On uniform nullcontrollability and blow-up estimates

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Abstract

The paper looks at some of the techniques (separation of variables, Fourier series, Carleman estimates) used to obtain observability and null-controllability results to determine qualitatively the asymptotic behavior of the estimates obtained with respect to relevant parameters. We are particularly concerned with blow-up as the control time becomes short ($T \to 0$).

1. Introduction

Most of the work done on observability or nullcontrollability for distributed parameter systems refers to single settings, but increasingly one is interested to consider the relation between a family of such problems depending on some parameter (as, e.g., the coefficients of the equations or the length of the time interval or a discretization mesh size). Our twin themes here — *uniformity* and *blow-up* — are, of course, just the two possible alternatives in an investigation of asymptotic behavior. Here, in looking at some of the techniques used to obtain observability and nullcontrollability results, we will examine them more carefully to determine the asymptotic behavior of the estimates obtained with respect to relevant parameters.

We will be particularly concerned with blow-up as the control time becomes short ($T \to 0$) and so will restrict our attention to problems, as for the heat equation, with no minimum control time. This question of blow-up as $T \to 0$ was apparently introduced for distributed parameter systems in [26] (and only later for finite dimensional systems in [30]) and has recently been the subject of greater interest (cf., e.g., [33], [2], [3], [4], [5]) partly in response to Da Prato's observation [7] that this is relevant to the analysis of corresponding stochastic differential equations.

A few of the examples presented here are new, but many are relevant historical examples revisited. These have largely been drawn from my own earlier work, simply because of my greater familiarity with that. On the other hand, the central position of references to [31] may be considered a shameless bit of advertising for a result in nonharmonic analysis specifically designed for its relation to the present concerns.

Consider an abstract linear autonomous system

$$u_t = \mathbf{A}u \tag{1.1}$$

where we include the homogeneous boundary conditions in specification of the domain of the operator $\mathbf{A}$. Given observation of

$$y(t) = \mathbf{B}u(t,\cdot) \qquad \text{for } 0 \le t \le T, \tag{1.2}$$

our principal concern here is with an *observability estimate*:

$$\|u(T,\cdot)\| \le C\|y(\cdot)\|. \tag{1.3}$$

We speak of *uniform observability* for a family $\{(\mathbf{A},\mathbf{B},T)\}$ of such problems if (1.3) holds in each of the instances with a fixed constant C used for all the problems considered.

It is well-known that (1.1), (1.2) is dual to the nullcontrollability problem for the adjoint equation (after a time reversal):

$$v_t = \mathbf{A}^*v + \mathbf{B}^*\varphi \qquad \text{with } v\Big|_{t=0} = \omega \tag{1.4}$$

> *Given ω, determine a nullcontrol φ on $[0,T]$ so that the solution of (1.4) will satisfy: $v\Big|_{t=T} = 0$.*

This duality means that one will have (1.3) if and only if for each ω one can choose such a nullcontrol with

$$\|\varphi\| \le C\|\omega\| \tag{1.5}$$

— using the same constant C as in (1.3). Indeed, we will restrict our attention here to settings in which the relevant spaces are Hilbert spaces and we can then choose each nullcontrol φ to minimize the norm; the mapping $\mathbf{C} : \omega \mapsto \varphi$ for each of the problems is then linear and continuous with operator norm $\|\mathbf{C}\| \le C$. We now speak of *uniform nullcontrollability* for a family of such problems if this holds in each instance with a fixed nullcontrollability bound C.

PROPOSITION 1 *Suppose we have a weak continuity property for the sequence of problems $\{(1.4)_\nu : \nu = 1, 2, \ldots; \infty\}$:*

$$\begin{gathered} \textit{using } \varphi_\nu \textit{ in each } (1.4)_\nu \textit{ to get a solution } v_\nu, \\ \textit{if } \varphi_\nu \rightharpoonup \varphi_\infty \textit{weakly, then also } v_\nu(T) \rightharpoonup v_\infty(T). \end{gathered} \tag{1.6}$$

Then, if the sequence $\{(1.4)_\nu : \nu = 1, 2, \ldots\}$ is uniformly nullcontrollable, it follows that the limit problem $(1.4)_\infty$ is also nullcontrollable, with the same bound.

PROOF: In a Hilbert space context, for any ω the uniform bound on the sequence of nullcontrols $\{\varphi_\nu\}$ gives a subsequence weakly convergent to some φ_0 and by (1.6) this φ_0 must be a nullcontrol for ω in $(1.4)_\infty$. ■

Complementary to this is an obvious blow-up result: if the limit problem $(1.4)_\infty$ is *not* nullcontrollable, then one must have blow-up: $C = C_\nu \to \infty$ in (1.5). In particular, suppose one were to take $\mathbf{A}_\nu, \mathbf{B}_\nu$ fixed in $(1.4)_\nu$, but vary the control time — more precisely, keeping the nominal time T fixed as there, but restricting support of the control φ_ν to $t \in [0, T_\nu]$ — then, if this can be done with $T_\nu \to 0$, any weakly convergent subsequence would necessarily give convergence to f_0 with empty support, clearly not a nullcontrol for any $\omega \neq 0$. Thus, if one does have nullcontrollability for arbitrarily short control times, this bound must blow up as the time goes to 0.

The *spectral approach* to the observability estimate (1.3) utilizes spectral decomposition of the operator $\mathbf{A}$ appearing in (1.1), making a spatial expansion in eigenfunctions so time dependence is given by an exponential series. For nullcontrollability, this is also known as the *method of moments*. This has been a useful tool for treating distributed parameter systems since the earliest days of the subject; note the papers cited here, especially the survey paper [23], and the book [6].

Since the paper [31] was motivated precisely by our present concerns regarding uniformity and blow-up in applying the spectral approach, we will devote the next section to describing the principal results obtained there. Section 3 is then devoted to discussing the spectral approach and some historical examples of how it works out while Section 4 considers blow-up, mostly utilizing the spectral approach but also noting the behavior of Carleman estimate techniques in relation to our thematic concerns. The final section then discusses some further recent results,

2. The 'window problem' for complex exponential series

Let Λ be a complex sequence $\{\lambda_k = \tau_k + i\sigma_k\}$ and consider functions of the form:

$$f(t) = \sum_k c_k e^{i\lambda_k t}. \tag{2.1}$$

We think of these as 'observed through the time window $[0, T]$' and topologize this set of functions $\mathcal{M} = \mathcal{M}_T(\Lambda)$ as a subset of $L^2(0, T)$. The 'window problem' we consider here is then to determine the sequence of terms (evaluated at $t = T$):

$$\mathbf{c}_T = \left\{ c_k e^{i\lambda_k T} \right\} \tag{2.2}$$

from observation of $f(\cdot)$ on $(0, T)$.

We will impose the following conditions on the exponent sequence Λ:

$$\sigma_k \geq 0, \tag{2.3}$$

$$\text{uniform separation: for some } r_0 > 0, \quad |\lambda_j - \lambda_k| \geq r_0 \ (j \neq k), \tag{2.4}$$

uniform sparsity: for some $a > 0$ and uniformly for $\lambda_* \in \Lambda$, one has

$$\#\{\lambda \in \Lambda : 0 < |\lambda - \lambda_*| \leq r\} \leq \nu(r) \equiv a\sqrt{r}. \tag{2.5}$$

The principal result of [31], somewhat specialized to this form of ν, is then:

THEOREM 2 *If the complex sequence Λ satisfies (2.3)–(2.5), then there is a constant $C = C(T, \Lambda)$ such that*

$$\sum_k \left|c_k e^{i\lambda_k T}\right|^2 \leq C^2 \int_0^T \left|\sum_k c_k e^{i\lambda_k t}\right|^2 dt \tag{2.6}$$

for all $f, \mathbf{c}_T$ as in (2.1), (2.2). For the special form $\nu(r) = a\sqrt{r}$ used here in (2.5), we have

$$C = C(T, \Lambda) \leq Ae^{B/T} \tag{2.7}$$

with positive constants A, B depending only on r_0, a.

Thus, we necessarily have uniformity of the estimate over families $\{\Lambda\}$ of such exponent sequences for which we can use fixed r_0, a and we have blow-up in the estimate exponential to the order of $1/T$ as $T \to 0$.

We note that the heart of the proof in [31] is a technical lemma.

LEMMA 3 *For any $T > 0$ and any $\nu(r) = a\sqrt{r}$ there exists an entire function $P(\cdot)$ such that*

- *$|P(z)| \leq 1$ on the upper half-plane $\mathbb{C}_+$ and is real and positive on the imaginary axis, with a somewhat technical a-dependent lower bound for $P(is)$ when $s \geq 0$*
- *P is of exponential type with*

$$\left|e^{-i(T/2)z} P(z)\right| \leq Ke^{(T/2)|z|} \qquad (z \in \mathbb{C}) \tag{2.8}$$

- *For real r one has a bound*

$$\left|P(r)e^{\nu(|r|)}\right| \leq C = C(a, T). \tag{2.9}$$

The constant C in (2.9) satisfies

$$C(a, T) \leq Ae^{B/T} \tag{2.10}$$

for T near 0 (with a-dependent A, B).

The paper [31] actually considers f as observed on an interval $[0, \delta]$ although the terms are evaluated at $t = T$, without necessarily taking $\delta = T$ as here. In this, as in some other respects, we are simplifying the description here of the results of [31] for our present convenience. In [31] the admissible functions $\nu(\cdot)$ are, more generally:

continuous and unboundedly increasing, but with $\nu(s)/s^2$ decreasing and integrable on $[r_0, \infty)$.

Obviously the statements of Theorem 2 and Lemma 3 become more complicated with more general admissible functions $\nu(\cdot)$ for (2.5). For full details, of course, see [31].

This approach is particularly effective for one-dimensional problems. We can restrict ourselves here to taking $\nu(r) = a\sqrt{r}$ in (2.5) because the sequences involved for our applications typically come from eigenvalues of Sturm-Liouville problems and so are quadratically distributed, i.e., asymptotically like ck^2. We note, for example, that for the exponent sequence $\{\lambda_k = ck^2\}$ one has an easy computation to obtain (2.5) with $\nu(r) = \sqrt{2r/|c|}$.

It should be noted that with the restriction to $\nu(r) = a\sqrt{r}$ results like Lemma 3 and Theorem 2 had been obtained earlier (cf., e.g., [20], [17], [19], [9]) — except for consideration of the asymptotics (2.10), (2.7). As compared, e.g., with [29], the particular innovation of Theorem 2 is the treatment of more general complex exponent sequences, used here for Theorems 6 and 10.

3. Spectral methods; some history

We have already noted that spectral methods have long provided a useful tool for treating control-theoretic questions for partial differential equations and we sketch here some historical examples. While the original treatments referred to a variety of background results on Dirichlet series and non-harmonic analysis by Schwartz [24], Redheffer [20], Luxembourg and Korevaar [17], etc. — as well as developing some additional theory themselves — it will be sufficient as well as more convenient here to refer only to Theorem 2 as described in the preceding section.

The 'spectral approach' to the observability estimate (1.3) for (1.1), (1.2) utilizes a spatial expansion in eigenfunctions so the time dependence is given by an exponential series. Thus, we assume the set of eigenfunctions $\{e_k(\cdot)\}$ of $\mathbf{A}$ is a Riesz basis so the solution u of (1.1) has an expansion:

$$u(t, \cdot) = \sum_k a_k e^{\alpha_k t} e_k(\cdot) \tag{3.1}$$

where $\{\alpha_k\}$ is the corresponding set of eigenvalues: $\mathbf{A}e_k = \alpha_k e_k$. Using (3.1) in (1.2) then gives an exponential series for the observation as a function of t:

$$y(t) = \sum_k c_k e^{\alpha_k t} \tag{3.2}$$

where

$$c_k = \beta_k a_k \qquad (\beta_k = \mathbf{B}e_k)\,. \tag{3.3}$$

Suppose, now, $y(\cdot)$ is scalar and one has a uniform lower bound

$$|\beta_k| \geq \kappa > 0 \text{ so } K_1 = \sup_k\{1/|\beta_k|\} \leq 1/\kappa < \infty \tag{3.4}$$

and also suppose the sequence $\{\alpha_k = i\lambda_k\}$ is such that Theorem 2 applies to the exponential series (3.2). Using (3.1) for $t = T$ and recalling (3.3) (so $a_k = c_k/\beta_k$) and that $\{e_k(\cdot)\}$ is a Riesz basis, we then have

$$\begin{aligned} \|u(T,\cdot)\|^2 &\le K^2 \sum_k \left|a_k e^{\alpha_k T}\right|^2 = K^2 \sum_k \left|\frac{c_k}{\beta_k} e^{i\lambda_k T}\right|^2 \\ &\le K^2 [\sup_k \{1/|\beta_k|\}]^2 \sum_k \left|c_k e^{i\lambda_k T}\right|^2 \\ &\le (CKK_1)^2 \|y(\cdot)\|^2_{L^2(0,T)} \end{aligned} \tag{3.5}$$

which is just (1.3) with the bound as given by (2.6), apart from the fixed constants K, K_1.

While our original consideration was the observability map: $y \mapsto u(T,\cdot)$, we note that the spectral method has factored this as

$$y \mapsto \mathbf{c}_T = \left(c_k e^{-\alpha_k T} : k = 1, \ldots\right) \mapsto \sum_k (1/\beta_k) c_k e^{-\alpha_k T} e_k = u(T,\cdot).$$

This use of Theorem 2 for (3.2) requires the sparsity condition (2.5), which is plausible only for one-dimensional settings. We do note, however, that a separable setting (e.g., for a cylindrical [product] region $\Omega = (0,1) \times \Omega_*$) can reduce the problem to a collection of one-dimensional problems. Suppose the eigenfunctions of $\mathbf{A}$ were to have the form of products $\{e_k(x) f_\ell(x_*)\}$ (with $\{f_\ell\}$ orthonormal for simplicity). We then can replace (3.1), (3.2) by

$$\begin{aligned} u(t,\cdot) &= \sum_{k,\ell} a_{k,\ell}\, e^{\alpha_{k,\ell} t}\, e_k(\cdot) f_\ell(\cdot) \\ y(t,\cdot) &= \sum_\ell y_\ell(t) f_\ell(\cdot) \\ &\qquad \text{with } y_\ell(t) = \sum_k c_{k,\ell}\, e^{\alpha_{k,\ell} t} \end{aligned} \tag{3.6}$$

where we assume $\mathbf{B}$ acts only at the base $\Gamma = \{0\} \times \Omega_*$ with $\mathbf{B}[e_k f_\ell] = [\mathbf{B} e_k] f_\ell = \beta_k f_\ell$ so (3.3) becomes $c_{k,\ell} = \beta_k a_{k,\ell}$ and we continue to assume (3.4). We may then consider each problem$_\ell$ separately: as in (3.5) we get

$$\|u_\ell(T,\cdot)\|^2 \le (C_\ell K K_1)^2 \|y_\ell(\cdot)\|^2_{L^2(0,T)}$$

with

$$u_\ell(t,\cdot) = \langle f_\ell, u(t,\cdot)\rangle = \sum_k a_{k,\ell} e^{\alpha_{k,\ell} t} e_k(\cdot)$$

so

$$\begin{aligned} \|u(T,\cdot)\|^2 &= \sum_\ell \|u_\ell(T,\cdot)\|^2 \\ &\le [(KK_1) \max_\ell \{C_\ell\}]^2 \sum_\ell \|y_\ell\|^2 \\ &= \hat{C}^2 \|y\|^2_{L^2([0,T]\times\Gamma)} \end{aligned} \tag{3.7}$$

with existence of $\max_\ell\{C_\ell\}$ giving $\hat{C}$ corresponding to a requirement of uniform observability for the family $\{\text{problem}_\ell\}$.

Going back to the 1960's, we begin by following [18] in considering boundary observation at $x = 0$ for the one-dimensional heat equation

$$u_t = u_{xx} \quad \text{on } (0,1) \text{ with } u = 0 \text{ at } x = 0,1 \tag{3.8}$$

In the absence of information about the initial state $\left(u\Big|_{t=0}\right)$, we observe the endpoint heat flux, $y(t) = u_x(t,0)$, for an interval $0 \le t \le T$ and wish to determine the terminal state $\left(u\Big|_{t=T}\right)$. More specifically, we seek an estimate

$$\int_0^1 |u(T,x)|^2\,dx \le C^2 \int_0^T |u_x(t,0)|^2\,dt \tag{3.9}$$

for solutions of (3.8), since that estimate ensures the observability. To obtain (3.9), we use the spectral approach as in the preceding section.

The operators here corresponding to $\mathbf{A}, \mathbf{B}$ in (1.1), (1.2) are the Sturm-Liouville operator $\mathbf{A} : z \mapsto z''$ with homogeneous Dirichlet boundary conditions (so the eigenvalue sequence is $\{\alpha_k = -\pi^2 k^2\}$ with the orthonormal basis of corresponding eigenfunctions $\{e_k(x) = (1/\sqrt{2}) \sin k\pi x\}$) and $\mathbf{B} : z \mapsto z'(0)$ (giving $\beta_k = k\pi/\sqrt{2}$ so we obviously have (3.4) with $K_1 = 1$). As was remarked in the previous section, the exponent sequence $\{\lambda_k = i\pi^2 k^2\}$ satisfies (2.3)–(2.5) so it follows immediately from Theorem 2 that we have (2.6). Thus we have (3.5), (1.3) — i.e., (3.9).[1]

Already in [18] the argument via (3.6) was noted for the heat equation on a cylindrical region $\Omega = (0,1) \times \Omega_*$— although the uniformity of the component one-dimensional problems was neither noted nor even noticed, since one had eigenvalues $\alpha_{k,\ell} = -\pi^2 k^2 - \hat{\alpha}_\ell$ where $\{-\hat{\alpha}_\ell\}$ is the eigenvalue sequence for the Laplacian on the cross-sectional region Ω_*, which permitted simple absorption of the factors $e^{-\hat{\alpha}_\ell T} < 1$ in the estimation. The first problem which explicitly required[2] consideration of uniformity for the estimates of a family of quadratically distributed exponent sequences was the treatment of the heat equation on a sphere [19], [9] by way of separation of variables for the spherical Laplacian. This is along the lines of (3.6) above, but required some concern for the zeroes of Bessel functions to verify the necessary uniformity in (2.4), (2.5).

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[1]This is so much easier now than it seemed in the late 1960's! Indeed, at that time the corresponding nullcontrollability result was obtained by an independent direct argument (cf., [8], solving a moment problem to construct the control), rather than by an appeal to the now-standard duality.

[2]In a sense this was not required, but was only an artifact of proofs by way of separation of variables for the Laplacian on a sphere along the lines of (3.6) above. At the time of [19], for example, one had available neither the deep arguments of [21], [22] nor even the more elementary observation [25] that (in the nullcontrollability context) one could obtain boundary nullcontrollability for a general bounded region Ω by embedding Ω in a large box or cylinder (for which the expansion could be treated as here) and then using as control the trace on $\partial\Omega$ of that nullcontrolled solution.

A rather different kind of spectral approach was used by Russell, avoiding the necessity to use spatial separation of variables to decompose into one-dimensional problems by deriving observability/nullcontrollability results for the heat equation from similar results for the corresponding wave equation (which were then obtained by using scattering theory, etc.).

THEOREM 4 *On a spatial domain Ω, consider a second-order (wave) equation for $w = w(\tilde{t}, x)$:*

$$w_{\tilde{t}\tilde{t}} + \mathbf{A}^2 w = 0 \qquad (0 \leq \tilde{t} \leq \tilde{T}) \tag{3.10}$$

where $\mathbf{A}$ is a positive operator (as, e.g., $(-\Delta)^{1/2}$) and assume that, for a suitable observation operator $\mathbf{B}$, one has an observability estimate

$$\|\mathbf{A}w(\tilde{T})\|^2 + \|w_t(\tilde{T})\|^2 \leq \tilde{C}^2 \int_0^{\tilde{T}} \|\mathbf{B}w(t)\|^2 \, dt \tag{3.11}$$

for solutions of (3.10). *If one considers the corresponding heat equation for the same Ω:*

$$u_t + \mathbf{A}^2 u = 0 \qquad (0 \leq t \leq T), \tag{3.12}$$

then one also has observability, using the same $\mathbf{B}$, for arbitrarily small $T > 0$ with a corresponding observability estimate

$$\|u(T)\|^2 \leq C^2 \int_0^T \|\mathbf{B}u(t)\|^2 \, dt. \tag{3.13}$$

Proof: See [21], [22]; compare also [27]. One notes that the Fourier transforms in t of (3.10), (3.12) are

$$-\tilde{\tau}^2 \hat{w} + \mathbf{A}^2 \hat{w} = 0, \qquad i\tau\hat{u} + \mathbf{A}^2\hat{u} = 0$$

so — formally — the equations can be related in the Fourier domain by substituting $\tau = \tilde{\tau}^2$. This permitted Russell to use Fourier transform techniques, especially the Paley-Wiener Theorem, to obtain (3.13) from (3.11). The technical lemma needed to justify the formal procedure was a version of Lemma 3, above, although without (2.10). Essentially, (2.9) justifies that the relevant functionals will transform legitimately (in L^2) while (2.8) ensures that the kernels have support in $[0, T]$.

The spectral approach (with reduction to one-dimensional problems) has also been used for partial differential equations other than the heat equation.

THEOREM 5 *For the Euler plate equation*

$$\begin{aligned} u_{tt} + \Delta^2 u = 0 \quad & \text{on } \Omega = (0,1)^2, \\ u_\nu = 0 = (\Delta u)_\nu \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.14}$$

one has boundary observability — and so controllability, by duality — for arbitrarily short times $T > 0$, using the observation of $y(t, x_2) = u(t, 0, x_2)$ for $0 < x_2 < 1$ and $0 < t < T$.

PROOF: See [14], [29]. The boundary conditions selected have the considerable advantage of making the problem separable and permitting explicit computation so this problem can be treated as in the double expansion (3.6), with

$$e_k(x_1) = (1/\sqrt{2})\cos \pi k x_1, \qquad f_\ell(x_2) = (1/\sqrt{2})\cos \pi \ell x_2.$$

We then obtain the expansions

$$\begin{aligned} u(t,\cdot) \quad &= \sum_\ell u_\ell(t,\cdot)\varphi_\ell(\cdot) \\ &\text{with } \ u_\ell(t,\cdot) = \sum_{k,\pm} a_{[k,\pm],\ell}\, e^{\pm i\pi^2[k^2+\ell^2]\,t} e_k(\cdot) \\ y(t,\cdot) \quad &= \sum_\ell y_\ell(t) f_\ell(\cdot) \\ &\text{with } \ y_\ell(t) = \sum_{k,\pm} c_{[k,\pm],\ell}\, e^{\pm i\pi^2[k^2+\ell^2]\,t}. \end{aligned}$$

where, for this observation, $c_{[k,\pm],\ell} = a_{[k,\pm],\ell}$. For each ℓ one has the exponent sequence

$$\Lambda_\ell = \left\{\lambda^\ell_{k,\pm} = \pm\pi^2\left[k^2+\ell^2\right]\right\}$$

and one easily verifies (2.4), (2.5) for these sequences, uniformly in ℓ. Thus Theorem 2 applies to give (3.7) and so the desired observability. ■

4. Blow-up

We have already noted as a corollary to Proposition 1 that nullcontrols associated to control times $T \to 0+$ cannot remain bounded. The question of determining the asymptotic *blow-up rate* (e.g., for boundary control of the one-dimensional heat equation) was raised as far back as the mid-1970's in [26] — although with the wildly optimistic conjecture that this blow-up rate was $\mathcal{O}(1/\sqrt{T})$ as $T \to 0$. [It is interesting that this question of blow-up rates was considered for distributed parameter systems before the corresponding question had been raised for finite dimensional control problems.] The finite dimensional case, however, now seems quite well understood [30], [32]; see also [33]. The infinite dimensional case remains fertile ground for further investigation.

By the mid-1980's the incorrect conjecture in [26] had been somewhat corrected: the paper [28] kept track of the relevant 'constants' in the treatment in [20] and obtained, as an $e^{\mathcal{O}(1/T)}$ upper bound on the blow-up rate for the 1-D heat equation (3.8). This was complemented by Güichal's computation [11] of a lower bound with the same asymptotic behavior. Thus, at least for (3.8), it is now known that the correct asymptotics as $T \to 0$ are precisely *'exponential to order of* $1/T$*'*. We may note that use of Theorem 2 in the analysis in the previous section already provides, through (2.7), an upper bound exponential to order of $1/T$ for each of the problems discussed there: for example, again for (3.8), we may note that the constant C appearing in (3.9)=(1.3) was exactly the constant obtained in (2.6) and so satisfies (2.7) by Theorem 2. The same use of (2.7) applies also to the other heat equation problems considered there. In particular, we note that, while Theorem 4 did not include any condition of the asymptotics as $T \to 0$, the inclusion of (2.10) in Lemma 3 now provides the blow-up rate

$$C(a,T) \le Ae^{B/T}\tilde{C} \qquad \text{in (3.13)} \tag{4.1}$$

(with a-dependent A, B). As an example other than the heat equation, we also recall the treatment above of the Euler plate equation (3.14). Apart from the necessary uniformity of the one-dimensional problems, the treatment above and in [29] gives the now-familiar $e^{\mathcal{O}(1/T)}$ blow-up as $T \to 0$.

We might next consider observability for the equation

$$\begin{aligned} u_{tt} - 2\kappa\Delta u_t + \Delta^2 u = 0 \quad &\text{on } \Omega = (0,1)^2 \\ u_\nu = 0 = (\Delta u)_\nu \quad &\text{on } \partial\Omega \end{aligned} \tag{4.2}$$

describing a structurally damped Euler plate; one can also write this as a first-order system

$$U_t = \mathbf{A}U \qquad \text{with } \mathbf{A} = (-\Delta)M,\ M = \begin{bmatrix} 0 & 1 \\ -1 & -2\kappa \end{bmatrix}. \tag{4.3}$$

Boundary control for this plate model problem was considered by Hansen in [12] — and is also described in Section 6 of [31], specifically devoted to some applications to distributed parameter system theory, since (apart from the facility of obtaining the required observability estimate from a general result) the use of Theorem 2 automatically provides the blow-up estimate (2.7) for this problem as for (3.14).

THEOREM 6 *For the structurally damped Euler plate model* (4.2) *with*[3] *damping coefficient* $0 < \kappa < 1$, *consider observation of* $y = u\Big|_{x_1=0}$ *for* $0 < t < T$. *Then we have observability with an observability estimate*

$$\left[\int_0^1\int_0^1 \left(|\Delta u|^2 + |u_t|^2\right)\,dx_1dx_2\right]^{1/2} \le Ae^{B/T}\left[\int_0^T\int_0^1 |u(t,0,x_2)|^2\,dx_2dt\right]^{1/2}. \tag{4.4}$$

Proof: Note that the eigenvalues of $\mathbf{A}$ are $\xi_\pm\pi^2[k^2+\ell^2]$ where the eigenvalues of M are $\xi_\pm = -\kappa \pm i\sqrt{1-\kappa^2}$. Much as for the treatment of (3.14), one gets a family of exponent sequences for (2.1)

$$\Lambda_\ell = \left\{\lambda^\ell_{k,\pm} = -i\xi_\pm\pi^2\left[k^2+\ell^2\right]\right\},$$

[3]There is no new difficulty as $\kappa \to 0$, for which one just gets (3.14) in the limit. We do note, however, that one must have blow-up as $\kappa \to 1$, when the eigenvalues and eigenvectors of M degenerate, but we will discuss here only the blow-up rate as $T \to 0$ and not for this.

but here this sequence no longer lies on the real axis: instead, each of the resulting one-dimensional problems here involves a complex exponent sequence with each Λ_ℓ the union of two copies of a quadratically distributed sequence placed along the rays $\{[\pm\sqrt{1-\kappa^2}+i\kappa]\tau : \tau > 0\}$. That geometry is discussed in [31] and one easily verifies (2.3) and that such a union continues to satisfy the conditions (2.4), (2.5) and that the relevant shifts leave the same $\nu(\cdot)$ uniformly applicable. Thus Theorem 2 gives the desired uniform estimate. That estimate was, of course, obtained by Hansen. Section 6 of [31] observes that use of Theorem 2 automatically provides the blow-up estimate (2.7) for this problem as was noted earlier for (3.14). ■

Other approaches Above we have been considering applications of the spectral expansion approach, using Theorem 2 to obtain blow-up estimates. We will not discuss this here, but note that weighted energy estimates have also been effectively used to this end: cf. [2], [3], [4], [5]. We note, at this point, that finite dimensional results of [30], [33] have also proved directly applicable for distributed parameter systems; cf., e.g., [16].

There are also, of course, a variety of situations in which observability results seem unavailable by any use of spectral expansions and have only been obtained through the use of Carleman estimates. For the heat equation $u_t = \Delta u$ alone, these situations include most of the known results about observability/nullcontrollability with interaction restricted to a small patch as well as for variants with variable coefficients. Three things become clear from, e.g., a look at the book [10]:

- this is a powerful approach to these observability/controllability problems,
- the dependence of the estimate on the coefficients is only through certain bounds on coefficients and their derivatives and so is uniform over relevant classes of equations, and
- the calculations of these Carleman estimates is messy enough[4] to make it difficult to track any time dependence so as to obtain an estimation of the blow-up rate.

Apparently no such estimation has previously been done, but, at least for the heat equation, we have verified [15] the blow-up rate for the patch control settings which rely on Carleman estimates.

THEOREM 7 *Consider the patch nullcontrollability problem for the heat equation:*

$$\begin{aligned} u_t - \Delta u = \varphi = \textit{control} \quad & \textit{on } \mathcal{Q} = [0,T]\times\Omega \\ u = 0 \quad & \textit{on } \Sigma = [0,T]\times\partial\Omega \\ u = u_0 \quad & \textit{on } \Omega \textit{ at } t=0 \end{aligned} \tag{4.5}$$

[4]The forthcoming treatment [1] makes the Carleman calculations somewhat more transparent in the case of the heat equation, but these still remain formidable.

with the spatial support of the control φ restricted to a specified patch ω (so $\omega \neq \emptyset$ is open with compact closure in the bounded, connected, open domain Ω). There are then constants A, B depending only on Ω, ω, T_* such that:

For each $0 < T < T_*$ and each initial state $u_0 \in L^2(\Omega)$ there exists a nullcontrol function $\varphi \in L^2([0,T] \times \omega)$ — i.e., the solution of (4.5) satisfies $u(T, \cdot) \equiv 0$ — such that

$$\|\varphi\|_{L^2([0,T]\times\omega)} \leq Ae^{B/T}\|u_0\|_{L^2(\Omega)}. \tag{4.6}$$

Proof: See [15]. As usual, one works with the dual observability problem. Following the Carleman calculations in [1], one can track the T-dependence with a careful T-dependent scaling of the parameters to obtain an estimate of Carleman type for the observability problem, with a constant independent of $T > 0$. It is in going from that bound to the observability estimate that we now obtain the anticipated estimate of the now familiar form: $e^{\mathcal{O}(1/T)}$. ■

5. Some more recent results

We begin this section by mentioning another result presented in Section 6 of [31]. The intent here was to be able to consider homogenization of a control problem — so we are interested in considering $q = q(x, x/\varepsilon)$ with $\varepsilon \to 0+$. Such a rapidly varying q might correspond physically to heat dissipation by a closely spaced array of fins. One expects, in this setting, that $q(\cdot, \cdot/\varepsilon) \rightharpoonup q_0(\cdot)$ so, as the weak continuity hypothesis is easy to verify here, Theorem 1 would apply. [Eventually one might wish to use techniques like those of [27] to show a stronger continuity for the controls as $\varepsilon \to 0$ so as to justify use of the limit control as a good approximation to the control associated with a problem involving a rapidly varying coefficient, but at present we only inquire as to uniform observability.]

THEOREM 8 *Consider the observation problems*

$$\begin{array}{cc} u_t = u_{xx} - qu & (0 < x < \ell) \\ u_x\Big|_{x=0} \equiv 0, \quad u\Big|_{x=\ell} \equiv 0 & \end{array} \tag{5.1}$$

with unspecified initial data. We now observe $z(t) = u(t, 0)$ for $0 \leq t \leq T$ and seek to determine $u(T, \cdot)$. With spatially varying coefficients $q(\cdot)$ (subject to a uniform bound: $|q| \leq M$) we have a uniformly observable family of observation problems — whence, also, we have uniform nullcontrollability for the corresponding family of dual boundary nullcontrol problems.

Proof: See [31]. Most of the effort consists of showing, by use of the Courant Minmax Theorem, that the bound $|q| \leq M$ ensures that the condition (2.5) holds uniformly and then showing (by a compactness argument) that (2.4) also holds uniformly. ■

Next we note a new result in the same spirit about finite difference approximations for the problem considered earlier for (3.8). We take an equally spaced mesh on $[0, 1]$ with $N-1$ interior nodes $\{x_j = x_j^N = j/N : j = 1, \ldots, N-1\}$ spaced $h = 1/N$ apart and let $\mathbf{u} = \mathbf{u}^N$ be the vector in $\mathbb{R}^{N-1}$ with entries $u_j = u_j^N$ intended to approximate the values $u(\cdot, x_j^N)$. Using the standard central difference approximation to the (spatial) second derivative but keeping time continuous, the partial differential equation (3.8) becomes a finite dimensional system of ordinary differential equations

$$\dot{u}_j = [u_{j-1} - 2u_j + u_{j+1}]/h^2 \qquad (j = 1, \ldots, N-1) \tag{5.2}$$

where, corresponding to the boundary conditions $u(t, 0) = 0 = u(t, 1)$, we are taking $u_0 = 0 = u_N$ — i.e., in (5.2) we take $u_{j-1} = 0$ for $j = 1$ and $u_{j+1} = 0$ for $j = N-1$.

While ultimately we might seek to show convergence for the controls, our present concern is only to show uniformity for the relevant family of dual observability problems:

> *Observe the 'boundary flux' $y = y^N = [u_1 - u_0]/h = Nu_1^N$ for $0 \leq t \leq T$ and, without knowledge of the initial data, reconstruct the terminal state $\mathbf{u}^N(T)$.*

As earlier (compare (3.9)), we seek an estimate

$$(1/N) \sum_{j=1}^{N-1} \left|u_j^N\right|^2 \leq C^2 \int_0^T \left|Nu_1^N(t)\right|^2 \, dt. \tag{5.3}$$

THEOREM 9 *For the finite difference approximations* (5.2) *to the observability problem for* (3.8)*, the estimate* (5.3) *holds uniformly, so C is independent of N, i.e., as the mesh spacing $h \to 0$.*

PROOF: The relevant exponent sequence Λ^N, here, is the finite sequence of eigenvalues of the standard tridiagonal matrix corresponding to the central difference scheme used in (5.2). It is not too difficult to obtain these eigenvalues and the corresponding eigenvectors explicitly: much as for the continuous problem (3.8) we have

$$\begin{aligned} (e_k^N)_j &= \alpha_N \sin k\pi j/N \qquad (j = 1, \ldots, N-1) \\ \sigma_k^N &= 2N^2(1 - \cos k\pi/N) \end{aligned} \tag{5.4}$$

for $k = 1, \ldots, N-1$ (with the normalizing constant $\alpha \approx 1/\sqrt{2}$).

Since Λ^N is a finite sequence, it is trivial that (2.3)–(2.5) will hold for each N and, indeed, we would have the stronger finite dimensional blow-up results of [30], [32]. Our concern is to verify that the separation condition (2.4) and the sparsity condition (2.5) hold *uniformly.* For (2.4) one need only bound $2N^2[\cos 2\pi/N - \cos \pi/N]$ away from 0, which is easy. For any choice of r in (2.5), we separately consider the two cases: $N^2 \leq 2r$ and $2r < N^2$. For the first case, since there are only $N-1$ eigenvalues altogether, we have

$$\nu^N(r) = \#\{\sigma_j^N \in [\sigma_* - r, \sigma_* + r]\} < N \leq \sqrt{2r}.$$

For the second case we see that $\nu^N(r) = k$ means that $\sigma_k^N \approx 2r$ so, setting $s = \sqrt{r}/N < 1/\sqrt{2}$, we have $(1 - \cos k\pi/N) \approx s^2$ and $\cos^{-1}(1-s^2) \approx k\pi/N = k\pi s/\sqrt{r}$ which gives

$$\nu^N(r) = k \approx \frac{\cos^{-1}(1-s^2)}{\pi s}\sqrt{r}.$$

Since $(1/s)\cos^{-1}(1-s^2)$ is bounded on $(0, 1/\sqrt{2}]$, we have a uniform bound on a appearing in (2.5) for either of the cases and our result then follows from Theorem 2. ∎

Finally, we announce a new result [16] for boundary observability of a thermoelastic plate, here taken to be governed by the system of coupled partial differential equations on $\mathcal{Q} = [0,T] \times \Omega$

$$\begin{array}{ll} \left.\begin{array}{rl} w_{tt} + \Delta^2 w - \alpha\Delta\vartheta & = 0 \\ \vartheta_t - \Delta\vartheta + \alpha\Delta w_t & = 0 \end{array}\right. & \text{on } \mathcal{Q} = [0,T] \times \Omega \\ \qquad\qquad w, \Delta w, \vartheta = 0 & \text{on } \Sigma = [0,T] \times \partial\Omega \end{array} \tag{5.5}$$

with coupling constant $\alpha > 0$. Much as for (4.3), this can be put in first-order form as

$$U_t = \mathbf{A}U \quad \text{with } \mathbf{A} = (-\Delta)M, \quad M = M(\alpha) = \begin{bmatrix} -1 & 0 & -\alpha \\ 0 & 0 & -1 \\ 1 & -\alpha & 0 \end{bmatrix} \tag{5.6}$$

for $U = [\vartheta, -\Delta w, w_t]^{\mathsf{T}}$, embedding the boundary conditions in specification of the Laplacian as an operator on $L^2(\Omega)$. We consider a cylindrical region $\Omega = [0,1] \times \Omega_*$ and, as with (3.8), wish to observe the boundary flux at the base of the cylinder $\Gamma = \{0\} \times \Omega_*$

$$y(t,\cdot) = [-\partial\vartheta(t,\cdot)/\partial x_1]\Big|_{x=0} = [-1,0,0] \cdot U_{x_1}(t,0,\cdot) \tag{5.7}$$

and use this to determine the full state U. This determination is clearly impossible when the equations are uncoupled ($\alpha = 0$), so we must have blow-up both when $\alpha \to 0$ and when $T \to 0$.

THEOREM 10 *For the coupled thermoelastic plate model* (5.5), (5.6) *with observation* (5.7)*, one has observability with an estimate*

$$\begin{aligned} &\left[\int_0^1 \int_0^1 \left(|\vartheta(T,\cdot)|^2 + |[\Delta w](T,\cdot)|^2 + |w_t(T,\cdot)|^2\right) dx_1 dx_2\right]^{1/2} \\ &\qquad \leq Ae^{B/T}\left[\int_0^T \int_0^1 |\vartheta_{x_1}(t,0,x_2)|^2\, dx_2 dt\right]^{1/2}. \end{aligned} \tag{5.8}$$

where $A = A(\alpha), B = B(\alpha)$ are bounded for α in compact subsets of $(0, \infty)$ with B bounded and $A = A(\alpha) = \mathcal{O}(1/\alpha)$ as $\alpha \to 0$.

PROOF: See [16]. We briefly sketch the argument, much along the lines noted above for Theorem 6. We here obtain the expansions

$$\begin{aligned} U(t,\cdot) &= \sum_\ell U_\ell(t,\cdot)\varphi_\ell(\cdot) \\ &\quad \text{with } U_\ell(t,\cdot) = \sum_{j,k} a_{[j,k],\ell} e^{\xi_j[\pi^2k^2+\mu_\ell]\,t} V_j e_k(\cdot) \\ y(t,\cdot) &= \sum_\ell y_\ell(t) f_\ell(\cdot) \\ &\quad \text{with } y_\ell(t) = \sum_{j,k} c_{[j,k],\ell} e^{\xi_j[\pi^2k^2+\mu_\ell]\,t}. \end{aligned} \tag{5.9}$$

where $\{(\xi_j, V_j) : j = 0, 1, 2\}$ are the eigenpairs for $M = M(\alpha)$ and $\{\mu_\ell\}$ is the spectrum of the cross-sectional Laplacian (i.e., $-\Delta$ for Ω_* so $\mu_\ell > 0$). Note that (5.7) gives, much as for (3.3),

$$c_{[j,k],\ell} = \beta_{[j,k]} a_{[j,k],\ell} \qquad \beta_{[j,k]} = \pi k[-1, 0, 0] \cdot V_j. \tag{5.10}$$

To proceed, one first shows that the eigenvalues of M are distinct — one real ($j = 0$) and a conjugate pair — with negative real parts. There is thus a basis of $\mathbb{C}^3$ consisting of eigenvectors of M— not orthonormal, since the matrix M is not normal, but a controllability/compactness argument gives uniformity in α of the norm equivalence of these coordinates to the usual Euclidean norm for $\mathbb{C}^3$. Since spectral asymptotics show the first components of V_1, V_2 are roughly proportional to α for α near 0, (5.10) gives

$$K_1 = K_1(\alpha) = \max_{[j,k]}\{1/|\beta_{[j,k]}|\} = \mathcal{O}(1/\alpha) \tag{5.11}$$

as $\alpha \to 0$. Each of the relevant exponent sequences now is the union of quadratically distributed sequences placed along the three rays in $\mathbb{C}$ given by $\{-i\tau\xi_j : \tau > 0\}$. Following the discussion in [31] it is again not difficult to verify (2.3) and that the conditions (2.4), (2.5) hold uniformly in ℓ. As with our previous results, this suffices for applicability of Theorem 2 to obtain (5.8), with the blow-up estimate: $A(\alpha) = \mathcal{O}(1/\alpha)$ as $\alpha \to 0$ following from (5.11). ■

In connection with (5.5) it is also interesting to consider the use of observation of ϑ (or other of the state components) in all of Ω, rather than only the boundary, for determination of the full state. This turns out to be much closer in its nature ($\mathcal{O}(T^{-[k+1/2]})$ estimate) to the finite dimensional analysis in [30]. A direct application of that analysis also appears in [16] (including discussion of the α asymptotics) along with a parallel application of the weighted energy approach introduced in [2]–[5] (which has the advantage here of applicability to more general boundary condition); note also the treatment of this problem in [33].

References

[1] P. Albano and P. Cannarsa, *Carleman Estimates for Second Order Parabolic Operators with Applications to Control Theory,* in preparation.

[2] G. Avalos and I. Lasiecka, *Optimal blowup rates for the minimum energy nullcontrol of the strongly damped abstract wave equation,* Ann. Scuola Norm. Sup. Pisa **5**, pp. 601–616 (2003).

[3] G. Avalos and I. Lasiecka, *Mechanical and thermal nullcontrollability of thermoelastic plates and singularity of the associated minimal energy function,* Control and Cybernetics, to appear.

[4] G. Avalos and I. Lasiecka, *The nullcontrollability of thermoelastic plates and singularity of the associated minimal energy function,* J. Math. Anal. Appl., to appear.

[5] G. Avalos and I. Lasiecka, *Asymptotic rates of blowup for the minimal energy function for the nullcontrollability of thermoelastic plates: the free case,* this volume.

[6] S.A. Avdonin and S.A. Ivanov, *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge Univ. Press, New York, 1995.

[7] G. Da Prato, 1997 personal communication to R. Triggiani, cited in [33].

[8] H.O. Fattorini and D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension,* Arch. Rat. Mech. Anal. **43**, pp. 271–292 (1971).

[9] H.O. Fattorini and D.L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with applications to the control theory of parabolic equations,* Quart. Appl. Math., pp. 45–69 (1974).

[10] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Note Series #34, Seoul National University, Seoul, 1996.

[11] E. Güichal, *A lower bound of the norm of the control operator for the heat equation* , J. Math. Anal. Appl. **110**, no.2, pp. 519–527 (1985).

[12] S.W. Hansen, *Bounds on functions biorthogonal to sets of complex exponentials; control of damped elastic systems*, J. Math. Anal. and Appl. **158**, pp. 487–508 (1991).

[13] W. Krabs, *On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes* (Lecture Notes in Control and Inf. Sci. #173), Springer-Verlag, New York, 1992.

[14] W. Krabs, G. Leugering, and T.I. Seidman, *On boundary controllability of a vibrating plate,* Appl. Math. Opt. **13**, pp. 205–229 (1985).

[15] I. Lasiecka and T.I. Seidman, *Carleman estimates and blow-up rate for nullcontrol of the heat equation,* in preparation.

[16] I. Lasiecka and T.I. Seidman, *Thermal boundary control of a thermoelastic system,* in preparation.

[17] W.A.J. Luxemburg and J. Korevaar, *Entire functions and Müntz–Szász type approximation*, Trans. Amer. Math. Soc. **157**, pp. 23–37 (1971).

[18] V.J. Mizel and T.I. Seidman, *Observation and prediction for the heat equation,* J. Math. Anal. Appl. **28**, pp. 303–312 (1969).

[19] V.J. Mizel and T.I. Seidman, *Observation and prediction for the heat equation, II,* J. Math. Anal. Appl. **38**, pp. 149–166 (1972).

[20] R. Redheffer, *Elementary remarks on completeness,* Duke Math. J. **35**, pp. 103–116 (1963).

[21] D.L. Russell, *A uniform boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Appl. Math. **52**(3), pp. 189–211 (1973).

[22] D.L. Russell, *Exact boundary value controllability theorems for wave and heat processes in star-complemented regions,* in *Differential Games and Control Theory* Marcel Dekker, New York, 1974.

[23] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions,* SIAM Review **20**, pp 639–739 (1978).

[24] L. Schwartz, *Étude des sommes d'exponentielles (2^{me} ed.)*, Hermann, Paris, 1959.

[25] T.I. Seidman, *Observation and prediction for the heat equation, III,* J. Diff. Eqns. **20**, pp. 18–27 (1976).

[26] T.I. Seidman, *Boundary observation and control for the heat equation,* pp. 321–351 in *Calculus of Variations and Control Theory* (D.L. Russell, edit.), Academic Press, N.Y. 1976.

[27] T.I. Seidman, *Exact boundary controllability for some evolution equations,* SIAM J. Control Opt. **16**, pp. 979–999 (1978).

[28] T.I. Seidman, *Two results on exact boundary control of parabolic equations,* Appl. Math. Opt. **11**, pp. 145–152 (1984).

[29] T.I. Seidman, *The coefficient map for certain exponential sums,* Nederl. Akad. Wetensch. Proc. Ser. A, **89** = Indag. Math. **48**, pp. 463–478 (1986).

[30] T.I. Seidman, *How violent are fast controls?*
Math. of Control, Signals, Syst. **1**, pp. 89–95 (1988).

[31] T.I. Seidman, S. Avdonin and S. Ivanov, *The 'window problem' for series of complex exponentials*, J. Fourier Anal. and Appl. **6**, pp. 235–254 (2000).

[32] T.I. Seidman and J. Yong, *How violent are fast controls, II*, Math. of Control, Signals, Systems **9**, pp. 327–340 (1997).

[33] R. Triggiani, *Optimal estimates of norms of fast controls in exact nullcontrollability of two non-classical abstract parabolic systems,* Adv. Diff. Eqs. **8**, pp. 189–229 (2003).