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An abstract ‘bang-bang principle’ and time-optimal boundary control of the heat equation¹

Victor J. Mizel² and Thomas I. Seidman³

ABSTRACT: A principal technical result of this paper is that the one-dimensional heat equation with boundary control is exactly null-controllable with control restricted to an arbitrary set $\mathcal{E} \subset [0, T]$ of positive measure. A general abstract argument is presented to show that this implies the ‘bang-bang’ property for time-optimal controls — i.e., such a control can take only extreme values of (the hull of) the constraint set — without imposing any condition regarding the target state, in contrast to previous results.

KEY WORDS: *bang-bang control, nullcontrol, reachability.*

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²Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15217; e-mail: {vm09+@andrew.cmu.edu}.

³Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21228; e-mail: {seidman@math.umbc.edu}.

1. Introduction

Our principal concern will be with the ‘bang-bang’ property for time-optimal boundary control of the 1-dimensional heat equation

$$\begin{aligned}
 (1.1) \quad & u_t = u_{xx} \quad (0 < t < T, \ 0 < x < 1) \\
 & u(\cdot, 0) = \varphi = \textit{control}, \quad u(\cdot, 1) = 0 \\
 & u(0, \cdot) = \omega_0 \in \mathcal{X}_0 = L^2(0, 1).
 \end{aligned}$$

[Here we will assume a pointwise control constraint and then will say that φ has the ‘bang-bang’ property if it takes only extremal values.] We note at this point (cf., Remark 4.1) that the previously known results [12], [8] on the ‘bang-bang’ property for time-optimal control of (1.1) are incomplete in that their hypotheses impose conditions on the target state which turn out to be extraneous for the ‘bang-bang’ property per se; our focal goal will be the removal of such restrictions.

To this end, we will introduce an abstract formulation of the problem, following [14] in spirit if not quite in detail, and prove a general abstract result (Theorem 2) which, in the context of (1.1), reduces the problem to a question of independent interest: *exact nullcontrollability for (1.1) when φ is restricted to $L^\infty(\mathcal{E})$ for an arbitrary set \mathcal{E} of positive measure in $[0, T]$* . The argument for the latter (Theorem 5) is based on a recent result in nonharmonic analysis by P. Borwein and T. Erdélyi [2], [3].

We begin with the observation that there are really two quite distinct versions of the time-optimality problem in control theory:

- immediately initiate control so as to reach the goal as early as possible;
- reach the goal by a fixed time T while delaying initiation of active control to as late as possible.

The first of these is usually taken as the standard statement of the problem but, much as in [14], it will be more convenient here to use the second version for our abstract formulation. We do observe that the two versions are clearly equivalent when the problem is autonomous (with suitable initial conditions). It is also worth emphasizing that (still when the problem is autonomous) the construction used in the proof of the ‘abstract bang-bang

principle', Theorem 2, below, could equally well be used directly for a proof of the 'bang-bang' property in the context of 'version 1', with no restriction on the initial data.

In the context of 'version 2', we adjoin to (1.1) the target condition that the profile at time T belong to a prescribed set

$$(1.2) \quad u(T, \cdot) \in \mathcal{S}_T$$

and formulate the time-optimality problem as finding a pair (φ, τ) which maximizes τ subject to the *admissibility constraints* that, for a given set-valued function $A : [0, T] \rightarrow 2^{\mathbb{R}}$, one has

$$(1.3) \quad \begin{aligned} (i) \quad & \varphi(t) = \varphi_*(t) = \text{given} \quad \text{for } 0 \leq t < \tau \\ (ii) \quad & \varphi(t) \in A(t) \quad \text{for } \tau \leq t \leq T \\ (iii) \quad & \text{the solution } u \text{ of (1.1) using this control } \varphi \\ & \text{satisfies (1.2).} \end{aligned}$$

We interpret φ_* as a 'trivial' or 'passive' control (e.g., $\varphi_* \equiv 0$) so τ represents the time at which we initiate 'active' control and maximizing τ is just minimizing the duration $(T - \tau)$ of the actively controlled interval. [If $\omega_0 = 0$, $\varphi_* \equiv 0$, then (1.1) gives $u(\tau, \cdot) = 0$ so, if also $A(t)$ were independent of t , we could translate $[\tau, T]$ to $[0, T - \tau]$ to get the usual 'earliest arrival' (version 1) for this autonomous problem.] Concerning the set function $A(\cdot)$, we assume that

$$(1.4) \quad \begin{aligned} & a(t), b(t) \in A(t) \quad \text{and } a(\cdot), b(\cdot) \in L^\infty(0, T) \\ & \text{for } a(t) := \min\{A(t)\}, \quad b(t) := \max\{A(t)\}. \end{aligned}$$

We emphasize that to obtain the 'bang-bang' property *we need impose no hypotheses whatsoever on the data $\{\omega_0, T, \mathcal{S}_T, \varphi_*\}$ beyond the implicit assumption that such a time-optimal control does exist.* We do note that the usual argument gives existence of an optimizer in the setting above with $\mathcal{S}_T = \{\omega_T\}$, provided only that the data are *compatible* (i.e., there is some control satisfying (1.3)) — and this 'reachability' is the *only* restriction to be imposed regarding the target state ω_T in $\mathcal{X}_T = L^2(0, 1)$.

For (1.1), (1.2) with $\mathcal{S}_T = \{\omega_T\}$, (1.3-ii) with (1.4), we will show that

pointwise ae on $[\tau, T]$, the values of any time-optimal control φ must be either $\varphi(t) = a(t)$ or $\varphi(t) = b(t)$, with a, b as in (1.4-ii),

and that this time-optimal control φ is unique. Note that in this situation the control φ is a *scalar* function of t and this strongly affects the ease with which we can use Theorems 1, 2 to obtain such a ‘*bang-bang*’ *property*. We will, however, comment in Section 4 on the related situation in which one has control at both ends of the interval so

$$(1.5) \quad u|_{x=0} = \varphi_1, \quad u|_{x=1} = \varphi_2$$

and the control $\varphi := (\varphi_1, \varphi_2)$ is then an \mathbb{R}^2 -valued function on $[0, T]$.

2. Evolutionary abstract control systems

We begin by recalling from [14], in slightly modified form, the notion of an *evolutionary abstract control system*. Consider $\mathcal{I}_0 := [0, T]$ as an order category, i.e., writing $\mathcal{I} = \mathcal{I}'\mathcal{I}''$ for $\mathcal{I} = [r, t]$ means $\mathcal{I}'' = [r, s]$, $\mathcal{I}' = [s, t]$ with $0 \leq r \leq s \leq t \leq T$; let $(\mathcal{X}, \mathbf{E})$ be a functor from \mathcal{I}_0 to the category of Banach spaces and continuous linear maps so $\mathcal{I} = [r, t]$ gives $\mathbf{E}_{\mathcal{I}} : \mathcal{X}_r \rightarrow \mathcal{X}_t$ with $\mathcal{I} = \mathcal{I}'\mathcal{I}''$ implying $\mathbf{E}_{\mathcal{I}} = \mathbf{E}_{\mathcal{I}'} \circ \mathbf{E}_{\mathcal{I}''}$ —i.e.,

$$(2.1) \quad \mathbf{E}_{r,t} = \mathbf{E}_{s,t} \mathbf{E}_{r,s} : \mathcal{X}_r \rightarrow \mathcal{X}_s \rightarrow \mathcal{X}_t \quad \text{for } r \leq s \leq t.$$

In general, \mathbf{E}_{\cdot} represents uncontrolled system evolution for a possibly non-autonomous well-posed problem. Next we associate *control spaces* $\mathcal{U}_{\mathcal{I}} = \mathcal{U}_{r,t}$ to the intervals $\mathcal{I} = [r, t] \subset \mathcal{I}_0$. We always think of each $\mathcal{U}_{\mathcal{I}}$ as a space of functions defined on \mathcal{I} — e.g., $\mathcal{U}_{\mathcal{I}} = L^2(\mathcal{I})$ — so when $\mathcal{I} = \mathcal{I}'\mathcal{I}''$ we may decompose $\varphi \in \mathcal{U}_{\mathcal{I}}$ into a pair of functions (φ', φ'') , defined on \mathcal{I}' and \mathcal{I}'' , respectively, by restriction maps $\mathbf{\Omega}', \mathbf{\Omega}''$. We then ask that $\varphi' = \mathbf{\Omega}'\varphi \in \mathcal{U}_{\mathcal{I}'}$ and $\varphi'' = \mathbf{\Omega}''\varphi \in \mathcal{U}_{\mathcal{I}''}$. The control maps $\mathbf{C}_{r,t} : \mathcal{U}_{r,t} \rightarrow \mathcal{X}_t$ must satisfy the obvious identity

$$(2.2) \quad \mathbf{C}_{r,t} = \mathbf{C}_{s,t} \mathbf{\Omega}'_s + \mathbf{E}_{s,t} \mathbf{C}_{r,s} \mathbf{\Omega}''_s$$

for any such decomposition (any choice of $s \in [r, t]$).

Given any Banach space \mathcal{V} with an injection $\mathbf{I}_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{U}_{\mathcal{I}}$ — so we may think of \mathcal{V} as consisting of functions with support in (some specified subset of) $\mathcal{I} = [t, T]$ — we say that \mathcal{V} has the *nullcontrollability property* and write $\mathcal{V} \in \mathcal{NC}_{t,T}$ if there is a nullcontrol $v \in \mathcal{V}$ for each ‘initial state’ in \mathcal{X}_t , i.e., if

$$(2.3) \quad \text{For each } x = x_t \in \mathcal{X}_t \text{ there is some } v \in \mathcal{V} \text{ such that} \\ \mathbf{E}_{t,T}x + \mathbf{C}_{\mathcal{V}}v = 0 \quad (\mathbf{C}_{\mathcal{V}} := \mathbf{C}_{t,T}\mathbf{I}_{\mathcal{V}}).$$

[Clearly, if $\mathcal{V} \in \mathcal{NC}_{t,T}$ for some $\mathcal{V} \hookrightarrow \mathcal{U}_T$, then $\mathcal{U}_T \in \mathcal{NC}_{t,T}$. We recall Theorem 1 of [14]: If $\mathcal{U}_{t,T} \in \mathcal{NC}_{t,T}$, then $\mathcal{K}_s = \mathcal{K}_t^0$ for all $s \leq t$, where $\mathcal{K}_s := \mathcal{R}(\mathbf{E}_{s,T}) + \mathcal{R}(\mathbf{C}_{s,T})$ and $\mathcal{K}_s^0 := \mathcal{R}(\mathbf{C}_{s,T})$.] Slightly more delicate than (2.3), but useful later, is the *restricted nullcontrollability property*: we write $\mathcal{V} \in \mathcal{NC}_{t,T}^r$ if

$$(2.4) \quad \text{For each } x \in \overline{\mathcal{R}(\mathbf{C}_{0,t})} \subset \mathcal{X}_t \text{ there is some } v \in \mathcal{V} \text{ such that} \\ \mathbf{E}_{t,T}x + \mathbf{C}_\mathcal{V}v = 0,$$

i.e., we are restricting initial states in (??) to $\overline{\mathcal{R}(\mathbf{C}_{0,t})}$.

We will need the following result, which we present in full although the argument is already known in somewhat different contexts.

THEOREM 1: *If $\mathcal{V} \in \mathcal{NC}_{t,T}$, (respectively, $\mathcal{V} \in \mathcal{NC}_{t,T}^r$) then there is a constant $K_\mathcal{V}$ such that v in (2.3) (respectively, in (2.4)) may be chosen with $\|v\|_\mathcal{V} \leq K\|x\|_{\mathcal{X}_t}$ for any $K > K_\mathcal{V}$; dually, if $\mathcal{V} \in \mathcal{NC}_{t,T}$ one has*

$$(2.5) \quad \|\mathbf{E}_{t,T}^* \xi\|_{\mathcal{X}_t^*} \leq K_\mathcal{V} \|\mathbf{C}_\mathcal{V}^* \xi\|$$

for $\xi \in \mathcal{X}_T^*$ with the \mathcal{V}^* -norm on the right. Conversely, if \mathcal{V} contains a dual space ($\mathcal{W}^* \subset \mathcal{V}$) and $\mathbf{C}_\mathcal{V}^* \xi \in \mathcal{W}$ for a dense set \mathcal{D} of $\xi \in \mathcal{X}_T^*$, then (2.5) — using the \mathcal{W} -norm on the right for $\xi \in \mathcal{D}$ — implies that $\mathcal{V} \in \mathcal{NC}_{t,T}$.

PROOF: For brevity we now write simply \mathbf{E} for $\mathbf{E}_{t,T}$ and \mathbf{C} for $\mathbf{C}_\mathcal{V} := \mathbf{C}_{t,T}\mathbf{I}_\mathcal{V}$. Clearly, $\mathcal{V} \in \mathcal{NC}_{t,T}$ is equivalent to range containment:

$$\mathcal{R}(\mathbf{E}) \subset \mathcal{R}(\mathbf{C}) =: \mathcal{K}^0(\mathcal{V}) \subset \mathcal{X}_T.$$

Set $\hat{\mathcal{V}} := \mathcal{V}/\mathcal{N}(\mathbf{C})$ with an injective induced map $\hat{\mathbf{C}} : \hat{\mathcal{V}} \rightarrow \mathcal{X}_T$ (i.e., $\hat{\mathbf{C}}v = \mathbf{C}_\mathcal{V}\hat{v}$ for $v \in \hat{v} \in \hat{\mathcal{V}}$) and let

$$\Gamma := \{(x, \hat{v}) : \mathbf{E}x + \hat{\mathbf{C}}\hat{v} = 0\} \subset \mathcal{X}_t \times \hat{\mathcal{V}}.$$

Note that Γ is a subspace and is the graph of a linear map $\mathbf{L} = \mathbf{L}_\mathcal{V} : \mathcal{X}_t \rightarrow \hat{\mathcal{V}}$ which is well-defined on all of \mathcal{X}_t by (2.3) and the injectivity of $\hat{\mathbf{C}}$. Since Γ is closed (as $\mathbf{E}, \hat{\mathbf{C}}$ are continuous), it follows that $\mathbf{L}_\mathcal{V}$ is bounded, by the Closed Graph Theorem, and the bound on $\|v\|$ with $K_\mathcal{V} = \|\mathbf{L}\|$ follows from the definition of the quotient space norm on $\hat{\mathcal{V}}$. Simply replacing \mathcal{X}_t by $\overline{\mathcal{R}(\mathbf{C}_{0,t})}$ in the argument above now gives the bound when $\mathcal{V} \in \mathcal{NC}_{t,T}^r$. To obtain (2.5) when $\mathcal{V} \in \mathcal{NC}_{t,T}^r$, we note that the construction of \mathbf{L} gives

$$\mathbf{E} = -\hat{\mathbf{C}}\mathbf{L} \quad \text{so, dually,} \quad \mathbf{E}^* = -\mathbf{L}^*\hat{\mathbf{C}}^*$$

with $\mathbf{L}^* : \hat{\mathcal{V}}^* \rightarrow \mathcal{X}_t^*$. We then have $\|\mathbf{L}^*\| = \|\mathbf{L}\| =: K_{\mathcal{V}}$ and, since $\langle \hat{\mathbf{C}}^* \xi, \hat{v} \rangle = \langle \mathbf{C}^* \xi, v \rangle$ for $v \in \hat{v} \in \hat{\mathcal{V}}$ and $\xi \in \mathcal{X}_T^*$, this gives (2.5).

For the converse, consider any $\eta \in \mathbf{C}^* \mathcal{D}$ — i.e., $\eta = \mathbf{C}^* \xi$ for some $\xi \in \mathcal{D} \subset \mathcal{X}_T^*$ — we can set $\zeta := -\mathbf{E}^* \xi$, noting that if ξ is non-unique (so also $\eta = \mathbf{C}^* \xi'$ with $\xi' \in \mathcal{D}$) then (2.5) ensures that $\|\mathbf{E}^*(\xi - \xi')\| = 0$ so ζ is well-defined. Now, arbitrarily fixing $x = x_t \in \mathcal{X}_t$, we consider

$$\Phi : \eta \longmapsto \langle x, \zeta \rangle = -\langle x, \mathbf{E}^* \xi \rangle$$

for such η . It is clear that the functional Φ is linear on $\mathbf{C}^* \mathcal{D} \subset \mathcal{W}$ and that

$$|\langle \Phi, \eta \rangle| = |\langle x, \zeta \rangle| \leq \|x\| \|\zeta\| \leq \|x\| K \|\eta\|.$$

Thus, Φ extends by continuity to the \mathcal{W} -closure $\overline{\mathbf{C}^* \mathcal{D}}$ and then, by the Hahn-Banach Theorem, to a linear functional v on \mathcal{W} (i.e., $v \in \mathcal{W}^* \subset \mathcal{V}$) without increase of norm so $\|v\| \leq K_{\mathcal{V}} \|x\|$. Since

$$\langle \mathbf{C}v, \xi \rangle = \langle v, \mathbf{C}^* \xi \rangle = -\langle x, \mathbf{E}^* \xi \rangle = \langle -\mathbf{E}x, \xi \rangle$$

for ξ dense in \mathcal{X}_T^* , it follows that $\mathbf{E}x + \mathbf{C}v = 0$ and $v \in \mathcal{V}$ is a nullcontrol for x . As $x \in \mathcal{X}_t$ was arbitrary, we have (2.3) so $\mathcal{V} \in \mathcal{NC}_{t,T}$ as asserted. \blacksquare

3. Time-optimality

We now turn to formulation of the abstract time-optimality problem. It will be convenient here to abuse notation slightly by thinking of $\mathcal{U} = \mathcal{U}_{0,T}$ as the common domain of the control maps $\mathbf{C}_{s,t} : \mathcal{U} \rightarrow \mathcal{X}_t$, omitting explicit indication of the $\mathbf{\Omega}$ operators; note that we think of $\mathbf{C}_{s,t}\varphi$ as depending only on ‘the part of φ between s and t ’ — so $\mathcal{N}(\mathbf{C}_{s,t}) \supset \mathcal{N}(\mathbf{\Omega}_{[s,t]})$ where, in the obvious notation,

$$\mathbf{\Omega}_{[s,t]} := \mathbf{\Omega}'_{s,[0,t]} \mathbf{\Omega}''_{t,[0,T]} = \mathbf{\Omega}''_{t,[s,T]} \mathbf{\Omega}'_{s,[0,T]}.$$

Fixing the *passive control* $\varphi_* \in \mathcal{U}$, a basic assumption is that for each $s \in (0, T)$ we have

$$(3.1) \quad \varphi \in \mathcal{U} \Rightarrow \mathbf{P}_s \varphi := \begin{cases} \varphi_* & \text{on } [0, s) \\ \varphi & \text{on } [s, T] \end{cases} \in \mathcal{U}$$

or, more formally, $\Omega'_s \mathbf{P}_s \varphi = \Omega'_s \varphi$ and $\Omega''_s \mathbf{P}_s \varphi = \Omega''_s \varphi_*$; note that \mathbf{P}_s will not generally be linear unless $\varphi_* \equiv 0$. We impose the continuity condition that

$$(3.2) \quad \mathbf{C}_{r,t} \mathbf{P}_s \varphi \rightarrow \mathbf{C}_{r,t} \varphi \text{ as } s \searrow r.$$

for $0 \leq r < t \leq T$ — which just says that changing φ on the vanishingly small interval $[r, s]$ has vanishingly small control effect at any $t > r$.

The data for the time-optimality problem will be

$$(3.3) \quad x_0 \in \mathcal{X}_0, \quad \varphi_* \in \mathcal{U}, \quad \mathcal{A} \subset \mathcal{U}, \quad \mathcal{S}_T \subset \mathcal{X}_T$$

where x_0 is an *initial state*, φ_* the ‘passive control’, \mathcal{A} is a *constraint set*, and \mathcal{S}_T the *target set*. We will require — to simplify our statement, rather than as a restriction on \mathcal{A} — that $\varphi \in \mathcal{A}$ implies $\mathbf{P}_s \varphi \in \mathcal{A}$ for each s . The set of *admissible pairs* $\mathcal{P} = \mathcal{P}(\mathcal{A}, \mathcal{S}_T; x_0, \varphi_*)$ is then defined as

$$(3.4) \quad \mathcal{P} := \{(\varphi, \tau) \in \mathcal{A} \times [0, T] : \varphi = \mathbf{P}_\tau \varphi, [\mathbf{E}_{0,T} x_0 + \mathbf{C}_{0,T} \varphi] \in \mathcal{S}_T\}$$

and we say that a control $\bar{\varphi}$ or, more precisely, an admissible pair $(\bar{\varphi}, \bar{\tau}) \in \mathcal{P}$ is *time-optimal* (with respect to this data) if it maximizes τ over $(\varphi, \tau) \in \mathcal{P}$.

We will say that $\varphi \in \mathcal{U}$ is ‘*slack with respect to $(\mathcal{A}, \mathcal{V})$* ’ (for a Banach space \mathcal{V} with $\mathbf{I}_\mathcal{V} : \mathcal{V} \rightarrow \mathcal{U}$) if there is some $\varepsilon > 0$ such that

$$(3.5) \quad [\varphi + \mathbf{I}_\mathcal{V} v] \in \mathcal{A} \text{ for all } v \in \mathcal{V} \text{ with } \|v\|_\mathcal{V} < \varepsilon.$$

At this point we may state and prove our ‘abstract bang-bang principle’

THEOREM 2: *Suppose $\varphi \in \mathcal{U}$ is slack with respect to $(\mathcal{A}, \mathcal{V})$ for some $\mathcal{V} \in \mathcal{NC}_{t,T}^r$. Then (φ, τ) with $\tau < t$ cannot be time-optimal with respect to any data set $(\mathcal{A}, \mathcal{S}_T; x_0, \varphi_*)$ involving this \mathcal{A} .*

PROOF: Note that, while we have written simply $\mathbf{I}_\mathcal{V} : \mathcal{V} \rightarrow \mathcal{U}$, the condition that $\mathcal{V} \in \mathcal{NC}_{t,T}^r$ includes the implication that $\mathcal{R}(\mathbf{I}_\mathcal{V})$ is actually in $\mathcal{U}_{t,T}$ so for $s \leq t$ one has

$$(3.6) \quad \mathbf{P}_s \varphi_s = \varphi_s \text{ for } \varphi_s := \mathbf{P}_s(\varphi + \mathbf{I}_\mathcal{V} v) \quad (\text{any } v \in \mathcal{V}).$$

Now let K^r be as $K_\mathcal{V}$ in Theorem 1 applied to this $\mathcal{V} \in \mathcal{NC}_{t,T}^r$ and let $\varepsilon > 0$ be as in (3.5). In view of (3.2) with $r = \tau < t$, we may choose $s =: \hat{\tau}$ close enough to τ (with $\tau < \hat{\tau} < t$) that

$$(3.7) \quad \tilde{x} := \mathbf{C}_{\tau,t} [\mathbf{P}_{\hat{\tau}} \varphi - \varphi] \text{ gives } \|\tilde{x}\| < \varepsilon / K^r,$$

noting that $\tilde{x} \in \mathcal{R}(\mathbf{C}_{0,t}) \subset \mathcal{X}_t$. By Theorem 1 we may then choose $v \in \mathcal{V}$ such that

$$(3.8) \quad \mathbf{E}_{t,T}\tilde{x} + \mathbf{C}_{t,T}v = 0 \text{ and } \|v\|_{\mathcal{V}} < \varepsilon.$$

Now set

$$(3.9) \quad \hat{\varphi} := \mathbf{P}_{\hat{\tau}}(\varphi + \mathbf{I}_{\mathcal{V}}v) \quad \text{--- i.e., } \hat{\varphi} = \begin{cases} \varphi_* & \text{on } [0, \tau) \\ \mathbf{P}_{\hat{\tau}}\varphi & \text{on } [\tau, t) \\ \varphi + \mathbf{I}_{\mathcal{V}}v & \text{on } [t, T]. \end{cases}$$

Since $\|v\|_{\mathcal{V}} < \varepsilon$, we have $[\varphi + \mathbf{I}_{\mathcal{V}}v] \in \mathcal{A}$ by (3.5) so also $\hat{\varphi} \in \mathcal{A}$; we have $\mathbf{P}_{\hat{\tau}}\hat{\varphi} = \hat{\varphi}$ by (3.6). Using (2.2) twice, splitting $[0, T]$ at τ and at t , we have

$$(3.10) \quad \begin{aligned} \mathbf{C}_{0,T}\varphi &= \mathbf{E}_{\tau,T}\mathbf{C}_{0,\tau}\varphi_* + \mathbf{C}_{\tau,T}\varphi \\ &\quad \text{as } \varphi = \varphi_* \text{ on } [0, \tau) \\ &= \mathbf{E}_{\tau,T}\mathbf{C}_{0,\tau}\varphi_* + \mathbf{E}_{t,T}\mathbf{C}_{\tau,t}\varphi + \mathbf{C}_{t,T}\varphi \end{aligned}$$

and, similarly, we have

$$(3.11) \quad \mathbf{C}_{0,T}\hat{\varphi} = \mathbf{E}_{\tau,T}\mathbf{C}_{0,\tau}\varphi_* + \mathbf{E}_{t,T}\mathbf{C}_{\tau,t}\mathbf{P}_{\hat{\tau}}\varphi + \mathbf{C}_{t,T}[\varphi + \mathbf{I}_{\mathcal{V}}v]$$

using (3.9). Comparing (3.11) to (3.10) gives (with $\mathbf{C}_{\mathcal{V}} := \mathbf{C}_{t,T}\mathbf{I}_{\mathcal{V}}$ as before)

$$(3.12) \quad \begin{aligned} \mathbf{C}_{0,T}\hat{\varphi} - \mathbf{C}_{0,T}\varphi &= \mathbf{E}_{t,T}\mathbf{C}_{\tau,t}[\mathbf{P}_{\hat{\tau}}\varphi - \varphi] + \mathbf{C}_{\mathcal{V}}v \\ &= \mathbf{E}_{t,T}\tilde{x} + \mathbf{C}_{\mathcal{V}}v = 0 \end{aligned}$$

by (3.7) and (3.8).

It follows that $(\hat{\varphi}, \hat{\tau}) \in \mathcal{P} = \mathcal{P}(\mathcal{A}, \mathcal{S}_T; x_0, \varphi_*)$ for any data which gives $(\varphi, \tau) \in \mathcal{P}$: if $\mathbf{E}_{0,T}x_0 + \mathbf{C}_{0,T}\varphi =: x_T \in \mathcal{S}_T$, then also $\mathbf{E}_{0,T}x_0 + \mathbf{C}_{0,T}\hat{\varphi} = x_T$ for the very same $x_T \in \mathcal{S}_T$. Since $\hat{\tau} > \tau$, it would then be impossible for τ to be maximal and φ could not be a time-optimal control. \blacksquare

To see why we refer to Theorem 2 as an ‘abstract bang-bang principle’, we note our motivating consequence. Observe, first, that in considering scalar controls with a uniform pointwise bound as in (1.4), there is some arbitrariness about the specification of the control space \mathcal{U} . We will, somewhat arbitrarily, take $\mathcal{U} := L^p(0, T)$ for some finite $p > 1$ (so, in particular, \mathcal{U} is reflexive) and assume that each of the operators $\mathbf{E}_{s,t}$, $\mathbf{C}_{s,t}$ is continuous for this choice of \mathcal{U} .

THEOREM 3: Consider a time-optimality problem, as above, with \mathcal{S}_T closed and convex in \mathcal{X}_T , scalar control (say, $\mathcal{U} = L^p(0, T)$ for some $p \geq 1$), and \mathcal{A} of the form:

$$(3.13) \quad \mathcal{A} := \{\varphi \in \mathcal{U} : \varphi(t) \in A(t) \text{ ae on } [0, T]\}$$

with $A(\cdot)$ as in (1.4) and $\varphi_* \in \mathcal{A}$. Assume

$$(3.14) \quad \begin{aligned} &\text{For each } t \in (0, T), \text{ each set } \mathcal{E} \subset (t, T) \text{ of positive measure,} \\ &\text{one has } L^\infty(\mathcal{E}) \in \mathcal{NC}_{t,T}^r. \end{aligned}$$

Then there is a unique time-optimal control $\bar{\varphi}$ and this necessarily has the ‘bang-bang’ property:

$$(3.15) \quad [\varphi(t) = a(t) \text{ or } \varphi(t) = b(t)] \text{ ae on } [\tau, T]$$

with a, b as in (1.4).

The key to this is that for (3.15) to fail one must have

$$(3.16) \quad a(t) + \varepsilon \leq \varphi(t) \leq b(t) - \varepsilon \quad \text{for } t \in \mathcal{E}$$

for some $\varepsilon > 0$ and some set \mathcal{E} of positive measure in $[\tau, T]$ — perhaps restricting to an intersection, we may assume this set \mathcal{E} is actually contained in some $[\bar{t}, T]$ with $\bar{t} > \tau$. We do note that the very existence of a time-optimal control is not immediately clear at this point since we have not even assumed that $A(t)$ should be a closed set.

PROOF: We first consider the situation with \mathcal{A} replaced by \mathcal{A}_* where

$$\mathcal{A}_* := \{\varphi \in \mathcal{U} : \varphi(t) \in [a(t), b(t)] =: A_*(t) \text{ ae on } [0, T]\}.$$

As \mathcal{A}_* is bounded, closed, and convex (hence weakly compact in $\mathcal{U} = L^p(0, T)$), the usual argument gives existence of a time-optimal control: Let (φ_ν) be an optimizing sequence so we may assume $\varphi_\nu \rightharpoonup \bar{\varphi}$ with $\tau_\nu \nearrow \tau$; noting that $\mathbf{C}_{0,T}\varphi_\nu \rightarrow \mathbf{C}_{0,T}\bar{\varphi}$, we must have $\mathbf{E}_{0,T}x_0 + \mathbf{C}_{0,T}\bar{\varphi} \in \mathcal{S}_T$ whence $(\bar{\varphi}, \tau)$ is admissible and so time-optimal.] By (3.14) and Theorem 2, we see that $\bar{\varphi}$ cannot be slack with respect to $(\mathcal{A}_*, \mathcal{V})$ for any $\mathcal{V} = L^\infty(\mathcal{E})$ with \mathcal{E} of positive measure in (\bar{t}, T) , $\bar{t} > \tau$. On the other hand, we have already noted that a failure of (3.15) would give (3.16), which would imply such slackness and give

a contradiction. Hence, $\bar{\varphi}$ must satisfy (3.15) so, by (1.4), we have $\bar{\varphi} \in \mathcal{A}$ and this pair $(\bar{\varphi}, \tau)$ is also admissible for the original problem. Since the problem using \mathcal{A}_* is a relaxed version of that, $(\bar{\varphi}, \tau)$ must be time-optimal for the original problem.

To see uniqueness, note that if $(\hat{\varphi}, \tau)$ were a different time-optimal pair for the original problem (necessarily with the same τ), then we may set $\tilde{\varphi} := (\bar{\varphi} + \hat{\varphi})/2$ and note that $(\tilde{\varphi}, \tau)$ is an admissible pair for the problem using \mathcal{A}_* , since the system is linear and $\mathcal{A}_*, \mathcal{S}_T$ are convex. Whether or not $\hat{\varphi}$ satisfies (3.15), it is clear that (3.15) cannot hold for $\tilde{\varphi}$ on the (assumed nonnull) set where $\hat{\varphi} \neq \bar{\varphi}$. As above, we then see that $(\tilde{\varphi}, \tau)$ cannot be time-optimal for that problem, contradicting the assumed maximality of τ . Thus, $\bar{\varphi}$ is the unique optimal control for the original problem. \blacksquare

For the finite dimensional case (state space \mathbb{R}^n) we see that the hypotheses above are easily established for control systems governed by

$$(3.17) \quad \dot{x} = Ax + \varphi \mathbf{b} \quad x(0) = x_0.$$

COROLLARY 4: *The results of Theorem 3 apply to finite dimensional time-optimality problems of the indicated form for (3.17), provided $A(\cdot), \mathbf{b}(\cdot)$ are real-analytic on $[0, T]$ when this is non-autonomous.*

PROOF: We need only verify the hypothesis (3.14) and for this it is convenient to take $\mathcal{X}_t := \mathcal{R}(\mathbf{C}_{0,t})$ for $t \in [0, T]$ so, in particular, $\mathbf{C} = \mathbf{C}_{0,T}$ is surjective to \mathcal{X}_T . The choice of control space \mathcal{U} is not very significant and we take, e.g., $\mathcal{U} := L^2(0, T)$. One easily verifies that the adjoint map \mathbf{C}^* is given, for $\eta \in \mathcal{X}_T^* (\subset \mathbb{R}^n)$, by $\mathbf{C}^* : \eta \mapsto \langle \mathbf{b}, y \rangle \in L^2(0, T)$ where

$$(3.18) \quad -\dot{y} = A^* y, \quad y(T) = \eta.$$

The range $\mathcal{R}(\mathbf{C}^*) = \{\langle \mathbf{b}, y \rangle\}$ is then finite dimensional — indeed, as \mathbf{C} is surjective, it follows that \mathbf{C}^* is injective and $\dim \mathcal{R}(\mathbf{C}^*) = \dim \mathcal{X}_T^* = \dim \mathcal{X}_T \leq n$. The analyticity assumptions on $A(\cdot), \mathbf{b}(\cdot)$ ensure that y and $\langle \mathbf{b}, y \rangle$ are real-analytic on $[0, T]$. Hence, if $\langle \mathbf{b}, y \rangle = 0$ on any set \mathcal{E} of positive measure, one must have $\langle \mathbf{b}, y \rangle \equiv 0$ on $[0, T]$. Thus, the map $\mathbf{L}_{\mathcal{E}} : \eta \mapsto \langle \mathbf{b}, y \rangle|_{\mathcal{E}} : \mathcal{X}_T^* \rightarrow \mathcal{R}(\mathbf{C}) \rightarrow \hat{\mathcal{W}}$ (where $\hat{\mathcal{W}}$ consists of the restrictions to \mathcal{E} of functions in $\mathcal{R}(\mathbf{C}^*)$) is injective and so invertible. Since $\hat{\mathcal{W}}$

is finite dimensional, $[\mathbf{L}_{\mathcal{E}}]^{-1}$ is continuous with $\hat{\mathcal{W}}$ normed as a subspace of $\mathcal{W} := L^1(\mathcal{E})$ (so $\mathcal{V} := L^\infty(0, T)$ is just \mathcal{W}^*) and (2.5) holds, giving (3.14) by Theorem 1. The conclusion is now immediate from Theorem 3. \blacksquare

This argument seems new, even for the finite dimensional case; we do note that it does not seem to be usefully related to the usual characterization of time-optimal controls as in the Pontrjagin Maximum Principle.

4. Boundary control of the heat equation

In this section we return to consideration of (1.1) as an example of the abstract formulation of Sections 2, 3. Our principal new result is exact boundary nullcontrollability from measurable sets — more precisely, that $L^\infty(\mathcal{E}) \in \mathcal{NC}_{t,T}$ for any set \mathcal{E} of positive measure in $[t, T]$. This is just (3.14) — one notes that $\mathcal{NC}_{t,T}$ and $\mathcal{NC}_{t,T}^r$ are equivalent here — so Theorem 3 then gives the desired ‘bang-bang’ property for time-optimal boundary control of (1.1).

We will take $\mathcal{X}_t = \mathcal{X} := L^2(0, 1)$ for each $t \in [0, T]$ and will, e.g., take $\mathcal{U} = L^2(0, T)$, so $\mathcal{U}_{\mathcal{I}} = L^2(\mathcal{I})$ with the obvious interpretations of the $\mathbf{\Omega}$ operators by restriction. For this autonomous situation one has $\mathbf{E}_{r,t} = \mathbf{S}(t - r)$ where $\mathbf{S}(\cdot)$ is the semigroup on $L^2(0, 1)$ corresponding to (1.1) with homogeneous boundary conditions. Then $\mathbf{C}_{s,t}$ is the control effect (so $\mathbf{C}_{s,t} : \varphi \mapsto u(t, \cdot)$ where u satisfies (1.1-*i, ii*) with $u(s, \cdot) = 0$) and it is standard (cf., e.g., [10]) that each $\mathbf{C}_{s,t}$ is continuous — indeed, compact — from $L^2(s, t)$ to $\mathcal{X} = L^2(0, 1)$. [We note in passing that there is a well-known explicit representation for this control mapping associated with (1.1) — using convolution with a fundamental solution, expressible in terms of a theta function; cf., e.g., [5] p. 171.] The identities (2.1), (2.2) are clear in this context. For this \mathcal{U} there is no difficulty in defining \mathbf{P}_s and the continuity condition (3.2) here follows *a fortiori* from the stronger fact that $\mathbf{P}_s \rightarrow \mathbf{P}_r$ (strongly on $\mathcal{U} = L^2(0, T)$) as $s \rightarrow r$.

To compute the adjoint maps $\mathbf{E}_{t,T}^*$, $\mathbf{C}_{t,T}^*$ we consider u satisfying (1.1) for

$\bar{t} < t \leq T$ with $u(\bar{t}, \cdot) \equiv 0$ and y satisfying

$$(4.1) \quad \begin{aligned} -y_t &= y_{xx} & (0 < t < T, \ 0 < x < 1) \\ y(T, \cdot) &= \eta \in \mathcal{X}_T^* = L^2(0, 1) \\ y(\cdot, 0) &\equiv 0 \equiv y(\cdot, 1). \end{aligned}$$

A simple computation involving (1.1) with $u(\bar{t}, \cdot) = 0$, (4.1), and an integration by parts gives the identity

$$\int_0^1 u y \, dx \Big|_{t=T} = \int_{\bar{t}}^T \varphi [y_x(\cdot, 0)] \, dt$$

and, since $u(T, \cdot) = \mathbf{C}_{\bar{t}, T}^* \varphi$ here, this gives

$$(4.2) \quad \begin{aligned} \mathbf{C}_{\bar{t}, T}^* : \mathcal{X}_T^* &\rightarrow L^2(\bar{t}, T) \subset L^2(0, T) \\ : \eta &\mapsto \psi := y_x(\cdot, 0) \Big|_{[\bar{t}, T]}. \end{aligned}$$

Even more simply, (4.1) gives

$$(4.3) \quad \mathbf{E}_{\bar{t}, T}^* : \mathcal{X}_T^* \rightarrow \mathcal{X}_{\bar{t}}^* = L^2(0, 1) : \eta \mapsto y(\bar{t}, \cdot).$$

It will be necessary to represent y in terms of the eigenfunctions and eigenvalues

$$(4.4) \quad \eta_k(x) := \sqrt{2} \sin \sqrt{\lambda_k} x, \quad \lambda_k := k^2 \pi^2$$

so that

$$(4.5) \quad \eta = \sum_k c_k \eta_k \text{ gives } \begin{cases} y = \sum_k c_k e^{-\lambda_k(T-t)} \eta_k \\ \psi = \sum_k \left[\sqrt{2\lambda_k} c_k \right] e^{-\lambda_k(T-t)}. \end{cases}$$

Our immediate observation is that

$$\eta \in \mathcal{D} := \text{span} \{ \eta_k \} \Rightarrow \mathbf{C}_{\bar{t}, T}^* \eta = \psi \in \mathcal{M} = \mathcal{M}(\Lambda) := \text{span} \{ e^{-\lambda_k(T-t)} \}$$

where $\Lambda := \{ \lambda_k : k = 1, 2, \dots \}$ with, looking to a somewhat more general setting, $0 < \lambda_1 < \lambda_2 < \dots$ such that $\sum_k 1/\lambda_k$ is convergent — as is obviously the case here.

Our starting point will be an inequality

$$(4.6) \quad \|y(0, \cdot)\|_{L^2(0, 1)} \leq M_{\bar{t}} \|y_x(\cdot, 0)\|_{L^2(0, \bar{t})}$$

for solutions of (4.1); it is sufficient to consider this only for $\eta \in \mathcal{D}$. We recognize this as (2.5) — giving (2.3) by Theorem 1 — corresponding to having $\mathcal{U}_{0,\bar{t}} \in \mathcal{NC}_{0,\bar{t}}$, replacing T by \bar{t} here. We will take this nullcontrollability as ‘well-known’ — but note that essentially this inequality (with time reversed and an interchange of Dirichlet and Neumann conditions) was the principal result of [11], with the nullcontrollability form given in [4]). From (4.6) with time reversed, one sees clearly the interpretation of (2.5) as asserting *well-posed observability*: predicting the terminal state from (boundary) observations without knowing the initial state.

Our major new resource is an inequality recently obtained by P. Borwein and T. Erdélyi; this is Theorem 5.6 of [2], but see also [1], [3].

THEOREM (BE): *Assume $\sum_k 1/\lambda_k < \infty$, etc. Then, for every $q > 0$, $s > 0$, $\rho \in (0, 1)$, there is a constant $c = c_q(s, \rho, \Lambda)$ such that:*

For every set $\mathcal{S} \subset [\rho, 1]$ with $\text{meas } \mathcal{S} \geq s$ one has

$$(4.7) \quad \|p\|_{L^\infty(0,\rho)} \leq c \|p\|_{L^q(\mathcal{S})}$$

for every ‘polynomial’ $p \in \mathcal{M}_0 = \mathcal{M}_0(\Lambda) := \{\sum_k a_k x^{\lambda_k}\}$.

For our present purposes, we make the substitution $x = e^{-(T-t)}$ and set $\rho = e^{-(T-\bar{t})}$ so $t \in [0, \bar{t}]$, $[\bar{t}, T]$, \mathcal{E} correspond, respectively, to $x \in [e^{-T}, \rho] \subset [0, \rho]$, $[\rho, 1]$, \mathcal{S} and \mathcal{M} corresponds to \mathcal{M}_0 ; noting that $\text{meas } \mathcal{S} \geq \rho \text{meas } \mathcal{E}$ for $\mathcal{E} \subset [\bar{t}, T]$, one easily sees that, specializing to $q = 1$, (4.7) gives just the inequality we will need:

$$(4.8) \quad \|\tilde{\psi}\|_{L^2(0,\bar{t})} \leq \tilde{c} \|\tilde{\psi}\|_{L^1(\mathcal{E})} \quad \text{for } \tilde{\psi} \in \mathcal{M}$$

with $\tilde{c} = \sqrt{\bar{t}} c_1(\rho \text{meas } \mathcal{E}, \rho, \Lambda)$ for any set \mathcal{E} of positive measure in $[\bar{t}, T]$.

At this point we are in a position to state and prove our second principal result, on exact boundary nullcontrollability of the one-dimensional heat equation from arbitrary sets of positive measure.

THEOREM 5: *Let $T > 0$ and suppose $\mathcal{E} \subset [0, T]$ has positive measure. Then there is a constant K such that:*

For every $\omega_0 \in \mathcal{X} = L^2(0, 1)$ there is a control φ such that

$$|\varphi(t)| \leq K \|\omega_0\|_{\mathcal{X}} \text{ for } t \in \mathcal{E}, \quad \varphi(t) = 0 \text{ for } t \notin \mathcal{E},$$

and the solution u of (1.1), using φ , has $u(T, \cdot) = 0$.

PROOF: This follows directly from the results we have already developed. Choose any $\bar{t} > 0$ such that $\hat{\mathcal{E}} \cap [\bar{t}, T]$ has positive measure; set $\mathcal{W} := L^1(\hat{\mathcal{E}})$

and $\mathcal{V} := \mathcal{W}^* = L^\infty(\hat{\mathcal{E}})$. Consider $y(0, \cdot) = \mathbf{E}_{0,T}^* \eta$ and $\tilde{\psi} = \psi = \mathbf{C}_{0,T}^* \eta$ for $\eta \in \mathcal{D}$. Then (4.6) and (4.8) with \mathcal{E} replaced by $\hat{\mathcal{E}}$ give $\|y(0, \cdot)\|_{L^2(0,1)} \leq M_{\tilde{t}} \tilde{c} \|\psi\|_{L^1(\hat{\mathcal{E}})}$ or, equivalently,

$$\|\mathbf{E}_{0,T}^* \eta\|_{\mathcal{X}^*} \leq K_{\mathcal{V}} \|\mathbf{C}_{\mathcal{V}}^* \eta\|_{\mathcal{W}}$$

which we recognize as (2.5). The second part of Theorem 1 then gives $\mathcal{V} \in \mathcal{NC}_{\tilde{t},T}$ which, since $\hat{\mathcal{E}} \subset \mathcal{E}$ so $\mathcal{V} \hookrightarrow L^\infty(\mathcal{E})$, gives precisely the conclusion of the present theorem. \blacksquare

COROLLARY 6: *The results of Theorem 3 apply to the time-optimality problem for (1.1).*

PROOF: Theorem 5 just gives the hypothesis (3.14) in this context so Theorem 3 applies. \blacksquare

The argument in Theorem 5 establishing that for each \mathcal{E} in $[t, T]$ of positive measure one has $L^\infty(\mathcal{E})$ in \mathcal{NC}^r and hence that Theorem 3 applies, shows (cf. Theorem V 1.1 of [7]) that the vector measure

$$m : B[0, T] \rightarrow \mathcal{X}_T = L^2(0, 1) : \mathcal{E} \mapsto C_{0,T}(\chi_{\mathcal{E}})$$

is a *Liapunov measure* — i.e., for each Borel set $\mathcal{F} \subset [0, T]$ of positive measure, the set $\{m(\mathcal{E}) : \mathcal{E} \subset \mathcal{F}\}$ is a convex, weakly compact subset of L^2 . The control-theoretic implications of this property of m are discussed in Chapters V and IX of [7].

REMARK 4.1: We remark that the ‘bang-bang’ property for time-optimal controls is classical for the finite-dimensional case, but has previously been shown in the context of boundary controls of the heat equation only with the imposition of a ‘slackness condition’ on the target state: the control constraint has the form $|\varphi| \leq M$ where it is to be known that the target is actually reachable (in *some* time) subject to $|\varphi| \leq M'$ with the slackness consisting of asking that $M > M'$. Some years ago, when [12] appeared, we felt that this condition might be an artifact of the proof technique and we attempted to demonstrate the ‘bang-bang’ property without it, i.e., for arbitrary (reachable) targets. We failed at that time: the gap in our argument

was the need for an estimate such as (4.7) and it is the recent availability of the result by P. Borwein and T. Erdélyi [1] which has enabled us now to return successfully to the problem, at least for 1 space dimension.

It should be noted that a newer proof of the ‘bang-bang’ property was presented in W. Krabs’ book [8], but this proof also imposes an auxiliary condition on the target state u_T . The result, Theorem 2.4.13 of [8], is formulated in terms of a moment problem, so some translation is necessary for comparison. Krabs requires that $\mathbf{c} \in W$ where $\mathbf{c} = (c_k)$ is the sequence of Fourier coefficients of the target u_* and the space W is such that this requirement is equivalent to asking that u_* is a limit — in the sense that differences are reachable by controls with L^∞ -norm approaching 0 — of targets of the special form $\tilde{u}(\varepsilon, \cdot)$ for $\varepsilon > 0$ and \tilde{u} satisfying the equation $\tilde{u}_t = \tilde{u}_{xx}$ with control vanishing on $[T - \varepsilon, T]$. Certainly the special targets then have $\tilde{u}(\varepsilon, x) = 0$ at $x = 0, 1$ so this, in particular, will also be true in the limit, i.e., for the targets to which this Theorem 2.4.13 would apply. Krabs also provides Theorem 2.4.14, explicitly following ideas of [12], giving the conclusion with essentially the same ‘slackness condition’ mentioned earlier; this condition certainly implies that $|\tilde{u}(0)| \leq M' < M$. Thus, neither of these theorems would apply to use as target, e.g., the trivially reachable state obtained by taking $\varphi \equiv M$ on $[0, T_*]$. In comparison, we emphasize that we have imposed *no* requirement on the target to get the ‘bang-bang’ property for a time-optimal control except as is implicit in the very existence of such a control.

The paper [12] considers the n -dimensional case (a bounded spatial region $\Omega \subset \mathbb{R}^n$ with control φ on $[0, T] \times \partial\Omega$) subject to a constraint of the form

$$(4.9) \quad |\varphi(t, x)| \leq M \quad \text{ae for } 0 \leq t \leq T, \ x \in \partial\Omega.$$

To use our present approach to prove the strong form of the ‘bang-bang’ property — that $|\varphi^*| = M$ ae on $[0, T^*] \times \partial\Omega$ — would require an n -dimensional form of Theorem 5, showing exact nullcontrollability with controls in $L^\infty(\mathcal{E})$ where \mathcal{E} is now an arbitrary subset of positive measure in $[0, T^*] \times \partial\Omega$. This seems well out of reach by currently available ideas — indeed, even the nullcontrollability from a patch ($\mathcal{E} = [0, T^*] \times \mathcal{P}$ with $\mathcal{P} \subset \partial\Omega$ open but small) has only recently been demonstrated ([9], compare [13]). On the other hand, it seems to be a tractable open problem to show the weaker ‘bang-bang’ property that $\|\varphi^*(t, \cdot)\|_{L^\infty(\partial\Omega)} = M$ ae on $[0, T^*]$ by showing nullcontrollabil-

ity from $L^\infty(\mathcal{E} \times \partial\Omega)$ with \mathcal{E} of positive measure in $[0, T^*]$ as earlier.

Note that each of the results above obtains the ‘bang-bang’ property by way of the adjoint characterization: $\varphi = \{M \text{ where } v_x \geq M; -M \text{ where } v_x \leq -M\}$ for some solution v of the adjoint problem. A plausible conjecture is that the additional restriction on the target state might be significant to ensure this characterization (so there might conceivably be examples for which this characterization fails in the absence of some such slackness condition; this could be a subject for future investigation), although we have seen that it is not necessary for the ‘bang-bang’ property itself.

REMARK 4.2: An essentially identical argument works if we replace the heat equation in (1.1) by

$$(4.10) \quad u_t = (pu_x)_x - qu$$

and/or replace the Dirichlet boundary conditions there by some alternative type of boundary control. For this case we let $\{\lambda_k, z_k\}$ be the eigenvalues and eigenfunctions of the Sturm-Liouville operator $\mathbf{A} : z \mapsto -(pz')' + qz$ whose (homogeneous) boundary conditions are those of the new form of boundary control.

Similarly, one could consider the problem with scalar control in the equation itself:

$$(4.11) \quad u_t = (pu_x)_x - qu + \varphi(t)b$$

for some specified $b(\cdot) \in \mathcal{X}$, using homogeneous boundary condition. In this connection one might note Henry’s example [6] of a problem with time-optimal control not of bang-bang form — as in (4.10), but effectively considering version 1 of the time-optimality problem with time-dependent constraints, so it does not correspond to the situation we have analyzed.

REMARK 4.3: We may consider the problem with a non-scalar control: $\boldsymbol{\varphi} = [\varphi_0, \varphi_1]$ so the boundary conditions in (1.1) are replaced by

$$(4.12) \quad u(\cdot, 0) = \varphi_1 \quad u(\cdot, 1) = \varphi_2$$

and in (1.3) we take $A(t) = \mathcal{K} \subset \mathbb{R}^2$ where \mathcal{K} is a closed, bounded, convex set. Here we may distinguish two forms of the ‘bang-bang’ property:

WEAK: *ae on $[0, T^*]$ one has $\varphi(t) \in \partial\mathcal{K}$*

STRONG: *ae on $[0, T^*]$ one has $\varphi(t)$ an extreme point of \mathcal{K} .*

The weak form is immediate from the previous arguments: if there were $\mathcal{E} \subset [0, T]$ with positive measure for which φ remained in the interior, then we could obtain a contradiction as in the proof of Theorem 2, perturbing only the component φ_0 as there. For the strong form one needs a modification of this to avoid the possibility that φ might remain interior to some face within $\partial\mathcal{K}$ so that one must consider perturbations with a linear restriction: $\tilde{\varphi}(t) = \hat{\varphi}(t)\mathbf{c}$ for some nonzero $\mathbf{c} \in \mathbb{R}^2$. What would be needed then is the appropriate modification of the inequality (4.6), obtainable along similar lines.

To illustrate the situation, consider first $\mathcal{K} = [0, 1] \times [0, 1]$. Any case where φ is weakly optimal but not strongly optimal can be reduced to the following: for a set E of positive measure in $[\tau, T]$ and some $\varepsilon > 0$, one has $\varphi_1(t) \in (\varepsilon, 1 - \varepsilon)$ with $\varphi_2(t) \in \{0, 1\}$ for all $t \in \mathcal{E}$. By selecting $\hat{\tau} > \tau$ but close, we can ensure that $\mathcal{E}' = E \cap [\tau, T]$ has positive measure and that the state $\omega'_{\hat{\tau}}$ produced at $\hat{\tau}$ by use of the modified controls $\varphi'_i(t) = \{\varphi_i(t)$ for $t < \tau$; $= 0$ for $t \in [\tau, \hat{\tau}]$ differs from the state $\omega_{\hat{\tau}}$ produced by the original $\varphi = (\varphi_1, \varphi_2)$ by less than ε/K^r , i.e., $\|\omega'_{\hat{\tau}} - \omega_{\hat{\tau}}\| < \varepsilon/K^r$. Consequently, by modifying φ'_1 by v supported on \mathcal{E}' and of supnorm $< \varepsilon$, we obtain — as in the proof of Theorem 2 — that $(\varphi'_1 + v, \varphi'_2)$ attains the same target ω_T as φ yet with τ replaced by the larger $\hat{\tau}$, contradicting the assumed optimality of τ . On the other hand, if we take $\mathcal{K} = \{(x, y) : x, y \geq 0, x + y \leq 1\}$, it is clear that such an argument is only available if one knows that pairs with $\varphi_1 + \varphi_2 = 0$ on \mathcal{E} are available as nullcontrols for the state perturbation. Such ‘odd’ control pairs only produce corresponding odd states and so can only compensate for odd state perturbations. Hence our argument cannot be expected to work in this setting, although we cannot on this basis conclude that the ‘bang-bang’ property fails.

Similar considerations apply if one would generalize (4.11) to

$$(4.13) \quad u_t = (pu_x)_x - qu + \sum_j \varphi_j(t)b_j$$

with pointwise constraints imposed on the vector control $\varphi = [\varphi_1, \dots, \varphi_J]$.

REMARK 4.4: There is little difficulty in generalizing the abstract Theorem 3 to treat state-dependent constraints. It is convenient to take a space $\mathfrak{X} = \{x(\cdot)\}$ of ‘controlled trajectories’, where the state trajectory is defined by $x(t) := \mathbf{C}_{0,t}\varphi$ for $t \in [0, T]$, $\varphi \in \mathcal{U}$; we assume the topology imposed on \mathfrak{X} is such that the linear map $\mathbf{X} : \varphi \mapsto x(\cdot) : \mathcal{U} \rightarrow \mathfrak{X}$ is continuous. By a ‘state-dependent constraint’ we mean a set-valued function

$$(4.14) \quad (t, x) \mapsto A(t, x) \subset \mathbb{R} \quad \text{for } t \in [0, T], x \in \mathfrak{X}$$

so the control restriction (1.3-ii) becomes

$$(4.15) \quad \varphi \in \mathcal{A} := \{\varphi \in \mathcal{U} : \varphi(t) \in A(t, \mathbf{X}\varphi) \text{ ae on } [0, T]\}.$$

We continue to take $\mathcal{U} = L^p(0, T)$ and to assume (1.4), now also writing $a(t) = a(t, x)$, $b(t) = b(t, x)$; we will further assume that one has uniform bounds: $a \leq a(t, x) \leq b(t, x) \leq b$ for all $x \in \mathfrak{X}$. Finally, we need a mild continuity condition⁴

$$(4.16) \quad \begin{aligned} &\varphi_n \rightharpoonup \bar{\varphi} \text{ (weak convergence in } \mathcal{U}) \text{ with (4.15) for each } n \\ &\text{implies:} \quad a(t, \mathbf{X}\bar{\varphi}) \leq \bar{\varphi}(t) \leq b(t, \mathbf{X}\bar{\varphi}) \text{ ae on } [0, T]. \end{aligned}$$

We may then argue much as in the proof of Theorem 3. If φ_n is an optimizing sequence for the time-optimality problem given by (4.15), we have $\varphi_n \rightharpoonup \bar{\varphi}$ — using our assumptions on $A(\cdot)$ and extracting a subsequence if necessary — so we may set $A_*(t) := [a(t, \mathbf{X}\bar{\varphi}), b(t, \mathbf{X}\bar{\varphi})]$ and have $\bar{\varphi}(t) \in A_*(t)$ ae on $[0, T]$. As in the proof of Theorem 3, we consider the time-optimality problem using A_* for \mathcal{A} to obtain a (unique) time-optimal control $\hat{\varphi}$. As there, $\hat{\varphi}$ has the ‘bang-bang’ property and so is also admissible for the original problem — whence the control times τ are the same and we can conclude that $\hat{\varphi} = \bar{\varphi}$ so that this is the unique time-optimal control for the original problem.

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⁴For example, it is not hard to see that (4.16) will hold if one can take \mathfrak{X} compact in $C([0, T] \rightarrow \mathcal{X})$ and if, with $A(t) = A(t, x(t))$ so $A : [0, T] \times \mathcal{X} \rightarrow 2^{\mathbb{R}}$, one has

$$r_n \rightarrow \bar{r}, z_n \rightarrow \bar{z}, r_n \in A(t, z_n) \quad \Rightarrow \quad a(t, \bar{z}) \leq \bar{r} \leq b(t, \bar{z}).$$

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