ASYMMETRIC GAMES FOR CONVOLUTION SYSTEMS WITH APPLICATIONS TO FEEDBACK CONTROL OF CONSTRAINED PARABOLIC EQUATIONS

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Dedicated to Bill Ames in honor of his 80th birthday

Abstract

The paper is devoted to the study of some classes of feedback control problems for linear parabolic equations subject to hard/pointwise constraints on both Dirichlet boundary controls and state dynamic/output functions in the presence of uncertain perturbations within given regions. The underlying problem under consideration, originally motivated by automatic control of the groundwater regime in irrigation networks, is formalized as a minimax problem of optimal control, where the control strategy is sought as a feedback law. Problems of this type are among the most important

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in control theory and applications — while most challenging and difficult. Based on the Maximum Principle for parabolic equations and on the time convolution structure, we reformulate the problems under consideration as certain asymmetric games, which become the main object of our study in this paper. We establish some simple conditions for the existence of winning and losing strategies for the game players, which then allow us to clarify controllability issues in the feedback control problem for such constrained parabolic systems.

Keywords: Asymmetric games; Convolutions; Parabolic systems; Pointwise control and state constraints; Uncertainties; Minimax design; Feedback control

1 Introduction

This paper concerns feedback control design of state-constrained linear parabolic systems functioning under uncertain disturbances/perturbations. The original motivating example [8] came from a practical application: automatic control of the groundwater regime in irrigation networks, where the main objective was to neutralize the adverse effect of uncertain weather and environmental conditions. We immediately note that this need not always be possible: obviously, for the system to be acceptable, we must be capable of handling the *worst perturbations*. In particular:

- We must have enough irrigation capacity to keep the water supply up to the minimally acceptable level, even if there might be a drought for the entire period under consideration.

- Conversely, even if it might rain for the entire period, we must be able to reduce the irrigation supply enough to avoid flooding.

Assuming it *is* possible to compensate adequately for adverse fluctuations in the weather, one might then seek to optimize the policy used. Problems of this type may be formulated as *optimal control* problems, unavoidably requiring the use of *closed-loop feedback* to obtain an appropriate control, since the external input (weather, etc.) is not known in advance. Indeed, we have *minimax design* problems, seeking to design the feedback to minimize some cost in the presence of possibly worst case external inputs. The particular control systems modeled in [8, 9] were parabolic partial differential equations with Dirichlet boundary control. Among the important specific features introduced there in order to meet practical requirements we mention the following:

- *distributed uncertain perturbations* — taking values within given closed areas with only *bounds* assumed to be known;

- *hard control constraints* — pointwise constraints on the control functions (here acting through Dirichlet boundary conditions, offering minimal regularity for the linear dynamics);

- *hard state constraints* — pointwise constraints on the acceptable values of the evolving state (with both perturbation and compensating control).

Problems with such features are among the most challenging and difficult in control theory but, at the same time, are among the most important for applications. To the best of our knowledge, a variety of approaches and results developed in the theories of differential games, H_∞-control, and Riccati's feedback synthesis are not applicable to such problems; see, e.g., [1, 4, 3, 6] and also [9, 10, 11, 12] with the discussions and references therein.

The approach developed in [9] for the case of one-dimensional heat/diffusion equations and then partly extended in [10, 11, 12] to multidimensional settings mainly concerns the system reaction to *extreme perturbations*, which (as suggested above) would seem to provide the 'worst case' scenarios in the original environmental situation [8, 9]. In this way the structure and parameters of the control functions are computed by using the Pontryagin maximum principle [14] for ODE approximating systems of optimal control, with some further adjustment to the parabolic dynamics and the exclusion of unstable vibrations.

Such an analysis assumes that Nature is not malicious. On the other hand, we will see that even if the control resources are adequate to maintain all the constraints in response to the nominal 'worst case' of extreme perturbations — corresponding to the afore-mentioned requirements — the necessary corresponding commitment of resources might preclude an adequate response to some other perturbations. Thus, without further analysis one may not be able to verify a capability to respond adequately to more subtle scenarios merely from consideration of responses to those extreme perturbations. This might leave it unclear whether the constrained problem has *any* global policy solution, certainly a crucial precondition for subsequent optimization.

The present paper is intended to make a start at providing exactly the 'further analysis' addressing this possibility, seeking techniques verify the *existence of admissible feedback policies* — meaning causal policies which ensure satisfaction of the specified constraints in response to *all* admissible perturbations — as a necessary preliminary to optimization. Even this question of the existence of admissible feedback policies turns out to be more difficult than one might think, and we will be unable to obtain simple *necessary and sufficient* conditions for existence, much less address the optimization problem in this context.

In this paper we suggest an approach to the minimax synthesis of (hard) constrained parabolic systems based on their reduction to *asymmetric games* whose dynamics are given by *time convolutions*; see Section 3. This approach, applying to the underlying parabolic dynamics, is based on certain fundamental properties of such systems, partly on the classical Maximum Principle for parabolic equations. The reduction eventually allows us to clarify — via establishing conditions for the existence of winning and losing strategies of the game players — some important characteristics of feasible and optimal feedback controls and perturbations in the minimax problems under consideration.

The rest of the paper is organized as follows. In Section 2 we consider the original motivating problem of automatic control of the groundwater regime in irrigation networks and formulate it as an asymmetric game via time convolutions. Following this interpretation, we introduce in Section 3 a general asymmetric convolution game of two players personalized as the *fox* and the *hound*. Section 4 is devoted to the analysis of the convolution game establishing necessary and sufficient conditions for the existence of winning strategies for the hounds. Finally, Section 5 contains various results and discussions related to the main thrust of the paper. These include: the reduction of a general class of linear parabolic equations to convolution systems, the justification of well-posedness of the convolution game, and the implication of the game analysis to the original irrigation problem.

2 A Motivating Problem: Irrigation

In this section we describe in somewhat more detail the groundwater management control problem of [8]. Here we will be directly controlling the supply in a pair of irrigation channels to regulate the *groundwater level* (GWL) in the *seepage region* between these channels — so called because of the seepage of water into the ground, approximately modeled as a diffusion. We treat this as spatially one-dimensional, neglecting effects parallel to the channels, which are taken at $s = \pm 1$. Letting u and w, respectively, be the deviations from the desired GWL and the averaged external input (difference between precipitation and evaporation), these satisfy the linear parabolic equation

$$u_t - au_{ss} = w \text{ on } \mathcal{Q} = \mathcal{Q}_T := (0,T] \times (-1,1)$$

with the Dirichlet boundary condition, which we take as our control: the scaled difference $z(t)$ between the channel water supply and a reference supply just sufficient to maintain the desired GWL in the presence of the nominal (averaged) input. It is a reasonable approximation to take the *disturbance* $w = w(t,\cdot)$ to be spatially constant on $\Omega := (-1,1)$ and subject to an given bound $|w(t)| \le \beta$ on $[0,T]$; so $y = w/\beta$ is a function only of t, satisfying

$$|y(t)| \le 1 \text{ for } 0 \le t \le T. \tag{2.1}$$

We take the supply rate to be the same in each of the channels; the possible deviation from the reference supply is necessarily bounded — for simplicity of exposition we assume symmetry in this bound. Appropriately choosing α, we will have $u(t,\pm 1) = \alpha\, z(t)$ with

$$|z(t)| \le 1 \text{ for } 0 \le t \le T. \tag{2.2}$$

Supposing the GWL is initially at its nominal level, our complete model is

$$\begin{aligned} &u_t - au_{ss} = \beta\, y(t) \text{ on } \mathcal{Q} = \mathcal{Q}_T = (0,T] \times (-1,1), \\ &u(t,-1) = u(t,1) = \alpha\, z(t), \qquad\qquad u(0,s) = 0 \end{aligned} \tag{2.3}$$

with $y(\cdot)$ unknown, subject to (2.1), and with control function $z(\cdot)$ to be chosen subject to (2.2); we may indicate the dependence of the solution on the inputs by writing $u = u^{y,z}(\cdot,\cdot)$.

Our control problem is to regulate the GWL in the presence of unpredictable fluctuations in precipitation/evaporation by choosing the supply rate — i.e., the control function $z(\cdot)$ — so as to ensure that the level never becomes either too high or too low. We take the fluctuation in water level as characterized by its value at the midpoint $s = 0$ and require that our control ensure that the GWL stays within the prescribed tolerance:

$$|u^{y,z}(t,0)| \leq \ell \;\text{ for }\; 0 \leq t \leq T. \tag{2.4}$$

We may view this as a *game played against Nature*. Thus, while we are not viewing Nature as a malicious opponent in the evolution of the disturbance $w = \beta\, y$, we do approach this with a 'worst case' attitude, avoiding any unsupportably optimistic assumption that this disturbance will be of any special form conveniently favorable for the analysis of our control policy. In this way (2.4) is to be taken as an imposed *state constraint*. Furthermore, as there is *no restriction on the external input* $y(\cdot)$ *other than* (2.1), this is taken as a *constraint on the control policy* determining the response $z(\cdot)$.

In this 'worst case' analysis we will view an inability to compensate for arbitrary admissible perturbations as being a *definite failure* for our control system.

Observe that there are typically some other constraint requirements — assuming one could consider them without permitting violation of the state constraint (2.4). For example, we might wish to conserve the supplied water (minimizing the integral $\int y\, dt$) or to simplify the regulatory effort (e.g., minimizing the variation in y). However, we will not address such concerns in this paper.

The following statement justifies the possibility of describing the dynamics of (2.3) via *time convolutions* with *nonnegative* functions.

Lemma 2.1. [Convolution Description of the GWL Dynamics.] *Let* $u(\cdot)$ *be the solution to the parabolic partial differential equation* (2.3)*. Then the dynamics for*

$$x(t) := u^{y,z}(t,0), \quad 0 \leq t \leq T,$$

are given by convolution:

$$x(t) = \int_0^t \left[\varphi(t-\tau)y(\tau) + \eta(t-\tau)z(\tau)\right]\, d\tau \tag{2.5}$$

with appropriate nonnegative functions φ and η.

Proof. This follows from the more general results established in Theorem 5.1, where the form (2.5) of these dynamics is justified with the expressions

$$\varphi(t) = u_t^{1,0}(t,0) \quad \text{and} \quad \eta(t) = u_t^{0,1}(t,0). \tag{2.6}$$

Furthermore, the crucial fact of the *positivity* $\varphi, \eta \geq 0$ is derived therein from the *Maximum Principle* for parabolic equations. We will also defer to the last subsection of Section 5 for our further investigation of specific characterizations for the appropriate φ and η in the particular GWL setting of (2.3). △

Thus we have effectively replaced (2.3) by (2.5) in modeling the control problem. In view of the comments above, we turn now to an analysis of this class of convolution games.

3 The Fox and the Hound: a Convolution Game

In this section we introduce the game $\mathfrak{G}$, which is the focus of our subsequent analysis. We will be considering scalar systems with *convolution dynamics* — much as in (2.5), except that it is now convenient to reverse the sign of z. Thus for each $t \in \mathcal{I} := [0,T]$ we have

$$x(t) = \int_0^t \left[\varphi(t-\tau)y(\tau) - \eta(t-\tau)z(\tau)\right] d\tau. \tag{3.1}$$

Note that the functions φ and η are given and y, z are to be *inputs.* One might well consider vector-valued versions of this, but for our present purposes it will be sufficient to restrict our attention, for simplicity, to systems (3.1) with $\varphi, \eta, x, y,$ and z scalar-valued although possibly infinite horizoned with $T = \infty$.

The fact that we have two input functions suggests thinking of (3.1) as the setting for a 'game'. We personalize this game somewhat by thinking of a *fox* and a *hound*, considered as moving points $f(\cdot)$ and $h(\cdot)$ in $\mathbb{R}$, given by the above *convolutions* so

$$f := \varphi * y \;\text{ and }\; h := \eta * z \;\text{ in }\; \mathbb{R},$$

i.e., controlled by providing the inputs y and z, respectively. Thus $x = f - h$ in (3.1).

We are here taking the functions φ and η to be the (fixed) *motion characteristics* of the fox and hound, respectively. [We ignore any physical anomalies associated with this as an image — e.g., we permit $x(\cdot)$ to cross 0, with the fox and hound apparently passing through each other.] As an example, if the fox were to move by exerting a force $F = F(t)$ and one had velocity-proportional friction, then her position $f(t)$ would satisfy

$$mf'' = F - \lambda f'.$$

If we write $F = y\,F_0$, where F_0 is the maximum force available — so $y = 1$ means "full power ahead" and $y = -1$ means "full power reverse" — then, starting from rest, we would get $f = \varphi * y$ with

$$\varphi(\tau) = (F_0/\lambda)\left[1 - e^{-(\lambda/m)\tau}\right].$$

This would make our interpretation some sort of 'pursuit game' in which control lies in the acceleration rather than the velocity. The history dependence implicit in the convolution dynamics is here related to inertia.

Apart from more detailed interpretation as in examples such as this and Lemma 2.1, we will be assuming throughout our discussion that

$$\varphi \;\text{ and }\; \eta \text{ are specified in } L^1_{loc}(0,\infty) \text{ with } \varphi, \eta \geq 0, \tag{3.2}$$

and that we are imposing the constraints

$$|y(t)| \leq 1 \;\text{ and }\; |z(t)| \leq 1 \;\text{ for all }\; t \in \mathcal{I} = [0,T]. \tag{3.3}$$

It will be convenient in what follows ro label φ and η as *impulse response functions* and to introduce their *integral characteristics*

$$F(t) := \int_0^t \varphi(\tau)\, d\tau \quad \text{and} \quad H(t) := \int_0^t \eta(\tau)\, d\tau; \tag{3.4}$$

so $\varphi = F'$ and $\eta = H'$. Note that F and H are *nondecreasing* by (3.2) and that the dynamics of (3.1) can equivalently be written as $x = f - h$ with

$$\begin{aligned} f(t) &= \int_0^t y(t-\tau)\varphi(\tau)\, d\tau = \int_0^t y(t-\tau)\, dF(\tau), \\ h(t) &= \int_0^t z(t-\tau)\eta(\tau)\, d\tau = \int_0^t z(t-\tau)\, dH(\tau). \end{aligned} \tag{3.5}$$

While a considerable variety of interesting games might be described in this setting by adjusting the payoffs, our principal concern will be with the game in which *the fox wins if she can ever 'escape'* — i.e., get $f(t)$ farther than ℓ from $h(t)$ at some time $t < T$ so that $|x(t)| > \ell$. Conversely, *the hound wins if he can 'track' successfully* — i.e., keep $h(t)$ no farther than ℓ from $f(t)$ during the entire interval $\mathcal{I}$, maintaining this deviation bound throughout the interval so that

$$|x(t)| \le \ell \quad \text{for all } t \text{ in } \ \mathcal{I} = [0, T]; \tag{3.6}$$

there are no ties. We are then taking this game to have the payoff $+\infty$ to the fox (and $-\infty$ to the hound) if she can force (3.6) to fail. [We *could* have a variable payoff to the hound when he can maintain (3.6); such a variable payoff would provide the framework for subsidiary optimization with (3.6) as an imposed constraint. However, in focusing attention on whether the constraint can be maintained, we simplify by taking the winning payoff to the hound to be always $+\infty$ if (3.6) is maintained with corresponding payoff of $-\infty$ for the fox.] Thus, once we have introduced (3.1), (3.3), and (3.6), the game is completely specified by giving the relevant parameters $\ell, T > 0$ and the impulse response functions $\varphi, \eta \in L^1(\mathcal{I})$: we refer to this as $\mathfrak{G} = \mathfrak{G}(\ell, T; \varphi, \eta)$.

Note that the game $\mathfrak{G}$ is asymmetric in its definition of a 'win', and *our primary concern will be seeking a winning strategy for the hound.* For this analysis we assume, in particular, the necessity for the hound of protecting against a 'worst case' $y(\cdot)$: if the hound knew that the fox generated y *stochastically* with a known probability distribution, then he might be able to take advantage of this (e.g., to maximize his *probability* of winning). However, in a game-theoretic context, this would be making the unsupportably optimistic assumption that the fox might occasionally forego an assured win.

4 Existence of Winning Strategies

Our analysis of the game primarily addresses the *two fundamental questions*:

- Does either player, the fox or the hound, have a winning strategy for the game?
- How does the answer to the question above depend on the parameters ℓ, T, φ, and η?

The first result provides verifiable conditions for winning the game expressed in terms of *integral characteristics* F and H from (3.4).

Theorem 4.1. [Integral Conditions for Winning the Game.] *The condition*

$$F(T) \leq \ell$$

is sufficient for the hound to have an effortless ensured win. The condition

$$F(t) \leq H(t) + \ell \quad \text{for every} \quad 0 < t < T \tag{4.1}$$

is necessary, but not sufficient, for the hound to have a winning strategy.

Proof. Note that the above assumption (3.3) ensures that

$$|f(t)| \le F(t) \le F(T) \text{ and } |h(t)| \le H(t) \le H(T) \tag{4.2}$$

— with *strict inequalities* unless $y \equiv \pm 1$ and $z \equiv \pm 1$, respectively.

If $F(T) \le \ell$, then taking $z \equiv 0$ would be a winning strategy for the hound, since that gives $x \equiv f$, and so (4.2) implies (3.6) — the hound can simply sit still, knowing that it is impossible for the fox to escape in time using any admissible y. However, in any other case the hound must use an active strategy to be able to win.

If (4.1) were false, then taking $y \equiv 1$ would be a winning control for the fox as a fixed strategy, since that gives $f \equiv F$, and then (4.2) shows that $x = f - g \ge F - H$; so the fox escapes — i.e., (3.6) fails — at the same $t \in (0, T)$ for which (4.1) would fail. Thus, (4.1) is *necessary* for the hound to have any chance at winning against the fox's *extreme control.*

To see that (4.1) is *insufficient* to ensure a win for the hound, we need only provide a single example. Take $T = 2$, $\ell = 1$ and suppose that

$$\varphi(t) = \begin{cases} 3 & \text{for } 0 \le t \le 1, \\ 0 & \text{else;} \end{cases} \qquad \eta(t) = \begin{cases} 2 & \text{for } 0 \le t \le 2, \\ 0 & \text{else.} \end{cases}$$

These impulse response functions generate by (3.4) the integral characteristics

$$F(t) = \begin{cases} 3t & \text{for } 0 \le t \le 1, \\ 3 & \text{for } 1 \le t \le 2 = T; \end{cases} \qquad H(t) = 2t \text{ for } 0 \le t \le T.$$

With $\ell = 1$ this gives the strict inequality $F(t) < H(t) + \ell$ for all $t \in [0, 1)$, except for the equality at $t = 1$; i.e., (4.1) holds, and just running away ($y \equiv 1$) does not enable the fox to escape from the hound who would take $z \equiv 1$.

However, suppose that instead of simply running away straight ahead with $y \equiv 1$, the fox were to double back at $t = 1$. Using the input function

$$y(t) = \begin{cases} 1 & \text{for } t \leq 1, \\ -1 & \text{for } t > 1, \end{cases}$$

the fox has $f(t) = F(t) = 3t$ for $0 \leq t \leq 1$ as before, but now

$$f(t) = \int_{t-1}^{t} 3z(s)\,ds = 3\big[(1 - [t-1]) - (t-1)\big] = 9 - 3t$$

for $1 \leq t \leq 2 = T$. Even knowing this in advance, what could the hound do? One would have $f(1) = 3$ and, if $z \not\equiv 1$ on $[0,1]$, one would have $h(1) < H(1) = 2$, whence $|x(1)| > 1 = \ell$ — i.e., a win for the fox. Avoiding this by keeping $y \equiv 1$ on $[0,1]$, one first considers the choice

$$z_*(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ -1 & \text{for } 1 \leq t \leq 2; \end{cases} \quad \text{so } h_*(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq 1. \\ 4 - 2t & \text{for } 1 \leq t \leq 2. \end{cases}$$

Comparing, we would have $x_* = f - h_* = 5 - 4t$ for $1 \leq t \leq T$ so $|x_*(t)| > \ell$ for $t > 3/2$ — a win for the fox. Any other input choice $z(\cdot)$ with $z \equiv 1$ on $[0,1]$ would necessarily give $z \geq z_*$, so $h \geq h_*$ and the fox also escapes.

Thus the fox has a *winning pure strategy* in this example, even though F and H do satisfy (4.1). Indeed, a modification of this example, changing the hound's impulse response function to

$$\eta(t) = \begin{cases} 2 & \text{for } 0 \leq t \leq 1, \\ 7 & \text{for } 1 \leq t \leq 2 = T \end{cases}$$

shows, by a similar calculation, that (4.1) cannot even ensure the hound's success against the fox's extreme 'running away' strategy. △

The moral to be drawn from the scenario above is the importance of *agility.* For present purposes, we need not provide any technical definition of this vague notion of comparative ‘agility’ while observing the competitive disadvantage of a large tail for the impulse response function, which acts as a form of *inertia*. In particular, current variations of the trajectory $h(t)$ may be dominated by *residual effects* of much earlier control actions $z(\tau)$ if the resource function $\eta(\sigma)$ would be large even when the time difference $\sigma = t - \tau$ becomes large.

Complementing Theorem 3.6, we now turn to a more positive result for a hound with his impulse response function η: he can successfully track any fox whose impulse response function φ lies within a distance ℓ from the segment in $L^1(0,T)$ joining η to the origin.

Theorem 4.2. [Impulse Response Function Conditions for Winning the Game]. *The L^1-norm condition*

$$\ell \geq \min_{0 \leq c \leq 1} \left\{ \left\| \varphi - c\eta \right\|_1 := \int_0^1 \left| \varphi(\tau) - c\eta(\tau) \right| \, d\tau \right\} \tag{4.3}$$

is sufficient, but not necessary, for the hound to have a winning strategy.

Proof. Given (4.3), the hound can choose $c \in [0,1]$ such that $\|\varphi - c\eta\|_1 \leq \ell$ and then, taking into account Theorem 5.2 presented below, can use the control

$$z(\tau) = cy(\tau). \tag{4.4}$$

With $c \leq 1$, the given constraint $|y| \leq 1$ ensures that one always has $|z| \leq 1$, so this control is admissible. We then have from (3.1) that

$$\begin{aligned} |x(t)| \leq & \int_0^t |\varphi(t-\tau)y(\tau) - \eta(t-\tau)z(\tau)| \, d\tau \\ = & \int_0^t |\varphi(\tau) - c\eta(\tau)| \cdot |y(t-\tau)| \, d\tau \\ \leq & \int_0^t |\varphi(\tau) - c\eta(\tau)| \, d\tau \ \leq \|\varphi - c\eta\|_1 \leq \ell \end{aligned}$$

for each $0 \leq t \leq T$ — i.e., one has (3.6), and thus the control policy (4.3) is a winning strategy for the hound.

Conversely, it is *necessary* to have $\|\varphi - c\eta\|_1 \le \ell$ for the hound to use (4.4) as a winning strategy. Indeed, if the fox knew (4.4), she could simply choose

$$y(t) = \operatorname{sgn}\left[\varphi(t) - c\eta(t)\right], \qquad t \in [0, T],$$

giving $x(T) = \|\varphi - c\eta\|_1$ — and with $\|\varphi - c\eta\|_1 > \ell$ this would be a loss for the hound.

On the other hand, there are strategies other than (4.4), and we now show that it may be possible for the hound to have a winning strategy even with (4.3) false. To see that (4.3) is *not a necessary condition* to ensure a win for the hound, we need only provide a single example. Take $\ell = 1$, $T = 3$ with

$$\varphi(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2, \\ 0 & \text{else;} \end{cases} \qquad \eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le 2, \\ \frac{3}{2} & \text{else.} \end{cases}$$

Thus for any $0 \le c \le 1$ we have

$$\begin{aligned} \|\varphi - c\eta\|_1 &= \int_0^3 \left|\varphi(\tau) - c\eta(\tau)\right| d\tau = \int_0^2 \left|\varphi(\tau) - c\eta(\tau)\right| d\tau + \int_2^3 \left|\varphi(\tau) - c\eta(\tau)\right| d\tau \\ &= 2(1 - c) + (3 - 2)\tfrac{3}{2}c \ = 2 - \tfrac{1}{2}c \ge \tfrac{3}{2} > 1 = \ell; \end{aligned}$$

so (4.3) is false for this example. [Observe, parenthetically, that with the same ℓ, φ, and η we *would* have (4.3) if we had taken $T \le 8/3$.]

We have seen that the strategy (4.4) now fails. On the other hand, suppose the hound modifies this linear strategy and uses instead the piecewise constant control

$$z(\tau) = \begin{cases} 0 & \text{on } [0, 1], \\ y(\tau) & \text{on } (2, 3]. \end{cases} \tag{4.5}$$

Then we easily have

$$|x(t)| = \left|\int_0^t y(\tau)\, d\tau\right| \le 1 = \ell \qquad \text{whenever } \ 0 \le t \le 1.$$

For $t \in (1,3]$ we observe that the conditions $1 \le \tau \le t$ give $t - \tau \in [0,2)$ and therefore $\varphi(t-\tau) = \eta(t-\tau)$. Thus for $t \in (1,3]$ we have

$$\begin{aligned} |x(t)| &= \left| \int_0^1 \varphi(t-\tau) y(\tau)\, d\tau + \int_1^t \big[\varphi(t-\tau) - \eta(t-\tau)\big] y(\tau)\, d\tau \right| \\ &= \left| \int_0^1 \varphi(t-\tau) y(\tau)\, d\tau \right| \le 1 = \ell. \end{aligned}$$

This shows that (4.5) is now a winning strategy: using it, constraint (3.6) always holds for the entire interval $[0,3]$, and so the hound wins. $\triangle$

At this time it remains an *open problem* to find verifiable conditions on the parameters ℓ, T, φ, and η, which are *both necessary and sufficient* for the hound to win. Even though we have only exhibited a single winning strategy for those cases where we have shown existence, it is important for questions of possible *subsidiary optimization* that we would expect the basic constraint (3.6) to provide uniqueness of the control policy only in very special cases.

5 Further Results and Discussions

In this section we present various results supporting and justifying the above game convolution approach and illustrating its applications to feedback control of parabolic systems. We split our discussions into three subsections.

5.1. Autonomous Linear Systems and Convolutions. The classical *variation of parameters formula* is the source of our convolution formulation so this is a quite general result for autonomous linear problems. Let us begin our considerations with the abstract *linear autonomous state equation*

$$\dot{u} = \mathbf{A}u + w, \qquad \text{with } u(0) = 0, \tag{5.1}$$

imposing homogeneous initial conditions and input. Assuming further that the linear operator $\mathbf{A}$ is the infinitesimal generator of a C_0 semigroup $\mathbf{S}(\cdot)$ on the state space $\mathcal{X}$, we have the standard *semigroup convolution representation*

$$u(t) = \int_0^t \mathbf{S}(t-\tau)w(\tau)\,d\tau \tag{5.2}$$

for *mild solutions* of the abstract differential equation (5.1); see, e.g., [2, 13] as general references for such semigroup formulations.

If we now have the *scalar linear observation*

$$x(t) = \langle \gamma, u(t) \rangle$$

for some suitable linear functional γ and take the input w to have the form

$$w = \sum_{j=1}^{n} y_j(t) w_j + \sum_{j=1}^{\nu} z(t)\omega_j \tag{5.3}$$

for fixed elements w_j and ω_j in $\mathcal{X}$, then substituting (5.3) into (5.2) and the latter into $x = \langle \gamma, u \rangle$ gives

$$x(t) = \int_0^t \left(\sum_{j=1}^{n} \varphi_j(t-\tau)\, y_j(\tau) + \sum_{j=1}^{\nu} \eta_j(t-\tau)\, z_j(\tau) \right) d\tau \tag{5.4}$$

with the *impulse response functions*

$$\varphi_j(t) := \langle \gamma, \mathbf{S}(t)\, w_j \rangle, \qquad \eta_j(t) := \langle \gamma, \mathbf{S}(t)\, \omega_j \rangle. \tag{5.5}$$

In particular, we have from (5.4) that $\varphi_j = \left\langle \gamma, u^{j,0} \right\rangle$, where $u^{j,0} = \mathbf{S}(\cdot) w_j$ is the solution to

$$\dot{u}^{j,0} = \mathbf{A} u^{j,0} \qquad \text{with } \left. u^{j,0} \right|_{t=0} = w_j, \qquad j = 1, \ldots, n.$$

Introducing F_j and H_j much as in (3.4) (so, e.g., F_j corresponds to taking $y_j \equiv 1$ with $y_k = 0$ for all $k \neq j$ and with $z_j = 0$ for all j — giving $\varphi_j = dF_j/dt$), we can write these functions also in terms of solutions:

$$F_j(t) = \left\langle \gamma, U^{j,0}(t,\cdot) \right\rangle, \qquad H_j(t) = \left\langle \gamma, U^{0,j}(t,\cdot) \right\rangle, \tag{5.6}$$

where $U^{j,0}$ and $U^{0,j}$ are the solutions to (5.1) with $w \equiv w_j$ and $w \equiv \omega_j$, respectively.

We will also be interested in considering similar cases in which one might not have w_j or ω_j in the state space $\mathcal{X}$ or in which the observation functional γ may not be in $\mathcal{X}^*$. Whether this might lead to a successful model would depend on details of *regularity theory* for the particular spaces and operators involved. In particular, we wish to treat *boundary control* and *point observation* for parabolic partial differential equations, relying on the considerable smoothing provided by the corresponding analytic semigroups. This is given in the next theorem, which directly relates to our original motivations and justifies the possibility to reduce feedback control problems for linear parabolic systems to the convolution game studied in Sections 3 and 4.

Theorem 5.1. [Convolution Representation of Linear Parabolic Systems]. *Let Ω be a bounded region in $\mathbb{R}^m$ with sufficiently smooth boundary $\partial\Omega$, let $A(\cdot)$ be a smooth positive definite symmetric matrix-valued function on the closure of Ω, and fix $s_* \in \Omega$. We consider a parabolic partial differential equation on $\mathcal{Q} = \mathcal{Q}_T = (0,T] \times \Omega$ with Dirichlet boundary conditions and homogeneous initial conditions:*

$$u_t = \nabla \cdot A\nabla u + \sum_{j=1}^{n} y_j(t)\, w_j, \qquad u\Big|_{\partial\Omega} = \sum_{j=1}^{\nu} y_j(t)\, \omega_j, \qquad u(0,\cdot) \equiv 0 \tag{5.7}$$

and observe $x(t) = u(t, s_*)$. Then, subject to some regularity considerations for w_j, ω_j, the point observation $x(\cdot)$ is given by (5.4) with

$$\varphi_j := \frac{\partial\, U^{j,0}(t, s_*)}{\partial t} \quad \text{and} \quad \eta_j := \frac{\partial\, U^{0,j}(t, s_*)}{\partial t},$$

where $U^{j,0}$ and $U^{0,j}$ are the solutions, respectively, to the particular cases of (5.7):

$$\begin{aligned} U_t^{j,0} &= \nabla \cdot A\nabla U^{j,0} + w_j, \qquad & U^{j,0}\Big|_{\partial\Omega} &= 0, \qquad & U^{j,0}(0,\cdot) &\equiv 0; \\ U_t^{0,j} &= \nabla \cdot A\nabla U^{0,j}, \qquad & U^{0,j}\Big|_{\partial\Omega} &= \omega_j, \qquad & U^{0,j}(0,\cdot) &\equiv 0. \end{aligned} \tag{5.8}$$

Finally, φ_j and η_j are nonnegative when w_j and ω_j are nonnegative.

Proof. Equations such as (5.7) can equivalently be interpreted by a variety of methods depending on the geometry of Ω and the regularity assumed for w_j, ω_j. It is well known (see, e.g., [5] — especially the estimate in Theorem 16.3 of Chapter IV, regarding localization) that if w_j is in $H^{-1}(\Omega)$ and moderately smooth near s_* and if ω_j is in $L^2(\partial\Omega)$, then (5.7) is solvable — say, for y_j and z_j in $L^\infty(0,T)$ — and will be *smooth enough* to permit point evaluation at s_*. Note that these conditions can be substantially weakened, but are adequate for our present purposes. Indeed, we need only look at the *regularity* for the solutions $U^{j,0}$ and $U^{0,j}$ to (5.8). The representation (5.4) and its consequences discussed above are then immediate.

We now employ the *Maximum Principle* for parabolic equations to verify the *nonnegativity* asserted in the theorem. While one could also work with the classical Maximum Principle for smooth classical solutions and then use density arguments, we here work with the *weak formulation* of (5.7) by employing arguments requiring minimal regularity, based on the following result by Stampacchia [16]: if $u^-(s) := u(s) \wedge 0 = \min\{u(s)\}$, then (writing ∂_* for an arbitrary first derivative) one has

$$\partial_* u^- = \begin{cases} 0 & \text{where } \ u^- = 0, \\ \partial_* u & \text{where } \ u^- \neq 0 \end{cases} \tag{5.9}$$

almost everywhere — e.g., one has a.e. that $\nabla u_- \cdot \nabla u = |\nabla u^-|^2$. We now fix j and, assuming $\omega_j \geq 0$, we let u be a *weak solution* to

$$u_t = \nabla \cdot A\nabla u, \qquad u\Big|_{\partial\Omega} = z_j(t)\,\omega_j, \qquad u(0,\cdot) \equiv 0$$

with $z_j(\cdot) \geq 0$. With u^- as test function, the weak version of this is:

$$\int_\Omega u^- u_t + \int_\Omega \nabla u^- \cdot A\nabla u = \int_{\partial\Omega} u^-[A\nabla u \cdot \mathbf{n}] \equiv 0,$$

since we have $u \geq 0$ on $\partial\Omega$ so $u^- \equiv 0$ there. From (5.9) we have

$$\nabla u^- \cdot A\nabla u = \nabla u^- \cdot A\nabla u^- \geq 0,$$

since A was assumed *positive definite*. We also have

$$u^- u_t = \frac{1}{2} d(u^-)^2/dt.$$

Integrating this (while noting that $u^-(0) = 0$) gives

$$\frac{1}{2}\|u^-(t)\|^2 \leq 0 \qquad \text{so } \ u^- \equiv 0$$

which means that $u \geq 0$ on $\mathcal{Q}$. Thus, in particular, one evaluates at s_* to obtain

$$0 \leq u(t, s_*) = \int_0^t \eta_j(t-\tau)\, z_j(\tau)\, d\tau$$

whenever $z_j \geq 0$ on $[0, t]$ (provided that $\omega_j \geq 0$). [It is easy to choose y_j to have a counterexample to this if $\eta_j < 0$ on any set of positive measure.] Therefore, we can conclude that $\eta_j \geq 0$ as asserted in the theorem.

The justification that $\varphi_j \geq 0$ when $w_j \geq 0$ is essentially similar. We now let u be the solution to the parabolic homogeneous initial boundary problem

$$u_t = \nabla \cdot A\nabla u + y_j(t) w_j, \qquad u\Big|_{\partial\Omega} = 0, \qquad u(0, \cdot) \equiv 0;$$

so the weak form of this gives

$$\int_\Omega u^- u_t + \int_\Omega \nabla u^- \cdot A\nabla u = \int_\Omega u^- y_j(t)\, w_j$$

with the right hand side being *nonpositive* as $u^- \leq 0$ and $y_j(t) w_j \geq 0$. Again we have $u^- \equiv 0$ on $\mathcal{Q}$ and use this (for each $y_j \geq 0$) to conclude that $\varphi_j \geq 0$. △

5.2. The Information Structure of the Game. We should clarify the *information structure* of the game from the viewpoint of the hound, noting that if $\mathfrak{G}$ may be viewed as 'a game with perfect information' (like chess), then *only deterministic strategies are relevant* — we need not then consider probabilistic mixed strategies. As a worst case, we may attribute to the fox perfect causal information about *both* inputs $y(\cdot)$ and $z(\cdot)$ — but, since only causally determined strategies can be admissible, we must ask what information the hound will have available at each τ in determining his response.

We begin with the assumption that the hound knows (and remembers) his own input $z(\cdot)$ — hence can compute the resulting motion $h = \eta * z$ — and has observed (and remembers) the relative position $x(\cdot) = f(\cdot) - h(\cdot)$ up to that time. However, it is clear that the future evolutions of f and h beyond τ include some *history dependence*—this was much of the point of our discussion of 'agility' following Theorem 4.1 — and, in constructing $z(\cdot)$, it would seem desirable for the hound also to know at least the past history of the fox's input y that has not been provided directly. The next result justifies this, in a sense justifying the *well-posedness* of the game under consideration by, e.g., validating the use of such strategies as (4.3).

Theorem 5.2. [Well-Posedness of the Game]. *Let φ and η be given in $L^1[0,T]$ with $\varphi \not\equiv 0$ near 0. Then the histories of z and of*

$$x = \varphi * y - \eta * z$$

on any subinterval $[0,\tau]$ as $\tau \leq T$ uniquely determine the past history of y on $[0,\tau]$.

Proof. Take φ_τ and η_τ to be the restrictions of φ and η to $[0,\tau]$, respectively, that are taken to vanish outside $[0,\tau]$ (since anything else is irrelevant up to time τ) and similarly define y_τ, and z_τ. Hence $x_\tau(\cdot)$, defined by the convolutions

$$x_\tau = \varphi_\tau * y_\tau - \eta_\tau * z_\tau,$$

coincides with $x = \varphi * y - \eta * z$ on $[0,\tau]$. Taking the *Fourier transforms* [15] of these functions (denoted by 'hat' as usual), the above convolutions become simply products. Thus, rearranging slightly, we have

$$\widehat{\varphi_\tau}\,\widehat{y_\tau} = \widehat{\eta_\tau}\,\widehat{z_\tau} + \widehat{x_\tau}.$$

Note that $\eta_\tau, z_\tau, x_\tau$ are known at time τ (by prescription, memory, and observation); so the product $\widehat{\eta_\tau}\,\widehat{z_\tau}$ is also known. Since φ_τ and y_τ have compact support $[0,\tau]$, each of the factors $\widehat{\varphi_\tau}$ and $\widehat{y_\tau}$ is entire *analytic* (by the Paley-Wiener Theorem; see, e.g., [15]) with $\widehat{\varphi_\tau} \not\equiv 0$ — hence vanishing at most at isolated points — so $\widehat{y_\tau}$ is uniquely determined. Hence, inverting the Fourier transform, y_τ is also uniquely determined as asserted. Note that this argument is independent of the horizon T, we might even take $T = \infty$. △

Thus, despite the nominal asymmetry of the suggested information structure, we may actually assume that at each $\tau \in [0,T]$ both the fox and the hound have *perfect causal information* knowing both input functions y and z on $[0,\tau]$ — as well, of course, as knowing the impulse response functions φ and η.

We may remark, in this connection, that we are here assuming *exact observation and computation,* ignoring for now any concern for continuity of the maps $z_\tau, x_\tau \mapsto y_\tau$ whose existence has been assured by Theorem 5.2. However, our discussion has justified the *admissibility* of strategies such as (4.4) or (4.5).

5.3. The Irrigation Problem: Reprise. We now wish to compute more specifically the functions φ and η for the special case of (2.3), which came from the original motivation.

As a particular case of our discussion in Subsection 5.1, we already know that $H(\cdot)$ for (2.3) can be obtained as $\alpha\, U^{0,1}(\cdot,0)$, where $U^{0,1}$ is the solution to

$$U^{0,1}_t - aU^{0,1}_{ss} = 0 \text{ on } \mathcal{Q}_T, \qquad U^{0,1}(t,-1) = U^{0,1}(t,1) = 1, \qquad U^{0,1}(0,\cdot) \equiv 0.$$

This function has singularities only at $(0,\pm 1)$ — it is *analytic* on $\mathcal{Q}_T = (0,T] \times (-1,1)$ and C^∞ across $t=0$ (while taking $U^{0,1}(t,\cdot) \equiv 0$ for $t<0$). As $t \to \infty$ we would have the monotone increasing convergence of $U^{0,1}$ to the steady state solution $\equiv 1$. The function $\eta = H'$ is positive and unimodal, decaying exponentially to 0 as $t \to \infty$; we have

$$\eta^{[k]}(0) = 0 \ \text{ for } \ k = 0, 1, \ldots \quad \text{and} \quad \int_0^\infty \eta(\tau) \cdot \tau = \alpha.$$

For later purposes we now introduce the solution V to

$$V_t - aV_{ss} = 0 \ \text{ on } \mathcal{Q}_T, \qquad V(t,-1) = V(t,1) = t, \qquad V(0,\cdot) \equiv 0$$

and, differentiating this with respect to t, observe that $V_t = U^{0,1}$ since it satisfies the same equation. Thus

$$\alpha\, V(t,1) = \int_0^t H(\tau)\, d\tau.$$

We know, similarly, that $F(t) = \beta U^{1,0}(t,0)$, where $U^{1,0}$ is the solution to

$$U_t^{1,0} - aU_{ss}^{1,0} = 1 \ \text{ on } \mathcal{Q}_T, \qquad U^{1,0}(t,-1) = U^{1,0}(t,1) = 0, \qquad U^{1,0}(0,\cdot) \equiv 0;$$

again this is analytic on $\mathcal{Q}_T = (0,T] \times (-1,1)$ although not C^∞ across $t=0$. Defining now $W := t - U^{1,0}$, we see that

$$W_t = 1 - U_t^{1,0} = 1 - [aU_{ss}^{1,0} + 1] = aW_{ss}$$

with $W(t,\pm 1) = t - 0 = t$ and $W(0,\cdot) \equiv 0$. Comparing, we then observe that W satisfies the same system as V by showing that $U^{1,0} = t - V$. Evaluating at $s=0$ and differentiating, the latter implies that

$$\varphi(t) = F'(t) = \beta[1 - H(t)/\alpha] \ \text{ and } \ \eta(t) = -(\alpha/\beta)\varphi'(t). \tag{5.10}$$

Thus we get from this that $\varphi(0) = \beta$ with φ decreasing exponentially to 0, and then $\varphi^{[k]}(0) = 0$ for all $k = 1, 2, \ldots$.

What ithen are the implications for the groundwater management control problem of Section 2, which motivated our analysis? It is clear that $\mathfrak{G}(\ell, T, \varphi, \eta)$ — with the functions φ and η we have just computed — corresponds precisely to the groundwater management problem, except for a formal sign reversal for the interpretation of the control function z. The information we have just gathered about φ, η and their relation shows that φ is comparatively more *agile* than η in the sense of Section 4, so we expect considerable difficulty regarding the feasibility of this control problem without a substantial tolerance ℓ. It would certainly be of interest to determine numerically the minimal ℓ for which condition (4.3) would hold here, with its dependence on T and α/β.

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