the support of ω ensures that $|x-y|$ is bounded away from 0 where this is relevant in (2.6) and, although (2.7) is only given for $s > 0$, it certainly then implies a uniform bound (and, in fact, convergence to 0) as $s \to 0$ with $|x-y| \ge \delta > 0$. ■

Although stronger statements could be made, these are sufficient for our purposes.

LEMMA 2 *Let the region Ω be smoothly bounded and suppose the coefficients A, q are smooth. Then there exists a continuous function $\tilde{\omega} \in \mathcal{X}$ such that*

$$0 < \tilde{\omega} \le \beta \text{ on } \Omega \text{ with } -\tilde{v}_\nu = \mathbf{B}^*\tilde{\omega} \ge 1 \text{ on } \Sigma. \tag{2.8}$$

Proof: Consider (2.1) with T replaced by some $T' > T$ and with some (fairly arbitrary) choice of positive initial data at T'. By the strong Maximum Principle (cf., e.g., [6][Chap. 3, Sect. 3] applied to $-v$ with $M = 0$) we see, for all $t < T'$, that the solution $v(t, \cdot)$ is strictly positive everywhere in the interior of Ω and that $-v_\nu$ is strictly positive everywhere on $\partial\Omega$. We will now let $\tilde{\omega}_1 = v(T, \cdot)$. Allowing for the change to T', we note that this $\tilde{\omega}_1$ is continuous on $\overline{\Omega}$ by (2.4) and that $-v_\nu$ is continuous on $\Sigma = [0, T] \times \partial\Omega$. Since Σ is compact and $-v_\nu$ is positive there, this gives $-v_\nu$ bounded away from 0 on Σ, say, $-v_\nu \geq c > 0$ on Σ. Taking $\tilde{\omega} = (1/c)\tilde{\omega}_1$ and then defining $\beta = \max_\Omega \tilde{\omega}$, we have (2.8). ■

LEMMA 3 *Let the region Ω be smoothly bounded and suppose the coefficients A, q are smooth. Consider any fixed support $\overline{\Omega}_*$ in the interior of Ω. If $\|\omega\|_{\mathcal{X}}$ is small enough, one then has*

$$|v_\nu| \leq 1 \leq \mathbf{B}^*\tilde{\omega} \quad \textit{pointwise on } \Sigma, \tag{2.9}$$

where $\tilde{v}$ is the solution of (2.1) corresponding to the initial data $\tilde{\omega}$ at time T as in Lemma 2.

Proof: Fixing Ω_*, we may consider $\mathcal{X}_* = L^2(\Omega_*)$ as a (closed) subspace of $\mathcal{X}$, extending functions ω on Ω_* as 0 on $\Omega \setminus \Omega_*$, and note that (2.5) gives $\mathbf{B}^*\omega$ in $C(\Sigma)$ for such ω. Thus, with this restriction, $\mathbf{B}^*$ determines a linear mapping from $\mathcal{X}_*$ to $C(\Sigma)$. By the Closed Graph Theorem this mapping is continuous so there is a constant C — depending, of course, on the particular choice of Ω_* — such that

$$\max\{|-v_\nu|\} \leq C\|\omega\|_{\mathcal{X}_*} = C\|\omega\|_{\mathcal{X}}.$$

Then (2.9) holds for $\|\omega\|_{\mathcal{X}} \leq 1/C$. ■

LEMMA 4 *Let the region Ω be smoothly bounded and suppose the coefficients A, q are smooth. If $\|\omega\|_{\mathcal{X}}$ is small enough one then has*

$$|\mathbf{S}\omega| \leq \mathbf{S}\tilde{\omega} \quad \textit{pointwise on } \Omega. \tag{2.10}$$

PROOF: By (2.4), $v(0,\cdot) = \mathbf{S}\omega$ is in $C^1(\overline{\Omega})$. Since $v(0,\cdot)$ necessarily vanishes at $\partial\Omega$, we must have $|v(0,x)| \le K|x - \partial\Omega|$ where K is a bound on $\nabla v(0,\cdot)$. Similarly, since (2.8) gives $-\tilde{v}(0,\cdot) \ge 1$ at $\partial\Omega$, we must have $\tilde{v}(0,\cdot) \ge |x - \partial\Omega|/2$ on some neighborhood of $\partial\Omega$ so $|v(0,\cdot)| \le 2K\tilde{v}(0,\cdot)$ on this neighborhood. Letting Ω_* be the complement of this neighborhood in Ω, we have $v(0,\cdot)$ bounded and $\tilde{v}(0,\cdot)$ bounded away from 0. Thus, defining

$$\begin{aligned} \|v\|_\ddagger &= \sup_{x\in\Omega}\{|v(x)|/\tilde{v}(0,x)\} \\ &= \inf_{c>0}\{c : |v(\cdot)| \le c\tilde{v}(0,\cdot)\}, \end{aligned}$$

we have $\|\mathbf{S}\omega\|_\ddagger < \infty$ for each $\omega \in \mathcal{X}$. It is easily seen that this $\|\cdot\|_\ddagger$ is a norm; indeed, this is what is called in [1] the *order unit norm* with respect to $e = \mathbf{S}\tilde{\omega}$. Introducing the Banach space $\mathcal{X}_\ddagger$ of functions $v \in C(\overline{\Omega})$ for which $\|v\|_\ddagger$ is finite, we have shown that $\mathbf{S} : \mathcal{X} \to \mathcal{X}_\ddagger$ and (2.10) now follows from the Closed Graph Theorem as above for (2.9). ■

3. Answering the questions

In this section we will provide negative answers to the questions raised in the Introduction. Except as explicitly noted, we assume throughout the regularity requirements: that $\partial\Omega$ and A, q are 'smooth enough'. We begin with consideration of Q1.

THEOREM 5 *There are always positive target states u_T which are bounded away from all states reachable from $u_0 = 0$ in* (1.1) *by nonnegative controls — i.e., $\mathcal{B}_+$ is* not *dense in $\mathcal{X}_+$.*

PROOF: Beginning with the function $\tilde{\omega}$ of Lemma 2, we construct an appropriate function $\omega \in \mathcal{X}$ to use in (2.1), (2.2). Let χ^ε be the characteristic function of an ε-ball in Ω — i.e., for some fixed $x_* \in \Omega$ set

$$\chi^\varepsilon(x) = \begin{cases} \beta + 1 & x \in \Omega_\varepsilon \\ 0 & x \in [\Omega \setminus \Omega_\varepsilon] \end{cases} \qquad \Omega_\varepsilon = \{x : |x - x_*| \le \varepsilon\}$$

for $\varepsilon > 0$ small enough that $\overline{\Omega_\varepsilon} \subset \Omega$. Clearly, we have $\chi^\varepsilon \to 0$ in $\mathcal{X}$ as $\varepsilon \to 0$ with supports bounded away from $\partial\Omega$ so, by Lemma 3, we can choose $\varepsilon > 0$

so that $0 \leq \mathbf{B}^*\chi^\varepsilon \leq \mathbf{B}^*\tilde{\omega} = -\tilde{v}_\nu$ pointwise on Σ. We then set $\omega = \tilde{\omega} - \chi^\varepsilon$ with this choice of ε and have

$$\omega \leq -1 \text{ on } \Omega_\varepsilon \subset \Omega, \qquad \mathbf{B}^*\omega \geq 0 \text{ on } \Sigma.$$

For any $u(T,\cdot) = \mathbf{B}\varphi \in \mathcal{B}_+$ obtained using a control $\varphi \in \Phi_+$ for (1.1), one then has

$$\langle \omega, u(T) \rangle_\Omega = \langle \omega, \mathbf{B}\varphi \rangle_\Omega = \langle \mathbf{B}^*\omega, \varphi \rangle_\Sigma \geq 0 \tag{3.1}$$

(whence also $\langle \omega, u \rangle_\Omega \geq 0$ for any $u \in \overline{\mathcal{B}_+}$). On the other hand, for any target $0 \not\equiv u_T \in \mathcal{X}_+$ with support in Ω_ε, we have

$$\langle \omega, u_T \rangle_\Omega = \langle \omega, u_T \rangle_{\Omega_\varepsilon} \leq -\int_{\Omega_\varepsilon} u_T < 0 \tag{3.2}$$

This shows that any nontrivial $u_T \in \mathcal{X}_+$ with support in Ω_ε must be bounded away from $\mathcal{B}_+ = \mathbf{B}(\Phi_+)$ so, as asserted, $\mathcal{B}_+$ is not dense in $\mathcal{X}_+$: nonnegative targets cannot always be (approximately) reached by using nonnegative controls. ■

Remark: The regularity requirement on $\partial\Omega$ can actually be omitted for consideration of Q1. We need only consider a smoothly bounded subregion $\Omega_* \subset \Omega$. If some target in $L^2(\Omega)$ were approximable by a control at $\partial\Omega$, then its restriction to Ω_* would be equally approximable by using as boundary control the trace of that solution at $\partial\Omega_*$. On the other hand, we note that the result holds for the smoothly bounded Ω_* so this cannot be possible for all targets.

We next turn to Q2, whose resolution is almost identical to that above.

THEOREM 6 *There are always positive target states u_T which are bounded away from all states resulting at T in the uncontrolled ($\varphi \equiv 0$) version of* (1.1) *from initial states $u_0 \in \mathcal{X}_+$ — i.e., $\mathcal{S}_+ = \mathbf{S}(\mathcal{X}_+)$ is* not *dense in $\mathcal{X}_+$.*

PROOF: As in the proof of Theorem 5, we set $\omega = \tilde{\omega} - \chi^\varepsilon$ with $\varepsilon > 0$ now chosen so that, following (2.10), we have $[\mathbf{S}^*\chi^\varepsilon](\cdot) \leq \tilde{v}(0,\cdot) = [\mathbf{S}\tilde{\omega}](\cdot)$ pointwise, making

$$\mathbf{S}\omega \geq 0 \text{ on } \Omega \text{ although } \omega \leq -1 \text{ on } \Omega_\varepsilon.$$

As with (3.1), (3.2), we then have $\langle \omega, u \rangle_\Omega \geq 0$ for every $u \in \overline{\mathcal{S}_+}$ while $\langle \omega, u \rangle_\Omega < 0$ for every nontrivial $u_T \in \mathcal{X}_+$ with support in Ω_ε. This shows that $\overline{\mathcal{S}_+}$ is bounded away from all such targets so, as asserted, $\mathcal{S}_+$ is not dense in $\mathcal{X}_+$: nonnegative targets cannot always be approximated by the evolution of nonnegative initial states. ■

Finally, we consider Q3. Since $u(T,\cdot) = \mathbf{S}u_0 + \mathbf{B}\varphi$ (and, for any $u_0 \in \mathcal{X}$ one can find a nullcontrol $\varphi \in \Phi$ so this is exactly 0), we ask whether a target $u_T \in \mathcal{S}_+$ — i.e., of the special form $-\mathbf{S}u_0$ with a non-positive initial state u_0 — might always be approximately reachable using nonnegative controls. Again we obtain a negative answer, although now with a restriction on T. For convenience we also now assume that the equation (1.1) is autonomous: A, q independent of t in (1.2).

THEOREM 7 *For some $T_* > 0$ and all $0 < T < T_*$ there are always non-positive initial states u_0 for which $u(T,\cdot)$ is bounded away from 0 uniformly for all nonnegative controls φ used in* (1.1) *— i.e., $\mathcal{S}_+$ is not contained in $\overline{\mathcal{B}_+}$.*

Proof: Begin with some (large) $T = \hat{T}$ and again set $\omega = \tilde{\omega} - \chi^\varepsilon$ — now with $\varepsilon > 0$ chosen as in the proof of Theorem 5, so $\mathbf{B}^*\omega \geq 0$ on Σ. From the Maximum Principle it follows that the solution v of (2.1) is continuous from $[0,\hat{T}]$ to $L^\infty(\Omega)$ so $v(t,\cdot) \to \omega(\cdot)$ uniformly on Ω as $t \to \hat{T}$ — whence, since $\omega \leq -1$ on Ω_ε, it follows that for any $\alpha \in (0,1)$ we have $v(t,\cdot) \leq -\alpha < 0$ on Ω_ε for t in some interval $[\hat{T} - T_*, \hat{T}]$. Using the assumed autonomy of the equation, this shows that for $T < T_*$ we may translate $[0,\hat{T}]$ to $[T - \hat{T}, T]$ and have $\mathbf{S}^*\omega = v(0,\cdot) \leq -\alpha < 0$ on Ω_ε for this choice of ω, still with $\mathbf{B}^*\omega \geq 0$ on $\Sigma = (0,T)\times\partial\Omega \subset (T-\hat{T},T)\times\partial\Omega$. For any nontrivial initial state $u_0 \leq 0$ with support in Ω_ε and any control $\varphi \in \Phi_+$ we then have

$$\begin{aligned}\langle \omega, u(T)\rangle_\Omega &= \langle \mathbf{S}^*\omega, u_0\rangle_\Omega + \langle \mathbf{B}^*\omega, \varphi\rangle_\Sigma \\ &\geq \langle \mathbf{S}^*\omega, u_0\rangle_\Omega = \langle \mathbf{S}^*\omega, u_0\rangle_{\Omega_\varepsilon} \\ &\geq \alpha \int_\Omega [-u_0] > 0.\end{aligned}$$

This shows, of course, that for such an initial state every nonnegative control leaves $\|u(T)\|$ greater than $\alpha \int |u_0|/\|\omega\|$, bounded away from 0 so u_0 is not even approximately nullcontrollable with nonnegative controls. ■

We remark that, at this juncture, it is not clear whether the restriction to $T \leq T_*$ is a genuine necessity or merely an artifact of our proof. Since $\overline{\mathcal{B}_+}$

is convex, it is not difficult to see that the conclusion requires existence of some $\omega \in \mathcal{X}$ for which (2.1) gives $-v_\nu \geq 0$ on Σ, but with $v(0,\cdot) < 0$ on some nontrivial (open) subset of Ω. What is not clear is whether, for large enough T, the requirement that $\mathbf{B}^*\omega \geq 0$ may imply $\mathbf{S}^*\omega \geq 0$. Thus the possibility of removal of this restriction remains open.

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