

The ‘Output Stabilization’ problem: a conjecture and counterexample

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Abstract. ‘Output stabilization’ here refers to feedback which drives the system *output* to 0, without concern for the behavior of the full state. Since everything of concern is automatically observable, it is reasonable to conjecture — subject, of course, to some controllability hypothesis — that this output stabilization should always be possible by some kind of feedback from the output, with no necessity for the usual sort of observability hypothesis. This is true for the finite-dimensional case, but we show, by example, that the conjecture need not hold in infinite-dimensional contexts.

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1 Introduction

In discussing stabilization or feedback stabilizability of a system, standard analyses impose hypotheses of observability and controllability, from which is deduced stabilizability: that the *state* $x(\cdot)$ of the system is to go to the stationary point 0 as $t \rightarrow \infty$. We will here heuristically adopt the idea that, “*What we don’t know can’t hurt us*” — i.e., any deviation of the state which ‘matters’ to us must necessarily show up by affecting something we observe so, conversely, any deviation with no observable effects can be neglected as irrelevant. From this viewpoint, the stabilizability of what we *do* observe should be a more appropriate concern than seeking stabilizability of invisible components of the full state so we have a notion of **output stabilization**, meaning that the observable output y should go to 0 without direct concern for the behavior of the full state x .

Throughout we consider the autonomous linear control system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \quad y = \mathbf{C}x \quad (1)$$

with the functions x, u, y taking values in spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y}$, respectively and with suitable operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$. We then denote the solution of (1) with initial data $\xi = x(0)$ by $x = x(\cdot; \xi, u)$ and the corresponding output $y = \mathbf{C}x$ by $y(\cdot; \xi, u)$.

Quite generally, we call a control system (1) **output stabilizable** if there is some mechanism to determine the control u dynamically (causally based on the observed output) in such a way that, for each initial state ξ , the output $y(t; \xi, u)$ of the controlled system tends to 0 as $t \rightarrow \infty$. This is a very weak notion of feedback stabilizability since, at this point, we are assuming neither that the feedback mechanism is autonomous nor that it has any particular continuity property.

Remark 1.1 Apart from the restriction there to finite-dimensional spaces, we note that the output stabilization problem we are considering is *not* the similarly named problem of [5] Section 4.4 — which also requires $y \rightarrow 0$ but permits full state feedback. Our requirement here is that the control u must be constructed as a feedback depending only on the observable output $y = \mathbf{C}x$ so this problem corresponds more closely to what is called in Chapter 6 of [5] the *restricted regulator problem* (RRP). However, we also explicitly distinguish our concerns from Wonham’s RRPIS which seeks also to keep the full state bounded, including its invisible component. Finally, we note that, in contrast to the thrust of the analyses in [5], we are willing to impose a strong controllability condition here so as to focus attention entirely on issues of observability.

On the other hand, the considerations of ‘regional controllability’ in, e.g., [3], [1] are closely related to our present concerns. ■

If we assume some sufficient degree of controllability, then it is reasonable to conjecture that this output stabilization should always be possible since the problem formulation ensures that anything which cannot be observed must be irrelevant to our concerns: it should thus be possible to omit any observability hypothesis.

Conjecture 1.1 *If the system pair $[\mathbf{A}, \mathbf{B}]$ is open loop stabilizable (a fortiori if it is controllable), then there is some causal feedback mechanism:*

$$M : [\text{output history}] \longmapsto [\text{control}] = u(\cdot) \quad (2)$$

which, when coupled with (1), ensures asymptotic stability of the output: that $y \rightarrow 0$ as $t \rightarrow \infty$ for every solution x , i.e., for each choice of the initial data $x(0) = \xi$.

While finite-dimensional results can only be suggestive here, we note that this conjecture is supported by known results from standard system theory; compare [5], [4].

Theorem 1.2 *Consider the autonomous linear control system (1) with the functions x, u, y taking values in $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, and $\mathcal{Y} = \mathbb{R}^k$, respectively.*

1. If the system pair $[\mathbf{A}, \mathbf{B}]$ is controllable, then there is an $n \times n$ matrix \mathbf{K} such that the ‘feedback’ $u = \mathbf{K}x$ (assuming the availability of the full state, effectively taking $m = n$ and $\mathbf{C} = \mathbf{I}$) stabilizes — i.e., every solution of $\dot{x} = [\mathbf{A} + \mathbf{BK}]x$ converges (exponentially) to 0 as $t \rightarrow \infty$.
2. If the system pair $[\mathbf{A}, \mathbf{C}]$ is observable, then there is an $n \times n$ matrix \mathbf{L} , defining a tracking observer z by coupling

$$\dot{z} = (\mathbf{A} - \mathbf{LC})z + \mathbf{B}u - \mathbf{L}y, \quad (3)$$

with (1), such that the tracking error $[z - x]$ converges (exponentially) to 0 as $t \rightarrow \infty$.

3. If both of the above, then the feedback $u = \mathbf{K}z$ with \mathbf{K} as in 1. and z as in 2. stabilizes the system — i.e., every solution of the coupled system

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B}u, & u &= \mathbf{K}z \\ \dot{z} &= (\mathbf{A} - \mathbf{LC} + \mathbf{BK})z - \mathbf{L}y & y &= \mathbf{C}x \end{aligned} \quad (4)$$

converges (exponentially) to 0 as $t \rightarrow \infty$.

Note that 3. gives full state stabilization and so, a fortiori, output stabilization. Our initial hope is that we may omit the observability assumption in part 3. of Theorem 1.2 if we forego the state stabilization and seek stabilization only of the resulting output.

Corollary 1.3 *Conjecture 1.1 holds in the finite-dimensional case:*

If we consider (1) with $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ finite-dimensional and the system pair $[\mathbf{A}, \mathbf{B}]$ is controllable, then there are matrices \mathbf{K}, \mathbf{L} such that every output $y(\cdot)$ of the coupled system (4) converges (exponentially) to 0 as $t \rightarrow \infty$.

PROOF: Consider the quotient $\hat{\mathcal{X}} = \mathcal{X}/\mathcal{N}$ where $\mathcal{N} := \{\xi : y(t; \xi, 0) \equiv 0\}$ is the subspace of invisible states: this is invariant under $e^{t\mathbf{A}}$ and there is a reduced system on $\hat{\mathcal{X}}$ with input/output identical to that of the original system. Since the definition of \mathcal{N} ensures that the this reduced system is observable, we may apply 3. of Theorem 1.2 to it and get stabilization there, hence exponentially decaying output. ■

While it is well-known how to formulate natural conditions to adjoin so as to make the hypotheses sufficient for output stabilizability, the conditions normally used are in terms of observability properties of $[\mathbf{A}, \mathbf{C}]$ — and the whole point of Conjecture 1.1 lies in the avoidance of any such explicit observability hypotheses. As indicated by its title, the point of the present note is that the infinite-dimensional situation is different: Conjecture 1.1 need not always hold and our principal result is the presentation in the next section of an example demonstrating this.

2 A counterexample

In this section we present our principal result: the failure of Conjecture 1.1 in a general infinite-dimensional context. Since this is a negative result, we wish to consider it with the weakest form of the conclusion while imposing the strongest possible hypotheses.

In showing by this example that stabilization by feedback is impossible, we show impossibility even with the most favorable setting:

- i. $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces.
 - ii. \mathbf{A} is the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$.
 - iii. $\mathbf{B} : \mathcal{U} \rightarrow \mathcal{X}$, and $\mathbf{C} : \mathcal{X} \rightarrow \mathcal{Y}$ are bounded operators.
 - iv. The system (1) is open-loop nullcontrollable, i.e.,
for some fixed $T > 0$, for each $\xi \in \mathcal{X}$:
there is a control $u(\cdot) \in L^2([0, T] \rightarrow \mathcal{U})$ such that
 $x(T, \xi, u(\cdot)) = 0$.
- (5)

On the other hand, we are willing to permit the weakest interpretation of ‘feedback’, allowing even some (highly artificial) nonlinear, nonautonomous, discontinuous way of selecting the control, so long as the mechanism used produces an admissible control for each initial state and is *causal* — meaning only that

$$\text{If } y_1 \equiv y_2 \text{ on } [0, \tau], \text{ then } u_1 \equiv u_2 \text{ on } [0, \tau] \quad (6)$$

for arbitrary $\tau \geq 0$.

Theorem 2.1 *For infinite-dimensional problems the Conjecture 1.1 need not hold, even adjoining the strong hypotheses (5) and restricting the feedback mechanism of (2) only by (6).*

PROOF: For this negative result it is only necessary to provide a single example. For our exposition here, we put forward as such a counterexample the partial differential equation

$$x_t = x_s + x + \hat{u} \text{ for } t > 0, s > 0 \text{ with } x(0, \cdot) = \xi(\cdot) \quad (7)$$

so the state $x(t)$ is a function $x(t, \cdot)$ on $\mathbb{R}_+ = [0, \infty)$ and similarly for the control effect \hat{u} with the restriction that $\hat{u} = \mathbf{B}u(t, \cdot)$ will vanish on $[0, 2]$.

For an operator representation as in (1), we will take $\mathcal{X} = L^2(\mathbb{R}_+)$, $\mathcal{U} = L^2(2, \infty)$, and $\mathcal{Y} = L^2([0, 1])$, viewing \mathcal{U} as a subspace of \mathcal{X} , and

$$\begin{aligned} \mathbf{A} : \xi &\mapsto (\xi' + \xi) \quad \text{with domain} \\ \mathcal{D} &= H^1(\mathbb{R}_+ \rightarrow \mathcal{X}) = \{\xi \in \mathcal{X} : \xi' = d\xi/ds \in \mathcal{X}\}. \end{aligned}$$

The control map \mathbf{B} will be injection: $\mathcal{U} \hookrightarrow \mathcal{X}$ (extension as 0 on $[0, 2]$) and the observation map \mathbf{C} will be restriction to $[0, 1]$ so

$$[\mathbf{B}\omega](s) = \omega(s) \text{ for } s > 2, \quad y(t, \cdot) = x(t, \cdot) \Big|_{[0, 1]}.$$

The partial differential equation (7) is then just the abstract differential equation: $\dot{x} = \mathbf{A}x + \mathbf{B}u$ of (1) with output y .

We proceed to verify the hypotheses (5) for this system. It is immediate that $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces and that \mathbf{B}, \mathbf{C} are bounded linear operators. It is easy to check, for smooth $\xi(\cdot)$, that $x(t, s) = e^t \xi(s + t)$ gives a solution of the uncontrolled (7) with $u \equiv 0$. Thus, \mathbf{A} is the infinitesimal generator of the exponentially weighted left shift semigroup on \mathcal{X} :

$$[\mathbf{S}(t)\xi](s) = e^t \xi(s + t). \quad (8)$$

Finally, we verify the nullcontrollability condition with any $T \geq 3$. Note that the mild solution of (7) is given in terms of (8) by

$$x(T, s) = e^T \xi(T, s + T) + \int_0^T e^{T-\tau} u(\tau, s + T - \tau) d\tau.$$

To have $x(T, \cdot) \equiv 0$, we must choose the control u — subject to having $u(\tau, \cdot) \in \mathcal{U}$ — so that

$$\int_0^T e^{-\tau} u(\tau, s + T - \tau) d\tau = -\xi(T, s + T). \quad (9)$$

This can be done, for example, by setting

$$u(\tau, \sigma) = \begin{cases} -e^\tau \xi(\sigma + \tau) & \text{if } \sigma > 2, 0 < \tau < 1 \\ 0 & \text{else} \end{cases} \quad (10)$$

so $e^{-\tau} u(\tau, s + T - \tau) \equiv -\xi(T, s + T)$ for $0 < \tau < 1$ and all $s \geq 0$, noting that this gives $s + T - \tau = \sigma > 2$.

Having verified the hypotheses, it remains now to demonstrate that no causal feedback can possibly stabilize the output $y := x|_{[0,1]}$ for (7).

Claim: *For every causal feedback map M there is some initial state ξ for which the solution of the controlled system — (1) with $u = M(y)$ — fails to give asymptotically stable output, so $y(t) = \mathbf{C}x(t) \not\rightarrow 0$ as $t \rightarrow \infty$.*

The argument is conceptually simple, although a bit messy in the details. We begin with the integral representation

$$\begin{aligned} y(t, s) = x(t, s) &= e^t \xi(s + t) + \eta(t, s) && \text{for } 0 \leq s \leq 1 \\ \text{with } \eta(t, s) &= \int_0^{t-1} e^{t-\tau} u(\tau, s + t - \tau) d\tau. \end{aligned} \quad (11)$$

[The upper limit of the integral comes from the observation that we will have no contribution to $y(t, s)$ from the control u unless $s + t - \tau \geq 2$.]

We now consider initial states $\xi(\cdot) \in \mathcal{X}$ of the special form

$$\xi(s) := \{\omega_k e^{-k} \varphi(s - 2k) \text{ for } 2k \leq s < 2k + 2, k = 0, 1, \dots\} \quad (12)$$

where $\varphi(\cdot)$ is a suitable (smooth, nontrivial) function with support in $(0, 1)$ and $\omega_k = \pm 1$. [We note that (12) gives $\|\xi\|^2 = \|\varphi\|^2 / (1 - e^{-2}) < \infty$ and, indeed, ξ is similarly bounded in any $H_0^n(\mathbb{R}_+)$ if φ is in $H_0^n([0, 1])$.]

Introducing $\psi(\tau, s) := e^\tau \varphi(s + \tau)$, we note that for $t \in [2k - 1, 2k + 1]$, $s \in [0, 1]$ (so $(s + t) \in [2k - 1, 2k + 2]$) we have that

$$e^t \xi(s + t) = \omega_k e^k \psi(\tau, s) \quad (\tau := t - 2k) \quad (13)$$

from our requirement that $\text{supp } \varphi \subset (0, 1)$ in (12). By induction we then have for each $j = 1, \dots$:

$$(*)_j \quad y \Big|_{t \leq 2j-1} \text{ depends only on } (\omega_0, \dots, \omega_{j-1}),$$

meaning that, whatever the particular causal feedback mechanism used, the outputs y, \hat{y} occurring for initial states $\xi, \hat{\xi}$ will coincide for $0 \leq t < 2j - 1$ if the first j sign choices $(\omega_0, \dots, \omega_{j-1})$ in (12) are the same for $\hat{\xi}$ as for ξ .

This is clearly true for $j = 1$, since (11) gives no contribution η when $t \leq 1$ and (13) with $k = j - 1 = 0$ then gives y in terms of ω_0 . For the induction, we assume $(*)_j$ for some j and proceed in two steps. For $2j - 1 \leq t < 2j$ the first term in (11) depends (only) on ω_j by (13) and the second term depends on $u(\tau, \cdot)$ only for $\tau \leq t - 1 \leq 2j - 1$ — whence, by (6), only on $y \Big|_{[0, 2j-1]}$ which depends only on $(\omega_0, \dots, \omega_{j-1})$. Thus, $y \Big|_{[0, 2j]}$ depends only on $(\omega_0, \dots, \omega_j)$. For $2j \leq t < 2j + 1$ we again have the first term in (11) depending (only) on ω_j by (13) and the second term depending on $u(\tau, \cdot)$ only for $\tau \leq t - 1 \leq 2j$ — and so, by (6), only on $y \Big|_{[0, 2j]}$ which we have just seen depends only on $(\omega_0, \dots, \omega_j)$. Thus, $y \Big|_{[0, 2j+1]}$ depends only on $(\omega_0, \dots, \omega_j)$, which is $(*)_j$ as desired.

We now let y_k, η_k be the restrictions of y, η , respectively, to $t \in [2k - 1, 2k]$ and $s \in [0, 1]$ and let $\tilde{\psi}$ be the restriction of ψ to $A = [-1, 0] \times [0, 1]$. It follows from our discussion above that η_k depends only on the choices of $(\omega_0, \dots, \omega_{k-1})$ and, abusing notation slightly to think of y_k, η_k as functions of $[\tau = t - 2k, s] \in A$, we have from (11) that

$$y_k = \eta_k + \omega_k e^k \tilde{\psi} \quad (14)$$

is in $L^2(A)$.

The final argument is then by contradiction: we show that the signs (ω_0, \dots) used in (12) can always be chosen so as to produce an initial state ξ for which the feedback-controlled output $y(t, \xi, M(y))$ does not go to 0 as $t \rightarrow \infty$.

For this construction ω_0 is arbitrary — say, $\omega_0 = +1$. Recursively, we then suppose $(\omega_0, \dots, \omega_{k-1})$ have already been chosen so η_k is already determined for (14) by (11) and we will have (14) once we choose ω_k . Denoting by y_k^\pm the alternatives corresponding to the two possible choices $\omega_k = \pm 1$, we see from (14) that $\|y_k^+ - y_k^-\| = 2e^k \|\tilde{\psi}\|$, taking norms in $L^2(A)$. Thus, regardless

of the feedback mechanism and the prior choices of $(\omega_0, \dots, \omega_{k-1})$, one (at least) of the alternatives for ω_k could be selected so as to give

$$\int_{2k-1}^{2k} \|y(t, \cdot)\|_{\mathcal{U}}^2 dt \geq e^{2k} \|\tilde{\psi}\|^2 \quad (15)$$

for the output. We recursively make our sequence of choices so (15) holds for each k ; it is clear that, for this initial state, the output does not go to 0 (and, indeed, is not even bounded) as $t \rightarrow \infty$. This then verifies the Claim above and so demonstrates that (7) does provide a counterexample to the Conjecture 1.1. \blacksquare

Remark 2.2 Once one understands the underlying idea of the example used above for Theorem 2.1:

By the time we have any observational information about a segment of the state (data corresponding to some spatial interval), it will be too late to do anything about it. (16)

it is easy to construct similar examples in a variety of ways. For example, an alternative construction using the same idea would replace (7) by the equation

$$x_t = x_s + \hat{u} \quad (17)$$

while modifying the state space to have the w -weighted L^2 -norm

$$\|x(\cdot)\|_{\mathcal{X}} = \left[\int_0^\infty |x(s)|^2 w(s) ds \right]^{1/2}$$

with, e.g., $w(s) = e^{-\alpha s}$, similarly modifying the control space \mathcal{U} to preserve the nullcontrollability property. The construction of Theorem 2.1 then proceeds without any change except the omission of the exponential weighting factors: e.g., (8) now involves the unweighted left shift semigroup. [One could not have used such a rapidly decaying weight as $w(s) = \exp[-s^2]$ for the norm since the left shift would then be unbounded.]

It should be clear that the same idea may also be used in the context of the standard wave equation

$$x_{tt} = x_{ss} + \hat{u} \quad \text{on } \mathbb{R}_+ \times \mathbb{R} \quad (18)$$

with, e.g., output given by observation of the restriction of $x(t, s)$ for $|s| < 1$ and control restricted to $|s| > 2$. [The state now includes x_t as well as x , but this causes no difficulty.] Once one recalls that solutions of (18) may be dissected as left- and right-moving waves, one can ignore the invisible waves which will not pass through the observation interval so this becomes essentially the previous example. \blacksquare

3 Further remarks

The negative result of Theorem 2.1 raises the possibility that Conjecture 1.1 must *always* fail in the infinite-dimensional case, i.e.,

$$\begin{aligned} & \text{for every } [\mathbf{A}, \mathbf{B}] \text{ as in (5), there is some bounded } \mathbf{C} \\ & \text{such that Conjecture 1.1 fails, even subject only to (6).} \end{aligned} \quad (19)$$

Remark 3.1 All the examples noted have depended on an interaction between the geometry and the semigroup evolution in time related to a limitation on the ‘speed of propagation’ within the model. Since time is a continuous variable, it has been significant, in each case, that the geometry has had a continuous, ordered structure. On the other hand, since the Hilbert space $L^2(\mathbb{R}_+)$ is isometrically isomorphic to the sequence space ℓ^2 , the example of Theorem 2.1 must have an isomorphic image in the sequence space. Especially since the construction there is really only concerned with a time sampled description, one should be able to proceed by embedding iteration of the weighted discrete left shift operator on ℓ^2 in an image of the continuous semigroup of (8). Unfortunately, there seems not to be any simply described image of (8) in ℓ^2 , but the existence of such an isomorphism makes it awkward to conjecture that a finite speed of propagation is intrinsic to the failure of Conjecture 1.1. ■

We see, however, that the counterconjecture (19) cannot be unrestrictedly true:

Theorem 3.2 *Consider (1) subject to a modified (5) — further assuming in ii. that the semigroup $\mathbf{S}(\cdot)$ is analytic, but weakening iv. to replace the null-controllability by open-loop stabilizability: existence of some output-stabilizing control for each initial state. Then, without special consideration of observability, there is a feedback mechanism M , causal in the sense of (2), (6), which stabilizes the system output.*

PROOF: We need only one such feedback mechanism M — causal, but without concern for such properties as autonomy, linearity, or continuity. For example, there is the following:

Letting u vanish on $[0, 1]$, choose any state $\hat{\xi}_1$ consistent with the observed output, i.e., for some $\hat{\xi}$ one has

$$\hat{\xi}_1 = x(1, \hat{\xi}, 0) = \mathbf{S}(1)\hat{\xi} \quad \text{while} \quad y(t, \hat{\xi}, 0) \equiv y(t) \quad \text{on } [0, 1].$$

By assumption, there is some output-stabilizing control \hat{u}_1 for the initial state $\hat{\xi}_1$ so we can now define our control u on $[1, \infty)$ to be \hat{u}_1 , suitably shifted.

We do not know the true initial state $\xi = x(0)$, but observe that linearity gives

$$\begin{aligned} x(t, \xi, u) &= x(t, \hat{\xi}, u) + \mathbf{S}(t)[\xi - \hat{\xi}] \quad \text{so} \\ y(t, \xi, u) &= y(t, \hat{\xi}, u) + \mathbf{C}\mathbf{S}(t)[\xi - \hat{\xi}] \end{aligned}$$

for any control u . Our choice of $\hat{\xi}_1$ ensured the identity

$$\mathbf{CS}(t)\hat{\xi} = y(t, \hat{\xi}, 0) \equiv y(t) = y(t, \xi, 0) = \mathbf{CS}(t)\xi$$

for $0 \leq t \leq 1$ and, by the analyticity of $\mathbf{S}(\cdot)$, this identity holds for all $t \geq 0$. Thus the output for the system using this control u is just $y(t, \xi, u) \equiv y(t, \hat{\xi}, u)$ and the relation between $\hat{\xi}, u$ and $\hat{\xi}_1, \hat{u}_1$ ensures that $y(t, \hat{\xi}, u) \equiv y(t-1, \hat{\xi}_1, \hat{u}_1)$ — which goes to 0 as $t \rightarrow \infty$ by the choice of \hat{u}_1 . ■

Remark 3.3 Comparing Corollary 1.3 with Theorem 2.1, Remark 3.1, and Theorem 3.2, we note the difficulty of formulating a satisfactory replacement for (5) as a plausible sufficient consideration for output stabilizability — especially if one would wish the feedback to have some additional properties of autonomy and continuity.

A particularly interesting test case for output stabilization of a distributed parameter system was suggested by E. Zuazua. Consider the usual wave equation on a disk Ω with Dirichlet boundary control so, open loop, one has exact nullcontrollability in fixed time T . Now let the observation operator \mathbf{C} give as output the suitably topologized pair $[u, u_t]$ restricted to some smaller concentric disk ω . We might expect that the behavior will be essentially as in the examples above since the geometry admits trapped waves within the annulus $\Omega \setminus \omega$. The role of the restrictions of the state to intervals $[2n, 2n+1]$ in Theorem 2.1 might now be played by components of the initial state approximating these trapped rays, closer and closer to the boundary (as, e.g., inscribed n -gons with increasing n) so it takes longer and longer for ‘leakage’ to reach ω . The difficulty in analyzing this situation is that we no longer have the precise separation property which made it comparatively easy in Theorem 2.1 to keep track of the interactions of control, output, and initial state. This problem remains open. ■

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