Some 'regional controllability' issues for the heat equation[1]

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ABSTRACT: We discuss evolution problems governed by parabolic systems when the target states to be reached are specified in a subregion. Approximate controllability and existence of a unique optimal control are proved; convergence is demonstrated for a computational procedure to approximate the optimal solution.

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1. Introduction

For controllability issues one normally considers a control system on a time interval $[0, T]$ and asks whether some particular target state ξ_d in the state space $\mathcal{X}$ is reachable (from initial state $\xi_0 = 0$) or whether these reachable targets form a dense set (so one can come arbitrarily close to an arbitrary target state: approximate controllability) or whether the trivial state 0 is reachable from every initial state $\xi_0 \in \mathcal{X}$ (nullcontrollability), etc.

For distributed parameter system theory, the term 'regional analysis' has been used to refer to control problems in which the target of interest is not fully specified as a state, but refers only to a smaller region $\hat{\Omega}$, a portion of the spatial domain Ω on which the governing partial differential equation is considered — i.e., one 'reaches' a target η_d if one reaches any state ξ on Ω whose restriction to $\hat{\Omega}$ is η_d. If, e.g., we would have $\mathcal{X} = L^2(\Omega)$ and $\mathcal{Y} = L^2(\hat{\Omega})$ and would denote by $\boldsymbol{\gamma} : \mathcal{X} \to \mathcal{Y}$ the restriction map, then 'reaching η_d' means that the terminal state $x(T)$ satisfies $\boldsymbol{\gamma}\, x(T) = \eta_d \in \mathcal{Y}$ or, equivalently, that the system reaches the (closed) set $\boldsymbol{\gamma}^{-1}(\eta_d)$: the component of $x(T)$ on $\Omega \setminus \hat{\Omega}$ is invisible.

Such problems of 'regional analysis' have been treated, e.g., in [8] and [26] for situations where the subregion $\hat{\Omega}$ of interest was interior to Ω. We note that these questions are particularly natural in a setting such as the wave equation where a limited propagation speed may make it obviously impossible (for some T) to affect all of Ω using a given control mechanism, and so suggest the plausibility of considering more local targets; our present concerns do not have the geometric character of that setting and we will, instead, take the controlled heat equation

$$u_t = \Delta u + \mathbf{B}\varphi \text{ on } \Omega \qquad \left(u_\nu\Big|_{\partial\Omega} = 0,\ u\Big|_{t=0} = u_0\right) \tag{1.1}$$

as our model.

It is also plausible in real problems that the target region of interest may be a portion of the domain boundary so the target η_d is specified only on a subset $\Gamma \subset \partial\Omega$, rather than on an actual subregion $\hat{\Omega}$. Technically, the distinguishing difficulty is that the relevant restriction map $\boldsymbol{\gamma}$ is then a trace map and cannot be expected to be continuous on $\mathcal{X}$. As a (typical) motivating example of this nature we note one already adduced in [8]:

> *Heat a parallelepipedal body Ω so as to reach a prescribed target temperature distribution on one of its faces Γ.*

This typically takes the form of considering (1.1) with the control $\varphi \in \boldsymbol{\mathcal{V}} = L^2([0,T] \to \mathcal{V})$ to be chosen so that

$$u(T,\cdot)\Big|_{\Gamma} \approx \eta_d = \text{ desired target } \in \mathcal{Y} := L^2(\Gamma). \tag{1.2}$$

The point of this paper is to consider, abstractly and in the context of (1.1), the relations between controllability questions for 'partial state' targets in $\mathcal{Y}$ and the corresponding questions for the original system with target states in $\mathcal{X}$. In the process we review various results already known for 'full state controllability' and we may take the survey of these results as a secondary point of the paper.

2. Formulation

Consider an autonomous linear control problem governed by the abstract system

$$\dot{x} = \mathbf{A}x + \mathbf{B}\varphi \qquad x(0) = \xi \tag{2.1}$$

with x taking values in the Banach space (state space) $\mathcal{X}$ and the control φ taking values in the Banach space $\mathcal{V}$ (so, nominally, $\mathbf{B} : \mathcal{V} \to \mathcal{X}$). Letting $\mathbf{S}(t)$ be the C_0 semigroup on $\mathcal{X}$ generated by $\mathbf{A}$, we then have for $x = x(\cdot;\xi,\varphi)$ the 'mild solution' representation:

$$x(t) = \mathbf{S}(t)\xi + \mathbf{L}_t\varphi \tag{2.2}$$

where we have set

$$\mathbf{L}_t\varphi := \int_0^t \mathbf{S}(t-s)\mathbf{B}\varphi(s)\,ds \tag{2.3}$$

for φ in the control space $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_p := L^p([0,T] \to \mathcal{V})$ and $t \in [0,T]$.

To correspond to 'regional control', we introduce a closed linear map $\boldsymbol{\gamma} : \mathcal{X} \supset \mathcal{D}(\boldsymbol{\gamma}) \to \mathcal{Y}$ for a suitable Banach space $\mathcal{Y}$, and wish to define $\hat{\mathbf{L}} : \varphi \mapsto \psi$ for $\varphi \in \boldsymbol{\mathcal{V}}$ by setting $\psi(t) = \hat{\mathbf{L}}_t\varphi := \boldsymbol{\gamma}\mathbf{L}_t\varphi$ for $t \in [0,T]$. We will assume throughout that the system satisfies the basic hypotheses:

$$\begin{array}{ll} (i) & \boldsymbol{\gamma}\mathbf{S}(t) : \mathcal{X} \to \mathcal{Y},\ \boldsymbol{\gamma}\mathbf{S}(t)\mathbf{B} : \mathcal{V} \to \mathcal{Y} \text{ are continuous for each } t > 0, \\ (ii) & \|\boldsymbol{\gamma}\mathbf{S}(t)\mathbf{B}\| \le \mu(t) \qquad \text{with } \mu(\cdot) \in L^q(0,T) \\ (iii) & \mathbf{S}(t) : \mathcal{X} \to \mathcal{X},\ \boldsymbol{\gamma}\mathbf{S}(t) : \mathcal{X} \to \mathcal{Y} \text{ have dense range for each } t > 0. \end{array} \tag{2.4}$$

where $q := p/(p-1)$ as usual in (ii) so $1/p + 1/q = 1$. Note that (2.4-i) just asserts that the range of $\mathbf{S}(t)$ (and also of $\mathbf{S}(t)\mathbf{B}$) lies in $\mathcal{D}(\boldsymbol{\gamma})$; by the semigroup property, if this holds for some $t = \delta \in (0, T)$, then it holds for $t > \delta$ with continuity in t for each $\boldsymbol{\gamma}\,\mathbf{S}(t)\xi$.

LEMMA 1: *If $\|\boldsymbol{\gamma}\,\mathbf{S}(\cdot)\mathbf{B}\| \in L^q(0,T)$ (i.e., (2.4-ii)), then $\hat{\mathbf{L}}_t = \boldsymbol{\gamma}\,\mathbf{L}_t : \boldsymbol{\mathcal{V}} \to \mathcal{Y}$ is continuous for each t in $[0,T]$ and $\hat{\mathbf{L}}$ is continuous from $\boldsymbol{\mathcal{V}}$ to $C([0,T] \to \mathcal{Y})$.*

PROOF: Apply $\boldsymbol{\gamma}$ to (2.3) to get $\psi(t) := \boldsymbol{\gamma}\,\mathbf{L}_t\varphi$; noting the hypotheses (2.4-i, ii), we have

$$\|\psi(t)\| = \left\| \int_0^t \boldsymbol{\gamma}\,\mathbf{S}(t-s)\mathbf{B}\varphi(s)\,ds \right\| \le \|\mu\|\,\|\varphi\|,$$

(using the L^q-norm for μ and the $\boldsymbol{\mathcal{V}}_p$-norm for φ) so $\hat{\mathbf{L}}_t$ is bounded for each t and $\hat{\mathbf{L}} : \boldsymbol{\mathcal{V}} \to L^\infty([0,T] \to \mathcal{Y})$ is bounded. If φ is continuous in t (so $\|\varphi(t+h) - \varphi(t)\| \le \varepsilon$ if $0 \le h \le \delta(\varepsilon)$ and $\|\varphi(s)\| \le M$), then

$$\begin{aligned} \|\psi(t+h) - \psi(t)\| \;&\le \left\| \int_0^h \boldsymbol{\gamma}\,\mathbf{S}(t+h-s)\mathbf{B}\varphi(s)\,ds \right\| \\ &\qquad + \left\| \int_0^t \boldsymbol{\gamma}\,\mathbf{S}(t-s)\mathbf{B}[\varphi(s+h) - \varphi(s)]\,ds \right\| \\ &\le \left| \int_0^h \mu(t+h-s)M\,ds \right| + \left| \int_0^t \mu(t-s)\varepsilon\,ds \right| \\ &\le [Mh^{1/p} + T^{1/p}\varepsilon]\|\mu\| \end{aligned}$$

for $h \le \delta(\varepsilon)$ so ψ is then also continuous. Since the continuous functions φ are dense in $\boldsymbol{\mathcal{V}}$, we actually have $\hat{\mathbf{L}} : \boldsymbol{\mathcal{V}} \to C([0,T] \to \mathcal{Y})$, as desired. ■

In particular, this shows that $\mathcal{R}(\mathbf{L}_t) \subset \mathcal{D}(\boldsymbol{\gamma})$. Note that the continuity in t is needed at $t = T$ for evaluation:

$$\eta = \eta(\varphi) : \boldsymbol{\gamma}\,x(T) = \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \hat{\mathbf{L}}_T\varphi \qquad \text{with } \hat{\mathbf{L}}_T = \boldsymbol{\gamma}\,\mathbf{L}_T \tag{2.5}$$

to define η meaningfully, enabling us to focus attention on the 'restricted terminal state' $\eta = \boldsymbol{\gamma}\,x(T) \in \mathcal{Y}$. Our controllability issues then involve the *reachable sets*:

$$\begin{aligned} \mathcal{K}_{\mathcal{X}}(\xi;T) \;&:= \{x(T;\xi,\varphi) = \mathbf{S}(T)\xi + \mathbf{L}_T\varphi : \varphi \in \boldsymbol{\mathcal{V}}\} \subset \mathcal{X}, \\ \mathcal{K}_{\mathcal{Y}}(\xi;T) \;&:= \{\eta = \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \hat{\mathbf{L}}_T\varphi : \varphi \in \boldsymbol{\mathcal{V}}\} \subset \mathcal{Y}; \end{aligned} \tag{2.6}$$

or simply $\mathcal{K}_{\mathcal{X}}(T)$ — even $\mathcal{K}_{\mathcal{X}}$ if T is clear — for $\mathcal{K}_{\mathcal{X}}(0,T)$; similarly for $\mathcal{K}_{\mathcal{Y}}$.

3. Verification of the basic hypotheses

We are concerned to verify the hypotheses (2.4), for some concrete control situations. To relate this abstract formulation to the heat equation (1.1), which we are taking as a model system, we begin by noting some key facts relating to the smoothing properties of (1.1). We will consider only the Hilbert space setting $\mathcal{X} := L^2(\Omega)$, etc. In the notation of Section 2, the operator $\mathbf{A}$ is the Laplacian Δ,[2] specified as a self-adjoint operator on $\mathcal{X}$ with domain $\mathcal{D}(\mathbf{A}) = \{u \in H^2(\Omega) : u_\nu = 0\}$. It is convenient to view the situation in terms of the spectral expansion of $\mathbf{A}$: letting $(e_\kappa, -\lambda_\kappa)$ be the eigenpairs so

$$-\Delta e_\kappa = \lambda_k e_\kappa \text{ on } \Omega; \qquad \partial e_\kappa/\partial\nu = 0 \text{ on } \partial\Omega. \tag{3.1}$$

Note that $\lambda_\kappa > 0$ (except that $\lambda_0 = 0$ for the eigenfunction $e_0 = \mathit{const.}$) with $\lambda_\kappa \to \infty$ and that we can take $\{e_\kappa\}$ to be an orthonormal basis of $\mathcal{X}$. The analytic semigroup $\mathbf{S}(\cdot)$ generated by $\mathbf{A}$ is then given by

$$\mathbf{S}(t) : u = \sum_\kappa \alpha_\kappa e_\kappa \longmapsto \mathbf{S}(t)u = \sum_\kappa e^{-\lambda_\kappa t}\alpha_\kappa e_\kappa. \tag{3.2}$$

It is easy to use this expansion also to define the fractional powers $[-\mathbf{A}]^{s/2}$ with $\mathcal{D}([-\mathbf{A}]^{s/2})$ consisting of functions $u = \sum_\kappa \alpha_\kappa e_\kappa$ such that[3]

$$\|u\|_{(s)} := \left[\sum_\kappa \max\{1, \lambda_\kappa\}^s |\alpha_\kappa|^2\right]^{1/2} < \infty. \tag{3.3}$$

From [12] or [10] we know that this norm is equivalent to the $H^s(\Omega)$-norm and that $\mathcal{D}([-\mathbf{A}]^{s/2}) = H^s_*(\Omega)$ where the subscript$_*$ on H^s indicates the imposition of suitable boundary conditions — with $H^s_*(\Omega) = H^s(\Omega)$ for $s < 3/2$ in the case of Neumann boundary conditions. It is a simple computation, maximizing over all $\lambda \geq 0$, to see that

$$\lambda^\sigma e^{-\lambda t} \leq \begin{cases} 1 & \text{if } \sigma \leq 0 \text{ (indeed, if } \sigma \leq e) \\ (\sigma/e)^\sigma t^{-\sigma} & \text{if } \sigma \geq 0 \end{cases}$$

[2]In the domain specification, u_ν denotes the (outward) normal derivative $\partial u/\partial\nu$ at the boundary $\partial\Omega$. Our considerations, as in (3.4), (3.5) below, can be generalized from $\mathbf{A} = \Delta$ to $\mathbf{A} : u \mapsto \nabla \cdot p\nabla u + qu$ for suitable coefficient functions p, q and to other homogeneous boundary conditions. With some minor modifications one may also include more general first-order terms, so $\mathbf{A}$ may no longer be self-adjoint.

[3]The use of $\max\{1, \lambda_\kappa\}$ rather than $|\lambda_\kappa|$ in (3.3) is simply a 'correction' to allow for $\lambda_0 = 0$, e.g., as in (3.6) below when $k = \ell = 0$.

for $t > 0$ and then (3.2), (3.3) immediately give

$$\text{(3.4)} \qquad \|\mathbf{S}(t)u\|_{(s)} \le Mt^{-\sigma}\|u\|_{(r)} \qquad \text{so } \|\mathbf{S}(t)\|_{H^r_*(\Omega)\to H^s_*(\Omega)} \le Mt^{-\sigma}$$

for $0 \le \sigma := (s-r)/2$.

This does not quite complete the story as to smoothing: we must also note that (3.4) *localizes*. If $\tilde\Omega_1$, $\tilde\Omega_2 \subset \Omega$ are separated (disjoint closures), then for u with support in $\tilde\Omega_1$ we may replace (3.4) by

$$\text{(3.5)} \qquad \|\mathbf{S}(t)u\|_{(s),\tilde\Omega_2} \le \mu_*(t)\|u\|_{(r),\tilde\Omega_1}$$

where $\mu_*(\cdot)$, depending on $r, s, \tilde\Omega_1, \tilde\Omega_2$, is smooth and vanishes to high order as $t \to 0+$; here $\|\cdot\|_{(s),\tilde\Omega}$ denotes the H^s-norm on $\tilde\Omega$ (arbitrary s). [In this setting, we will actually have $\mathbf{S}(t)u$ real-analytic on $\tilde\Omega_2$ for $t > 0$, independent of the global regularity of u since u is analytic (i.e., 0) on a neighborhood of any point in $\tilde\Omega_2$.]

For our examples we will focus specifically on $\mathbf{A} = \Delta$ with Neumann boundary conditions for the unit square $\Omega = \Omega_* := (0,1)\times(0,1)$ with $\mathcal{X} = L^2(\Omega_*)$, etc. For future reference we note that in this setting the Laplace operator is separable and we have an orthonormal basis of product eigenfunctions

$$\text{(3.6)} \qquad \begin{aligned} e_\kappa(x,y) &= \hat e_k(x)\hat e_\ell(y) \text{ for } \kappa = (k,\ell) \\ \text{with} \quad \hat e_k(x) &= \begin{cases} 1 & \text{for } k = 0 \\ \sqrt{2}\cos k\pi x & \text{for } k = 1,2,\ldots; \end{cases} \\ \lambda_\kappa &= (k^2+\ell^2)\,\pi^2. \end{aligned}$$

There are three particularly interesting cases for $\boldsymbol\gamma$: we may have $\hat\Omega$ an interior region (open in Ω), we may have $\hat\Omega = \Gamma$ a boundary region (open in $\partial\Omega$), or, finally, we may have $\hat\Omega$ a finite set $\{p_n : n = 1,\ldots,N\} \subset \overline{\Omega}$ ('sentinels', cf., e.g., [15], [16]). For the first of these cases, we take[4] $\mathcal{Y} = L^2(\hat\Omega)$ so the restriction map $\boldsymbol\gamma : \mathcal{X} \to \mathcal{Y}$ is bounded. For targets on the boundary, we are taking $\mathcal{Y} = L^2(\Gamma)$ so $\boldsymbol\gamma$ is a trace map; it is well-known (cf., e.g., [1]) that such a (Dirichlet) trace map is continuous from $H^s(\Omega)$ to $\mathcal{Y} = L^2(\Gamma)$ for any $s > 1/2$. For the third case, $\mathcal{Y} = \mathbb{R}^N$ and $\boldsymbol\gamma$ corresponds to point evaluation; it is well-known (again cf., e.g., [1]) that $H^s(\Omega) \supset C(\overline{\Omega})$, making point evaluation continuous, for any $s >$ dim/2 — i.e., for $s > 1$

[4]Somewhat more generally, we might take $\mathcal{Y} = H^s(\hat\Omega)$ for suitable s.

in our 2-dim examples, but requiring larger s if we were to consider higher-dimensional examples.

Note that we have just associated a choice $(0, 1/2+, 1+)$ of the parameter s with each of our choices for $\hat{\Omega}$ so as to make $\boldsymbol{\gamma} : H^s(\Omega) \to \mathcal{Y}$ continuous. We will similarly associate a choice of the parameter r with each of our choices of the 'control mechanism' determined by $\mathbf{B}$ and $\mathcal{V}$ by asking that $\mathbf{B} : \mathcal{V} \to H^r_*(\Omega)$ should be continuous. We then would have, using (3.4),

$$\begin{aligned} \|\boldsymbol{\gamma}\,\mathbf{S}(t)\mathbf{B}\|_{\mathcal{V}\to\mathcal{Y}} &\leq \|\boldsymbol{\gamma}\|_{H^s\to\mathcal{Y}}\|\mathbf{S}(t)\|_{H^r\to H^s}\|\mathbf{B}\|_{\mathcal{V}\to H^r} \\ &\leq Mt^{-(s-r)/2} =: \mu(t). \end{aligned} \tag{3.7}$$

For applicability of Lemma 1, assuming we are taking $p = 2$ to have a Hilbert space setting and so also have $q = 2$, this would mean having $s - r < 1$ in each of our examples to satisfy (2.4-ii): $\mu \in L^2(0, T)$.

EXAMPLE 1: [interior (patch) control] Here $\hat{\Omega}$ is taken to be an open[5] set in Ω with $\mathcal{V} = L^2(\tilde{\Omega})$ and $\mathbf{B}$ the usual embedding as 'extension by 0'. Then $\mathbf{B} : \mathcal{V} \to H^r_*(\Omega)$ is continuous for $r \leq 0$ and we have $s - r < 1$ when we consider $\hat{\Omega}$ open in Ω $(s = 0)$ or $\hat{\Omega} = \Gamma \subset \partial\Omega$ $(s = \frac{1}{2}+)$, but not when we consider sentinels (point evaluations, so $s = 1+$) unless the set of sentinels has a neighborhood $\tilde{\Omega}_*$ separated from $\tilde{\Omega}$ (disjoint closures in $\overline{\Omega}$) so we can apply (3.5) instead of (3.4) — or unless we make somewhat different choices for $\mathcal{V}$, p.

EXAMPLE 2: Next we consider $\mathcal{V} = \ell^2$ and, for a specified sequence of functions $\{b_j\}$, set $[\mathbf{B}\varphi](\cdot) = \sum_j \varphi_j b_j(\cdot)$. The result depends on having $\{b_j\} \subset H^r_*(\Omega)$ for some r with

$$\left\| \sum_j \alpha_j b_j \right\|_{(r)} \leq M\|\boldsymbol{\alpha}\|_{\ell^2} \qquad \boldsymbol{\alpha} = (\alpha_j) \tag{3.8}$$

and we note that either

(a) $\sum_j \|b_j\|^2_{H^r} < \infty$ or

(b) $\{b_j\}$ a Riesz basis for its range in $H^r_*(\Omega)$, e.g., orthonormal.

[5] More generally, one might take $\hat{\Omega}$ to be a subset of positive measure in Ω, but nothing is known at present about controllability for that setting.

would be sufficient for this.

Our results would be as in Example 1 if $r = 0$ in (3.8) and for $r > 0$ we could even include the case of sentinels. Alternatively, we could allow point source controls by taking $r < -1$, which would need applicability of (3.5) — i.e., separation of the control supports (or, at least, their singular supports) from the target region $\hat{\Omega}$ — even to consider the case of $\hat{\Omega}$ open in Ω, but we then get our other cases for $\boldsymbol{\gamma}$ as well.

EXAMPLE 3: [boundary control] This is, of course, a most plausible setting for physical implementation, since it requires access to Ω only at its boundary. E.g., for Neumann control we would be replacing (1.1) by

$$u_t = \Delta u \text{ on } \Omega \qquad u_\nu\Big|_{\partial\Omega} = \mathbf{B}\varphi,\ u\Big|_{t=0} = u_0. \tag{3.9}$$

To consider this, however, we must modify (2.4) since the definition (2.3) is then replaced in the 'mild solution' representation (2.2) by[6]

$$\mathbf{L}_t\varphi := \int_0^t [-\mathbf{A}]\mathbf{S}(t-s)\mathbf{G}\mathbf{B}\varphi(s)\,ds \tag{3.10}$$

where we have introduced the *Green's operator* $\mathbf{G}$ associated with $\mathbf{A}$ and the involvement of nonhomogeneous boundary conditions. For (3.9), $\mathbf{G} : \psi \mapsto v$ is given by the Laplace equation:

$$\Delta v = 0 \text{ on } \Omega \qquad v_\nu\Big|_{\partial\Omega} = \psi.$$

For this setting, — asking, e.g., that the boundary controls are in $L^2(\tilde{\Gamma})$ with $\tilde{\Gamma} \subset \partial\Omega$ — we note that, for any $r < 3/2$, this map $\mathbf{G}$ is continuous to $H^r_*(\Omega)$ from $L^2(\tilde{\Gamma})$ (or even from $H^{r'}(\tilde{\Gamma})$ with $r' = r - 3/2 < 0$; cf., e.g., [17])).

The effect of this modification is that the condition (2.4-*ii*) should be replaced here by

$$(ii') \quad \|\boldsymbol{\gamma}\,\mathbf{A}\mathbf{S}(t)\mathbf{B}\|_{\mathcal{V}\to\mathcal{Y}} \le \mu(t) \qquad \text{with } \mu(\cdot) \in L^q(0,T) \tag{3.11}$$

[6]To see this, let $v := \mathbf{G}\mathbf{B}\varphi$ and note that $(u-v)_t = \mathbf{A}u - v_t = \mathbf{A}[u-v] - v_t$ so, as in (2.2), we have

$$\begin{aligned} [u-v](t) &= \mathbf{S}(t)[u-v](0) - \int_0^t \mathbf{S}(t-s)v_t(s)\,ds \\ &= \mathbf{S}(t)u_0 - v(t) - \int_0^t \frac{d\,\mathbf{S}(t-s)}{ds} v(s)\,ds \end{aligned}$$

with $\mathbf{S}' = \mathbf{A}\mathbf{S}$ so we get a new version of (2.2) with $\mathbf{L}_t$ as in (3.10); cf., [25] or [4].

so (3.7) would be replaced by a bound on $\|\boldsymbol{\gamma}\,\mathbf{AS}(t)\mathbf{B}\|$:

$$\|\boldsymbol{\gamma}\,\mathbf{AS}(t)\mathbf{B}\| \le Mt^{-\sigma} =: \mu(t) \quad \text{with } \sigma := 1 + (s-r)/2 \text{ for } r < 3/2,$$

assuming continuity of $\mathbf{B} : \mathcal{V} \to L^2(\tilde{\Gamma})$ and using (3.4); for $p = 2$, this gives (3.11) provided $s < 1/2$ and we can handle interior targets — but this just barely fails for regional control with targets at the boundary, which we have noted requires $s > 1/2$ and certainly fails for consideration of sentinels. On the other hand, if $\tilde{\Gamma}$ has a neighborhood disjoint from the region $\hat{\Omega}$, we can again apply (3.5) instead of (3.4) to handle the difficulty. We can also consider control functions as in Example 2, except taking the $\{b_j\}$ now to be functions on $\tilde{\Gamma} \subset \partial\Omega$ so we are considering boundary control. The merging of these considerations should be clear.

4. Controllability issues

Returning to the abstract system of Section 2, we will refer to controllability issues for the *original* system (2.1) with target states in $\mathcal{X}$ — or, more properly, (2.2) for $t = T$ — as '**O**-issues' and to such issues for the *regional* or *restricted* system with targets in $\mathcal{Y}$ as '**R**-issues' , referring respectively to the *reachable sets* of (2.6).

Thus, '**O**-nullcontrollability' (for time T) means that: $\forall \xi \in \mathcal{X}$ one has $0 \in \mathcal{K}_{\mathcal{X}}(\xi; T)$ (so there is some $\varphi \in \mathcal{V}$ such that $x(T; \xi, \varphi) = 0$; from (2.2) this is equivalent to the range inclusion: $\mathcal{R}(\mathbf{S}(T)) \subset \mathcal{R}(\mathbf{L}_T))$ in $\mathcal{X}$) while '**R**-nullcontrollability' means that $\forall \xi \in \mathcal{X}$ one has $0 \in \mathcal{K}_{\mathcal{Y}}(\xi; T)$ or, equivalently in $\mathcal{X}$, that one has $\mathcal{K}_{\mathcal{X}}(\xi; T) \bigcap \mathcal{N} \neq \emptyset$ where $\mathcal{N}$ is the nullspace $\mathcal{N}(\boldsymbol{\gamma})$ (equivalent to the range inclusion: $\mathcal{R}(\boldsymbol{\gamma}\,\mathbf{S}(T)) \subset \mathcal{R}(\hat{\mathbf{L}}_T))$ in $\mathcal{Y}$). In this case, '**O**-nullcontrollability' obviously implies '**R**-nullcontrollability' since we may use the same control, noting that $x(T) = 0$ certainly implies that $\eta = \boldsymbol{\gamma}\,x(T) = 0$. Note that the trace operator $\boldsymbol{\gamma}$ is not injective, so this argument would not be reversible and we should not expect, in general, the converse of this implication. On the other hand, a 'unique continuation' property — essentially that the (nontrivial) nullspace of $\boldsymbol{\gamma}$ intersects trivially with the **O**-reachable set — would provide the injectivity where needed and suffices to give this converse; note Example 4.

We now mention a 'strong nullcontrollability' property: that the controlled trajectory can be made to rest at 0 for an interval (in t). It is clear that **O**-nullcontrollability implies strong **O**-nullcontrollability, since one can extend the nullcontrol φ as 0 beyond $t = T$. On the other hand, η is only a partial state and if the 'regional nullcontrol' φ gives $0 \neq x(T) \in \mathcal{N} = \mathcal{N}(\boldsymbol{\gamma})$ we would have $\eta = 0$, but it is far from clear that there need be any extension of φ which would keep $x(\cdot) \in \mathcal{N}$ to give $y(\cdot) \equiv 0$ for the 'regional trajectory' $y(t) = \boldsymbol{\gamma}\, x(t)$ for $t > T$: without **O**-nullcontrollability, we cannot expect that **R**-nullcontrollability would necessarily imply strong **R**-nullcontrollability.

We now show that, as is already well-known for **O**-nullcontrollability, the **R**-nullcontrollability property implies existence of a continuous 'control operator' $\mathbf{C} : \mathcal{X} \to \boldsymbol{\mathcal{V}}$.

THEOREM 2: *Suppose, for the system described above, $\boldsymbol{\gamma}\,\mathbf{S}$, $\boldsymbol{\gamma}\,\mathbf{SB}$ are continuous as in (2.4-i), $\hat{\mathbf{L}}$ is continuous: $\boldsymbol{\mathcal{V}} \to C([0,T] \to \mathcal{Y})$ (i.e., the conclusion of Lemma 1) and one has **R**-nullcontrollability (i.e., $0 \in \mathcal{K}_{\mathcal{Y}}(\xi;T)$ for each $\xi \in \mathcal{X}$). It follows that:*

(a) There is a continuous map $\mathbf{C} : \mathcal{X} \to \boldsymbol{\mathcal{V}}$ such that $\varphi = \mathbf{C}\xi$ is a (regional) nullcontrol from the initial state $\xi \in \mathcal{X}$, i.e., such that $\hat{\mathbf{L}}_T\mathbf{C}\xi = -\boldsymbol{\gamma}\,\mathbf{S}(T)\xi$; if $\boldsymbol{\mathcal{V}}$ is a Hilbert space, then $\mathbf{C}$ can also be taken as linear.

(b) [We suppose, for simplicity, that $\mathcal{X}$ is a Hilbert space with $\mathbf{A}$ linear (corresponding to homogeneous BC) self-adjoint and that $\boldsymbol{\mathcal{V}}^* = L^q(\mathcal{V}^*)$.] There is some M such that, for all solutions v of $v_t = \mathbf{A}v$ with $\zeta := v(0)$ in the range of $\boldsymbol{\gamma}^*$, the terminal state $v(T) \in \mathcal{X}$ is uniquely determinable from observation of $\mathbf{B}^*v \in \boldsymbol{\mathcal{V}}^*$ (without specific knowledge of $\zeta \in \mathcal{R}(\boldsymbol{\gamma}^*)$) with a Lipschitz estimate

$$\|v(T)\| \leq M\|\mathbf{B}^*v\|. \tag{4.1}$$

(c) The reachable set $\mathcal{K}_{\mathcal{Y}}(\xi;T')$ is independent of ξ and of T' for $T' \geq T$.

PROOF: We begin by letting $\mathcal{N}$ be the nullspace of $\hat{\mathbf{L}}_T$, necessarily a closed subspace of $\boldsymbol{\mathcal{V}}$ by Lemma 1, and then letting $\tilde{\boldsymbol{\mathcal{V}}}$ be the quotient space $\boldsymbol{\mathcal{V}}/\mathcal{N}$. Next, note that $\hat{\mathbf{L}}_T$ induces a linear operator $\tilde{\mathbf{L}} : \tilde{\boldsymbol{\mathcal{V}}} \longmapsto \mathcal{Y}$ with $\tilde{\mathbf{L}}[\varphi] = \hat{\mathbf{L}}_T\varphi$ where $[\varphi]$ is the coset of φ in $\tilde{\boldsymbol{\mathcal{V}}} = \boldsymbol{\mathcal{V}}/\mathcal{N}$ (equivalently, $\hat{\mathbf{L}}_T\varphi = \tilde{\mathbf{L}}[\varphi]$ for each φ in any coset $[\varphi]$); one easily sees that $\tilde{\mathbf{L}}$ is injective and, from the definition of the quotient space norm, that $\|\tilde{\mathbf{L}}\| = \|\hat{\mathbf{L}}_T\|$. Then nullcontrollability means solvability (for each given $\xi \in \mathcal{X}$) of the equation: $\eta := \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \hat{\mathbf{L}}_T\varphi = 0$ for $\varphi \in \boldsymbol{\mathcal{V}}$ and so solvability of $\boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \tilde{\mathbf{L}}[\varphi] = 0$ for $[\varphi] \in \tilde{\boldsymbol{\mathcal{V}}}$ (taking the coset $[\varphi]$ of any original solution $\varphi \in \boldsymbol{\mathcal{V}}$) — and we note that this solution $[\varphi]$ is unique by the injectivity of $\tilde{\mathbf{L}}$. This defines a map $\tilde{\mathbf{C}} : \xi \mapsto [\varphi]$ and one easily verifies that $\tilde{\mathbf{C}} : \mathcal{X} \to \tilde{\boldsymbol{\mathcal{V}}}$ is linear. Note that the graph of the operator $\tilde{\mathbf{C}}$ is just the nullspace of the continuous linear map

$$\mathcal{X} \times \tilde{\boldsymbol{\mathcal{V}}} \longrightarrow \mathcal{Y} : (\xi, [\varphi]) \longmapsto \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \tilde{\mathbf{L}}[\varphi]$$

and so is closed in $\mathcal{X} \times \tilde{\boldsymbol{\mathcal{V}}}$. By the Closed Graph Theorem it then follows that $\tilde{\mathbf{C}}$ is continuous.

When $\boldsymbol{\mathcal{V}}$ is a Hilbert space one can identify $\tilde{\boldsymbol{\mathcal{V}}}$ with the orthogonal complement of $\mathcal{N}$, i.e., the subspace $\mathcal{N}^\perp \subset \boldsymbol{\mathcal{V}}$; we are then thinking of $\tilde{\mathbf{C}}$ as a linear operator $\mathbf{C} : \mathcal{X} \to \boldsymbol{\mathcal{V}}$ with range in $\mathcal{N}^\perp$. It is interesting to note that this just corresponds to use of the minimum norm nullcontrol in each case, selecting from the coset $\tilde{\mathbf{C}}\xi \in \tilde{\boldsymbol{\mathcal{V}}}$ the element having minimum $\boldsymbol{\mathcal{V}}$-norm:

$$\mathbf{C}\xi := \operatorname{argmin}\{\|\varphi\| : \varphi \in \tilde{\mathbf{C}}\xi \text{ so } \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \hat{\mathbf{L}}_T\varphi = 0\}. \tag{4.2}$$

For the general case, (4.2) need not be well-defined, will not be linear, and need not be continuous in ξ. The closest we can come to this seems to be an appeal to the Michael Selection Theorem [21], which promises existence of a continuous selection — i.e., a right inverse $\boldsymbol{\sigma} : \tilde{\boldsymbol{\mathcal{V}}} = \boldsymbol{\mathcal{V}}/\mathcal{N} \to \boldsymbol{\mathcal{V}}$ of the canonical projection: $\boldsymbol{\mathcal{V}} \to \tilde{\boldsymbol{\mathcal{V}}}$ so $\boldsymbol{\sigma}[\varphi] \in [\varphi]$ for each coset $[\varphi] \in \tilde{\boldsymbol{\mathcal{V}}}$) — which is nonlinear, but continuous, of linear growth, and *almost* norm minimizing. We then set $\mathbf{C} := \boldsymbol{\sigma} \circ \tilde{\mathbf{C}} : \mathcal{X} \to \boldsymbol{\mathcal{V}}$ and the proof of (a) is complete.

The equivalence of $\mathbf{O}$-nullcontrollability to an estimate of the form (4.1) is fairly standard: what is new here is the restriction of $v(0)$ to $\mathcal{R}(\boldsymbol{\gamma}^*)$ in the case of $\mathbf{R}$-nullcontrollability. To see (4.1) — with $M = \|\tilde{\mathbf{C}}\|$ as above — we note that there is $\xi \in \mathcal{X}^* = \mathcal{X}$ such that $\|\xi\| = 1$ and $\|v(0)\| = \langle v(0), \xi\rangle$ and then that there is some nullcontrol $\varphi \in \boldsymbol{\mathcal{V}}$ (so $\varphi \in \tilde{\mathbf{C}}\xi$) such that $\|\varphi\|$ is arbitrarily close to $\|\tilde{\mathbf{C}}\xi\| \le \|\tilde{\mathbf{C}}\| =: M$; let $x(\cdot)$ be the corresponding controlled solution and set $y(t) := x(T-t)$, so $-\dot{y} = \mathbf{A}y + \mathbf{B}\varphi(T-t)$ and $y(T) = x(0) = \xi$, $\boldsymbol{\gamma}\,y(0) = 0$. Since $\boldsymbol{\gamma}\,x(T) = 0$ and $\zeta = \boldsymbol{\gamma}^*\eta$ for some η, we have

$$\langle y, v\rangle\Big|_{t=0} = \langle x(T), \zeta\rangle = \langle x(T), \boldsymbol{\gamma}^*\eta\rangle = \langle \boldsymbol{\gamma}\,x(T), \eta\rangle = 0$$

whence

$$\|v(T)\| = \langle \xi, v(T)\rangle = \langle y, v\rangle\Big|_0^T = \int_0^T \langle y, v\rangle^{\boldsymbol{\cdot}}\,dt = -\int_0^T \langle \mathbf{B}\varphi(T-t), v(t)\rangle\,dt,$$

giving (4.1) since $\|\varphi\| \le \|\tilde{\mathbf{C}}\| =: M$.

For (c) we follow [24]. Suppose the target η_d is in $\mathcal{K}_{\mathcal{Y}}(\xi_1; T)$, i.e., there is some $\varphi_1 \in \boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\gamma}\,\mathbf{S}(T)\xi_1 + \hat{\mathbf{L}}_T\varphi_1 = \eta_d$. For any other initial state ξ_2, setting $\varphi_2 := \varphi_1 + \mathbf{C}[\xi_2 - \xi_1]$ gives

$$\begin{aligned}
&\boldsymbol{\gamma}\,\mathbf{S}(T)\xi_2 + \hat{\mathbf{L}}_T\varphi_2 \\
&\quad = \Big[\boldsymbol{\gamma}\,\mathbf{S}(T)\xi_1 + \hat{\mathbf{L}}_T\varphi_1\Big] + \boldsymbol{\gamma}\,\mathbf{S}(T)[\xi_2 - \xi_1] + \hat{\mathbf{L}}_T[\varphi_2 - \varphi_1] \\
&\quad = \eta_d + 0 = \eta_d
\end{aligned}$$

so $\mathcal{K}_{\mathcal{Y}}(\xi_1;T) \subset \mathcal{K}_{\mathcal{Y}}(\xi_2;T)$. By symmetry we have equality and we may simply write $\mathcal{K}_{\mathcal{Y}} = \mathcal{K}_{\mathcal{Y}}(T)$ for the reachable set.

Finally, suppose $T' > T$ (so $T' - T = \delta > 0$). If we have $\eta_d \in \mathcal{K}_{\mathcal{Y}}(T) = \mathcal{K}_{\mathcal{Y}}(0;T) = \mathcal{R}(\hat{\mathbf{L}}_T)$, i.e., $\eta_d = \hat{\mathbf{L}}_T\varphi_1$ for some $\varphi_1 \in \boldsymbol{\mathcal{V}}$, we may let

$$\varphi_2(t) := \begin{cases} 0 & \text{for } 0 \le t < \delta \\ \varphi_1(t-\delta) & \text{for } \delta \le t \le T' \end{cases}$$

and easily see from the definition — $\hat{\mathbf{L}}_t = \boldsymbol{\gamma}\,\mathbf{L}_t$ with (2.3) — that $\hat{\mathbf{L}}_{T'}\varphi_2 = \hat{\mathbf{L}}_T\varphi_1 = \eta_d$. Thus, $\mathcal{R}(\hat{\mathbf{L}}_T) \subset \mathcal{R}(\hat{\mathbf{L}}_{T'})$. Since $\mathbf{S}(T') = \mathbf{S}(T)\mathbf{S}(\delta)$ we have $\mathcal{R}(\boldsymbol{\gamma}\,\mathbf{S}(T')) \subset \mathcal{R}(\boldsymbol{\gamma}\,\mathbf{S}(T)) \subset \mathcal{R}(\hat{\mathbf{L}}_T)$ and this gives $\mathbf{R}$-nullcontrollability for time T'. On the other hand, if $\eta_d \in \mathcal{R}(\hat{\mathbf{L}}_{T'}) = \mathcal{K}_{\mathcal{Y}}(T')$, then $\eta_d = \hat{\mathbf{L}}_{T'}\varphi_2$ for some $\varphi_2 \in \boldsymbol{\mathcal{V}}$ and we can set $\xi_* := x(\delta;0,\varphi_2)$ and $\varphi_*(\cdot) := \varphi_2(\cdot+\delta)$ on $[0,T]$. One easily sees, using the autonomy of the system, that $\eta_d = \hat{\mathbf{L}}_{T'}\varphi_2 = \boldsymbol{\gamma}\,\mathbf{S}(T)\xi_* + \hat{\mathbf{L}}_T\varphi_* \in \mathcal{K}_Y(\xi_*;T) = \mathcal{K}_{\mathcal{Y}}(T)$. We have shown that $\mathcal{K}_{\mathcal{Y}}(T') = \mathcal{K}_{\mathcal{Y}}(T)$. ■

It is worth noting that our argument for (b) was essentially an interpretation of the adjoint $\tilde{\mathbf{C}}^*$ so it also provides the converse:

(4.1) *implies* $\mathbf{R}$*-nullcontrollability.*

We note here that, much as in our argument for (a) above, there is always a unique 'minimum-norm control' to any reachable target when $\boldsymbol{\mathcal{V}}$ is a Hilbert space. Indeed, we note for future reference that if

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_p \text{ with } 1 < p < \infty \text{ and } V \text{ uniformly convex,} \tag{4.3}$$

then $\boldsymbol{\mathcal{V}}$ is uniformly convex and so has the 'Efimov–Stečkin property'

$$[\varphi_j \rightharpoonup \bar{\varphi} \text{ plus } \|\varphi_j\| \to \|\bar{\varphi}\|\,] \quad \Rightarrow \quad \varphi_j \to \bar{\varphi}. \tag{4.4}$$

Since the coset $\tilde{\mathbf{C}}\xi := \{\varphi \in \boldsymbol{\mathcal{V}} : \boldsymbol{\gamma}\,\mathbf{S}(T)\xi + \hat{\mathbf{L}}_T\varphi = 0\}$ is a convex, closed (hence weakly closed) set, it then necessarily contains a unique element φ^* of minimum norm.

Similarly to our discussion of nullcontrollability, '**O**-approximate controllability' means that one can reach arbitrarily near to every target state, i.e.,

$$\forall \xi_d \in \mathcal{X}, \forall \varepsilon > 0 : \quad \exists \varphi \in \boldsymbol{\mathcal{V}} \ni \quad \|\xi_d - x(T;\xi,\varphi)\| \le \varepsilon, \tag{4.5}$$

i.e., $\mathcal{K}_{\mathcal{X}}(\xi;T)$ is dense in $\mathcal{X}$; correspondingly, '**R**-approximate controllability' means that $\mathcal{K}_{\mathcal{Y}}(\xi;T)$ is dense in $\mathcal{Y}$. These notions are independent of the initial state ξ since we may compensate for replacement of ξ_1 by ξ_2 in (4.5) by considering the target state $\xi_d + \mathbf{S}(T)[\xi_2 - \xi_1]$ instead of ξ_d. The range densities (in $\mathcal{X}$ for $\mathcal{R}(\mathbf{S}(\cdot))$ and in $\mathcal{Y}$ for $\mathcal{R}(\boldsymbol{\gamma}\,\mathbf{S}(\cdot))$) assumed in (2.4-*iii*) are obviously necessary for these even to be possibilities. Trivially, subject to (2.4), **O**-nullcontrollability implies **O**-approximate controllability, since we already know that **O**-nullcontrollability means the dense set $\mathcal{R}(\mathbf{S}(\cdot))$ is reachable. Since in (2.4-*iii*) we have also assumed that $\mathcal{R}(\boldsymbol{\gamma}\,\mathbf{S})$ is dense in $\mathcal{Y}$, we know that, similarly, **R**-nullcontrollability implies **R**-approximate controllability. Finally, we note the following.

LEMMA 3: *Assume the basic set of hypotheses (2.4). Then* **O**-*approximate controllability for some T' implies* **R**-*approximate controllability for any time $T > T'$.*

PROOF: Taking $\delta := T - T' > 0$, by (2.4-*iii*) we may approximate any target η_d arbitrarily closely by some $\eta_* = \boldsymbol{\gamma}\,\mathbf{S}(\delta)\xi_*$. The **O**-approximate controllability means that we may then approximate ξ_* arbitrarily closely in $\mathcal{X}$ using some control φ_* on $[0,T']$. Setting $\varphi(t) := \{\varphi_*(t) \text{ for } 0 \le t \le T';\ 0 \text{ for } t > T'\}$, we note from (2.2) and the semigroup property that $x(T;\xi,\varphi) = \mathbf{S}(\delta)x(T';\xi,\varphi_*)$ so (2.5) with (2.4-*i*) gives $\eta = \boldsymbol{\gamma}\,\mathbf{S}(\delta)x(T';\xi,\varphi_*) \approx \boldsymbol{\gamma}\,\mathbf{S}(\delta)\xi_* \approx \eta_d$. ■

5. Some examples

We consider several examples in which, as for Section 3, we are considering control for the heat equation in the square $\Omega_* = (0,1) \times (0,1)$ with regional targets.

For Example 1 (patch control) it has comparatively recently become known (cf., e.g., [14], [11]) that one always has **O**-nullcontrollability for (1.1) in this case. By our earlier discussion, this implies both **R**-nullcontrollability and **R**-approximate controllability for all the cases we are considering here —

in particular, the cases of open $\hat{\Omega} \subset \Omega$ and boundary targets on $\hat{\Omega} = \Gamma \subset \partial\Omega$, provided the control set $\tilde{\Omega}$ is separated from Γ (perhaps if we restrict our attention to a smaller control patch).

The cited results of [14], [11] do not depend for validity on the choice of $\Omega = \Omega_*$, and provide a clever argument[7] to show that one also obtains **O**-nullcontrollability from a boundary patch $\tilde{\Gamma}$ as in Example 3 (where, as noted there, one necessarily assumes separation of $\tilde{\Gamma}$ from $\hat{\Omega}$).

If, e.g., we were to take $\hat{\Omega} = \Gamma$ to be a side of $\partial\Omega_*$ and take the heat flux at the side $\tilde{\Gamma}$ opposite to Γ as a boundary control, this would somewhat resemble the *inverse heat conduction problem* (IHCP; cf., e.g., [5], [9]) in which one is trying to infer that heat flux (otherwise unknown, so treated as a control) from observation at the accessible face Γ. The significant distinction is that for the IHCP the observation is taken over the full time interval $t \in [0, T]$ and one wishes to determine the input flux uniquely, whereas in our present problem the observation/specification of the target is to be made only at the terminal time $t = T$ and we seek only to find *some* suitable control, without expectation of uniqueness — unless with the aid of such an auxiliary selection criterion as minimization of the control norm. Uniqueness for the IHCP is thus more closely related to the notion of strong nullcontrollability.

EXAMPLE 4: We do note that for boundary control (e.g., with a set $\{b_j\}$ of boundary control functions comparable to Example 2, so we do not already know **O**-nullcontrollability from [14], [11] as above) and regional control for an open patch $\hat{\Omega} \subset \Omega$ we have a unique continuation property: the real analyticity of solutions of the heat equation, mentioned earlier, implies that a solution which vanishes on $\hat{\Omega}$ then necessarily vanishes on all of Ω for this setting. Thus, **R**-nullcontrollability, which means vanishing at $t = T$ on $\hat{\Omega}$, implies **O**-nullcontrollability — and so also strong **O**-nullcontrollability and strong **R**-nullcontrollability.

EXAMPLE 5: This is a counterexample for the converse of Lemma 3: one has **R**-nullcontrollability here for all $T > 0$, but does not have **O**-approximate controllability at all.

[7]Given $\tilde{\Gamma} \subset \partial\Omega_*$, adjoin to Ω_* a small 'bulge' attached at $\tilde{\Gamma}$ to get $\Omega_{**} \supset \Omega_*$. Taking an interior patch $\tilde{\Omega}_{**}$ within this extra piece, we know there is a nullcontrol for Ω_{**} supported in $\tilde{\Omega}_{**}$. The corresponding solution takes *some* (smooth) data for the flux on $\tilde{\Gamma}$ and that may be taken as the Neumann data (boundary nullcontrol!) for the problem on Ω_*.

Consider (1.1) with $\mathcal{V} = L^2(0,1)$ and, for a fixed function $b(\cdot)$, set

$$[\mathbf{B}\varphi](x,y) := b(x)\varphi(y) \qquad \text{for } (x,y) \in \Omega_*,\ \varphi \in \boldsymbol{\mathcal{V}}$$

so we have

$$u_t = \Delta u + b(x)\varphi(t,y) \qquad u_\nu = 0 \text{ at } \partial\Omega_*. \tag{5.1}$$

The particular results to be obtained depend on the choice of $b(\cdot)$ and it is convenient to relate this to the corresponding 1-dimensional problem with scalar control:

$$w_t = w_{xx} + b(x)\psi(t) \qquad w_x = 0 \text{ at } x = 0,1. \tag{5.2}$$

We begin with the observation that we may consider solutions w_j of (5.2) with $\psi = \psi_j$ for $j = 0, \dots$ and then u satisfies (5.1) with

$$u(t,x,y) := \sum_j e^{-\lambda_j t} w_j(t,x) e_j(y), \quad \varphi(t,y) := \sum_j e^{-\lambda_j t}\psi_j(t) e_j(y). \tag{5.3}$$

It will also be convenient to introduce the solution $z = z(t,x)$ of

$$z_t = z_{xx} \qquad z_x\Big|_{x=0,1} = 0 \quad z\Big|_{t=0} = b, \tag{5.4}$$

noting that

$$\hat{w}(t,x) = \int_0^t z(t-s,x)\psi(s)\,ds \tag{5.5}$$

is then the solution of (5.2) with 0 initial data.

For targets restricted to $\Gamma = \Gamma_{\hat{x}} := \{(\hat{x},y) : y \in (0,1)\} \subset \Omega_*$, i.e., taking $\boldsymbol{\gamma}$ to be the trace to some choice of the segment $\Gamma_{\hat{x}}$, choosing $\hat{x} \in (0,1)$, we note that (5.3) makes $\eta(y) := u(T,\hat{x},y)$ vanish just when each ψ_j controls so each $w_j(T,\hat{x}) = 0$. Letting ω_j be the value at $(T,\hat{x})$ of the solution of $w_t = w_{xx}$ with initial data $w_j(0,\cdot) = \int u(0,\cdot,y)e_j(y)\,dy$, we then have $w_j(T,\hat{x}) = 0$ precisely if

$$\int_0^T \zeta(s)\psi_j(s)\,ds = -\omega_j \tag{5.6}$$

where $\zeta(t) := z(T-t,\hat{x})$. Note that the smoothing properties of the homogeneous heat equation ensure that $|\omega_j| \le K\|w_j(0,\cdot)\|$ for some fixed K.

Setting $\varphi_j(t) := e^{-\lambda_j t}\psi_j(t)$, we have $\|\varphi\|^2 = \sum_j \|\varphi_j\|^2$. Rewriting (5.6) as

$$\int_0^T \left[\zeta(s)e^{\lambda_j s}\right]\varphi_j(s)\,ds = -\omega_j,$$

we see that we can solve this with minimal $\|\varphi_j\|$, assuming that $\zeta \not\equiv 0$ on $(0,T)$, by taking

$$
\begin{array}{rrl}
 & \varphi_j(s) & := \dfrac{-\omega_j}{\|\zeta_j\|^2}\zeta_j(s) \\
(5.7) \qquad \text{with} & \zeta_j(s) & := \zeta(s)e^{\lambda_j s} = e^{\lambda_j s} z(T-s,\hat{x}) \\
\text{so} & \|\varphi_j\| & := \dfrac{|\omega_j|}{\|\zeta_j\|} \le \dfrac{K\|w_j(0,\cdot)\|}{\left[\int \zeta^2(s)e^{2\lambda_j s}\,ds\right]^{1/2}}.
\end{array}
$$

It is clear that $0 \ne \|\zeta_j\| \to \infty$ as $j = 0,1,\ldots \to \infty$ so $\|\varphi_j\| \le M\|w_j(0,\cdot)\|$ and $\|\varphi\| \le M\|u(0,\cdot,\cdot)\|$. This shows that (5.1) is **R**-nullcontrollable (so one has, *a fortiori*, **R**-approximate controllability) for essentially arbitrary choices of $b(\cdot)$ and $\hat{x}$ — subject only to the minimal assumption that $z(\cdot,\hat{x}) \not\equiv 0$.

On the other hand, we may observe the obvious fact that one does *not* have nullcontrollability or even approximate controllability for (5.1) if one takes, e.g., $b = \hat{e}_\kappa$ for any[8] fixed κ; in fact, the **O**-reachable set $\mathcal{K}_\mathcal{X}$ is then necessarily contained in the subspace

$$\{u(x,y) = \hat{e}_\kappa(x)g(y) : g \in L^2(0,1)\} \subset \boldsymbol{\mathcal{X}} = L^2(\Omega_*)$$

— and only such initial states can be controlled to 0. Thus, one does not have **O**-approximate controllability at all for such a choice of b.

While the principal point of this Example was to provide a counterexample for the converse of Lemma 3, we include a bit more information as to when (i.e., with a different choice of b) one *does* have **O**-nullcontrollability. Suppose one has nullcontrollability for the corresponding 1-dimensional equation (5.2). Given initial data $u_0 = u_0(x,y)$ for (5.1), consider (5.2) with initial data $\omega_j = \omega_j(x) := \int u_0(x,y)e_j(y)\,dy$ and take each ψ_j to be a corresponding nullcontrol — with a choice of ψ_j giving $\|\psi_j\| \le M\|\omega_j\|$ as is possible by Theorem 2: say $\psi_j = \mathbf{C}\omega_j$ with $\mathbf{C} = \mathbf{C}_{(5.2)}$ and $M = \|\mathbf{C}\|$. For u_0 in $L^2(\Omega_*)$, this gives $\|u_0\|^2 = \sum_j \|\omega_j\|^2$ and, much as earlier, $\|\varphi\| \le M\|u_0\| < \infty$. Thus, this φ is in $\boldsymbol{\mathcal{V}} = L^2([0,T] \to L^2[0,1]) = L^2([0,T] \times [0,1])$, so admissible, and is obviously a nullcontrol in (5.1) on setting $t = T$ in (5.3). This shows that nullcontrollability for (5.2) implies nullcontrollability for (5.1) with the same $b(\cdot)$; the converse here is immediate.

[8] Here $\hat{e}_\kappa$ is as given in (3.6) and one asks, of course, that $\hat{x}$ should not be one of the zeroes of $\cos \kappa\pi x$, while noting that in this case $z(t,x) = e^{-\lambda_\kappa t}\hat{e}_\kappa(x)$.

As in Theorem 2, we note the equivalence of this nullcontrollability for (5.2) and so for (5.1) to an estimate

$$\sum_j \omega_j^2 \leq M^2 \int_0^T \left[\sum_j \left(\beta_j e^{\lambda_j T} \omega_j \right) e^{-\lambda_j (T-t)} \right]^2 dt$$

corresponding to $\|v(0)\| \leq M \|\langle v, b \rangle\|$ for solutions of $-v_t = v_{xx}$. We remark without proof that one will have this if and only if none of the expansion coefficients $\{\beta_j := \langle b, e_j \rangle : j = 0, \ldots\}$ vanish and, for some $\varepsilon > 0$, one has[9]

$$|\beta_j| \geq \varepsilon e^{-\lambda_j T}. \tag{5.8}$$

6. Some additional results

We comment next on variational approximation in which we penalize residual errors, rather than directly fixing the terminal state as an imposed constraint.

THEOREM 4: *Consider a control system as described above for which $\boldsymbol{\mathcal{V}}$ is as in (4.3) and we have (2.4-i) and the conclusion of Lemma 1 — e.g., (2.4-ii) — or, for Example 3, (3.11). Fixing the initial ξ, write $\eta(\varphi)$ as in (2.5). Given a reachable target $\eta_d \in \mathcal{Y}$, we construct an approximating control sequence $\{\varphi_j\}$ either by*

$$\begin{array}{ll} \mathcal{J}(\varphi_j, \lambda_j) \leq \overline{\mathcal{J}}(\lambda_j) + \varepsilon_j & \quad \textit{where} \\ \mathcal{J}(\varphi, \lambda) := \|\varphi\|^2 + \lambda^2 \|\eta(\varphi) - \eta_d\|^2 & \quad \overline{\mathcal{J}}(\lambda) := \inf_\varphi \{\mathcal{J}(\varphi, \lambda\} \end{array} \tag{6.1}$$

with $\lambda_j \to \infty$ and $\varepsilon_j \to 0$ or, alternatively, by

$$\begin{array}{ll} \varphi_j \in \mathcal{S}(\varepsilon_j), \quad \|f_j\| \leq \sigma(\varepsilon_j) + \varepsilon_j & \quad \textit{where} \\ \mathcal{S}(\varepsilon) := \{\varphi \in \boldsymbol{\mathcal{V}} : \|\eta(\varphi) - \eta_d\| \leq \varepsilon\} & \quad \sigma(\varepsilon) := \inf\{\|\varphi\| : \varphi \in \mathcal{S}(\varepsilon)\} \end{array} \tag{6.2}$$

with $\varepsilon_j \to 0$.
For either method of approximation, we then have $\varphi_j \to \varphi^*$ where φ^* is the

[9] The sufficiency of this condition follows from a variant of the Müntz-Szász Theorem (cf., e.g., [18], [6], [2]), giving $\|(c_j)\|_{\ell^2} \leq M \| \sum_j c_j e^{-\lambda_j \cdot} \|_{L^2[0,T]}$ since $\lambda_j \sim j^2$. It is amusing to note, following [7], [23], that for $b(x) = \delta(x - \bar{x})$ (so $\beta_j \sim \cos j\pi\bar{x}$) the condition (5.8) prohibits $\bar{x}$ which are rational or are too rapidly approximable by rationals — a set of measure 0, but uncountable in every subinterval.

(unique) minimum norm control attaining η_d.

This is much more a result about variational approximation than about control theory and we proceed by way of an abstract lemma.

LEMMA 5: *Let $\mathbb{V}$ be a reflexive Banach space and let $\Gamma \subset \mathbb{V} \times \mathbb{R}_+$ satisfy*

$$(6.3)\qquad \begin{array}{ll} (i) & \text{for } \varepsilon > 0 \text{ there exists } (\omega, r) \in \Gamma \text{ with } r < \varepsilon, \\ (ii) & (\omega_j, r_j) \in \Gamma, \omega_j \rightharpoonup \bar{\omega}, r_j \to 0 \quad \Rightarrow \quad (\bar{\omega}, 0) \in \Gamma. \end{array}$$

(a) There is some ω^* minimizing $\|\omega\|$ in $\mathcal{S}_0 := \{\omega \in \mathbb{V} : (\omega, 0) \in \Gamma\} \neq \emptyset$.

Now, with $\varepsilon_j \to 0$, construct a sequence $(\omega_j, r_j) \in \Gamma$ such that

$$(6.4)\quad \|\omega_j\| \leq \nu_j + \varepsilon + j, r_j \leq \varepsilon_j \qquad [\nu_j := \inf\{\|\omega\| : (\omega, r) \in \Gamma, r \leq \varepsilon_j\}]$$

or, almost equivalently, taking $\lambda_j \to \infty$, and requiring

$$(6.5)\qquad \begin{array}{ll} F(\|\omega_j\|, \lambda_j r_j) & \leq \overline{\mathcal{J}}(\lambda_j) + \varepsilon_j \\ & [\overline{\mathcal{J}}(\lambda) := \inf\{F(\|\omega\|, \lambda r) : (\omega, r) \in \Gamma\}] \end{array}$$

where the function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is lsc, continuous, coercive, and nondecreasing in each argument with $F(\cdot, 0)$ strictly increasing.

(b) There is then a subsequence with $\omega_j \rightharpoonup \omega^*$ where ω^* is as in (a); if (a) characterizes ω^* uniquely, this is convergence of the full sequence. Further, we have $\|\omega_j\| \to \nu^* := \|\omega^*\|$ so, if $\mathbb{V}$ has the Efimov–Stečkin property (4.4), this is strong convergence: $\omega_j \to \omega^*$.

Proof: By (i) there is a sequence satisfying the hypotheses of (ii), so $\mathcal{S}_0$ is nonempty. Any minimizing sequence for $\|\omega\|$ over $\mathcal{S}_0$, must be bounded so there is a weakly convergent subsequence $\omega_j \to \omega^*$ and (ii) again applies to show $\omega^* \in \mathcal{S}_0$. As the norm is lsc for the weak topology, we have (a).

Using (a) we note that in (6.4) we have each $\nu_j \leq \nu_*$ and, of course, $\{\nu_j\}$ nondecreasing. The sequence $\{\omega_j\}$ is thus bounded and contains a subsequence: $\omega_j \rightharpoonup \bar{\omega}$ weakly convergent in the reflexive space $\mathbb{V}$ with

$$\|\bar{\omega}\| \leq \liminf \|\omega_j\| \leq \liminf \nu_j \leq \nu^*.$$

Now (ii) applies to give $\bar{\omega} \in \mathcal{S}_0$ so we may use $\bar{\omega}$ as ω^* in (a); with $\|\bar{\omega}\| \leq \nu^*$; necessarily, $\|\bar{\omega}\| \geq \nu^*$ so $\|\omega_j\| \to \|\bar{\omega}\| = \nu^*$ and we have (b).

Alternatively, given (6.5) we note that $\overline{\mathcal{J}}(\lambda_j) \leq F(\|\omega^*\|, 0)$ for each j so $\{F(\|\omega_j\|, \lambda_j r_j)\}$ is bounded. Hence, by coercivity, $\{\omega_j\}$ is bounded and $r_j \to 0$. As above, we can extract a subsequence $\omega_j \rightharpoonup \bar{\omega}$ and (ii) applies to give $\bar{\omega} \in \mathcal{S}_0$. We have

$$\begin{aligned} F(\|\bar{\omega}\|, 0) &\leq F(\liminf \|\omega_j\|, 0) \leq \liminf F(\|\omega_j\|, 0) \\ &\leq \liminf F(\|\omega_j\|, \lambda_j r_j) \leq \liminf \overline{\mathcal{J}}(\lambda_j) \\ &\leq F(\|\omega^*\|, 0) \end{aligned}$$

so, as $F(\cdot, 0)$ is strictly increasing, $\|\bar{\omega}\| \leq \|\omega^*\|$ (necessarily '=') and again we have (b). ■

Proof (of Theorem 4): We will take $\Gamma := \{(\varphi, r) \in \boldsymbol{\mathcal{V}} \times \mathbb{R}_+ : \|\eta(\varphi) - \eta_d\| \leq r\}$ and can then apply Lemma 5, noting that (6.3-i) is just approximate reachability of the target and that $\mathcal{S}_0$ is just the set of exact controls to η_d. The assumption of (4.3) ensures that $\boldsymbol{\mathcal{V}}$ is reflexive and (4.4) holds; we have already noted that Lemma 5(a) holds in the present context with uniqueness of the minimum norm control in cS_0, provided that η_d is (exactly) reachable as assumed here. Later we take $F(s, r) := s^2 + r^2$, which certainly satisfies the conditions imposed with (6.5).

We need only show (6.3-ii). As $\hat{\mathbf{L}}_T$ is continuous, its graph is closed in $\boldsymbol{\mathcal{V}} \times \mathcal{Y}$ (hence, by convexity, still closed using the weak topology of $\boldsymbol{\mathcal{V}}$ for the product). Given $\varphi_j \rightharpoonup \bar{\varphi}$ and $\hat{\mathbf{L}}_T \varphi_j \to [\eta_d - \boldsymbol{\gamma}\,\mathbf{S}(T)\xi]$ (since $r_j \to 0$), we then have $(\bar{\varphi}, \eta_d - \boldsymbol{\gamma}\,\mathbf{S}(T)\xi)$ in the graph so $\hat{\mathbf{L}}_T \bar{\varphi} = \eta_d - \boldsymbol{\gamma}\,\mathbf{S}(T)\xi$, i.e., $\eta(\bar{\varphi}) = \eta_d$ and $(\bar{\varphi}, 0) \in \Gamma$. The hypotheses of Lemma 5 are thus satisfied and the conclusions of Theorem 4 follow. ■

Finally, we turn to consideration of **R**-approximate controllability for certain quasilinear control systems in which (2.1) is perturbed by a nonlinearity:

$$\dot{x} = \mathbf{A}x + f(x) + \mathbf{B}\varphi \qquad x(0) = \xi. \tag{6.6}$$

As an **O**-issue this has been the subject of a considerable body of activity, for which we mention, e.g., [22], [13], [27], [28]; we may note that [27] addresses, in some sense, the **R**-issue with $\boldsymbol{\gamma}$ a projection to a finite-dimensional space rather than a trace operator as here. That the relevant results for **R**-approximate controllability and for **O**-approximate controllability are not immediately comparable will be clear from Example 6, below.

Much as for (2.2), (2.3), we use the 'mild solution' representation

$$x(t) = \mathbf{S}(t)\xi + \int_0^t \mathbf{S}(t-s) f(x(s))\, ds + \mathbf{L}_t\varphi, \tag{6.7}$$

which is here a Volterra integral equation for $x(\cdot)$. Adding to our hypotheses, for example, that f is Lipschitzian ($\|f(x_1)-f(x_2)\| \leq L\|x_1-x_2\|$), the standard Picard iteration argument shows that (6.7) will have a unique solution for each $\xi \in \mathcal{X}$, $\varphi \in \boldsymbol{\mathcal{V}}$ with

$$\varphi \mapsto x_f(\cdot;\xi,\varphi) : \boldsymbol{\mathcal{V}} \to \mathcal{C}([0,T] \to \mathcal{X}) \text{ continuous;} \tag{6.8}$$

it will also be convenient to write $\mathbf{F} : \varphi \mapsto f(x(\cdot)) = f(x_f(\cdot;\xi,\varphi))$. If we strengthen (2.4-*ii*) slightly to ask also that $\|\boldsymbol{\gamma}\,\mathbf{S}(t)\| \leq \mu(t)$, then

$$\begin{aligned} \varphi \mapsto \eta_f(\varphi) :=\ & \boldsymbol{\gamma}\, x_f(T;\xi,\varphi) : \boldsymbol{\mathcal{V}} \to \mathcal{Y} \text{ continuous} \\ =\ & \eta_0(\varphi) + \tilde{\mathbf{L}}_T\mathbf{F}(\varphi) \qquad \text{with} \\ \tilde{\mathbf{L}}_t :\ & g \longmapsto \textstyle\int_0^t \boldsymbol{\gamma}\,\mathbf{S}(t-s)g(s)\, ds. \end{aligned} \tag{6.9}$$

[Note that η_0, defined for $f \equiv 0$, is just $\eta(\cdot)$ as given in (2.5) and that (6.9) gives $\tilde{\mathbf{L}}_T = \eta_f - \eta_0$.]

We are now ready to extend to the context of **R**-approximate controllability a result from [22]. The hypotheses on $\mathbf{S}(\cdot)$ and $\boldsymbol{\gamma}\,\mathbf{S}(\cdot)$ here can easily be verified for Examples 1,2,3, above.

THEOREM 6: *Assume, in addition to (2.4-ii), that* $\mathbf{S}(\tau)$ *is a compact operator for small* $\tau > 0$. *Let* f *be a Lipschitzian nonlinear perturbation, giving (6.9) for (6.6) and assume* f *is bounded:* $\|f(\xi)\| \leq \beta$, *uniformly for* $\xi \in \mathcal{X}$. *Construct an approximating control sequence* $\{\varphi_j\}$ *as in Theorem 4, either by (6.1) or (6.2) — replacing the use of* $\eta(\varphi)$ *there by* $\eta_f(\varphi)$ *as in (6.9).*

(a) If the linear problem (2.1) is **R**-*approximately controllable, then the quasilinear control problem (6.6) is also* **R**-*approximately controllable.*

(b) If the target $\eta_d \in \mathcal{Y}$ *is reachable, then (for either method of approximation) one has* $\varphi_j \to \varphi^*$ *for a subsequence, where* φ^* *is a minimum norm control attaining* η_d.

PROOF: Suppose we fix (any) $\xi \in \mathcal{X}$ and $\eta_d \in \mathcal{Y}$. Then, for (a), given any $\varepsilon > 0$ we must show existence of $\bar{\varphi} \in \boldsymbol{\mathcal{V}}$ giving $\|\eta_f(\bar{\varphi}) - \eta_d\| \leq \varepsilon$ — which we write in the equivalent form $\|\eta_0(\bar{\varphi}) - [\eta_d - \boldsymbol{\gamma}\,\tilde{\mathbf{L}}_T\mathbf{F}(\bar{\varphi})]\| \leq \varepsilon$.

Note that $\mathbf{F} : \mathcal{V} \to \mathcal{G}_\beta = \{g : \|g(s)\| \le \beta \,\text{a.e.}\}$ and we observe that $\mathcal{R} := \eta_d - \tilde{\mathbf{L}}_T \mathcal{G}_\beta$ is convex and totally bounded in $\mathcal{Y}$. It is sufficient, for the latter, to show that, for any $\hat{\varepsilon} > 0$, $\mathcal{R}$ is within $\hat{\varepsilon}$ of some precompact set. To see this, choose $\delta > 0$ so $\beta \int_0^\delta \mu(\tau)\,d\tau < \hat{\varepsilon}$ and then, for $g \in \mathcal{G}_\beta$, set

$$g_\delta(s) := \{g(s) \text{ for } 0 < s < T - \delta;\ 0 \text{ for } s > T - \delta\}$$

and note that

$$\|\tilde{\mathbf{L}}_T g - \tilde{\mathbf{L}}_T g_\delta\| = \left\| \int_{T-\delta}^{T} \boldsymbol{\gamma}\,\mathbf{S}(T-s)g(s)\,ds \right\| \le \beta \int_0^\delta \mu(\tau)\,d\tau < \hat{\varepsilon}.$$

Thus, $\overline{\mathcal{R}}$ is within $\hat{\varepsilon}$ of the set

$$\eta_d - \boldsymbol{\gamma}\,\mathbf{S}(\delta)\{\int_0^{T-\delta} \mathbf{S}(T-\delta-s)g(s)\,ds : g \in \mathcal{G}_\beta\},$$

which is precompact — as $\mathbf{S}(\tau)$ is compact making $\boldsymbol{\gamma}\,\mathbf{S}(\delta)$ a compact operator for $\boldsymbol{\gamma}\,\mathbf{S}(\tau - \delta)$ continuous as in (2.4-ii).

Since $\mathcal{R}$ is totally bounded and (2.1) is $\mathbf{R}$-approximately controllable (so $\eta_0(\cdot)$ has dense range) there is a finite set $\{\bar{\eta}_j = \eta_0(\varphi_j) : j = 1, \ldots, J\}$ in the range of η_0 such that:

$$\forall \eta \in \overline{\mathcal{R}} \ \ \exists j \ni \quad \|\eta - \bar{\eta}_j\| < \varepsilon. \tag{6.10}$$

We may then define a 'partition of unity' for $\eta \in \mathcal{R}$ by

$$\psi_j(\eta) := \max\{0, \varepsilon - \|\eta - \bar{\eta}_j\|\}, \qquad \hat{\psi}_j(\eta) := \frac{\psi_j(\eta)}{\sum_k \psi_k(\eta)}$$

and use this to define

$$\Phi(\eta) := \sum_j \hat{\psi}_j(\eta)\varphi_j.$$

Since $\sum_j \hat{\psi}_j \equiv 1$ on $\overline{\mathcal{R}}$ with $\hat{\psi}_j(\eta) \ne 0$ only when $\|\eta - \eta_0(\varphi_j)\| \le \varepsilon$, we have

$$\|\eta_0(\Phi(\eta)) - \eta\| \le \varepsilon \qquad (\eta \in \overline{\mathcal{R}}). \tag{6.11}$$

The map $E : \eta \mapsto \varphi = \Phi(\eta) \mapsto [\eta_d - \tilde{\mathbf{L}}_T \mathbf{F}(\varphi)]$ is well-defined for $\eta \in \overline{\mathcal{R}}$ and, by our hypotheses, continuously maps: $\overline{\mathcal{R}} \to \mathcal{R}$. By the Schauder Theorem, there is then a fixpoint $\bar{\eta} = E(\bar{\eta}) \in \mathcal{R}$ and we let $\bar{\varphi} := \Phi(\bar{\eta})$ so $\bar{\eta} = \eta_d - \tilde{\mathbf{L}}_T \mathbf{F}(\bar{\varphi}) = [\eta_d - \eta_f(\bar{\varphi})] + \eta_0(\bar{\varphi})$. Then $\|\eta_f(\bar{\varphi}) - \eta_d\| = \|\eta_0(\bar{\varphi}) - \bar{\eta}\| \le \varepsilon$, by (6.11), so $\bar{\varphi}$ is the desired control of (6.6) to within ε of η_d from ξ. This completes the proof of (a).

To show (b), we proceed as for Theorem 4: we need only show (6.3-*ii*) for $\Gamma := \{(\varphi, r) \in \boldsymbol{\mathcal{V}} \times \mathbb{R}_+ : \|\eta_f(\varphi) - \eta_d\| \leq r\}$, to apply Lemma 5. Here, this amounts to showing that $\varphi_j \rightharpoonup \bar{\varphi}$ with $\eta_f(\varphi_j) \to \eta_d$ implies that $\eta_f(\bar{\varphi}) = \eta_d$ and we actually show that $\eta_f(\cdot)$ is continuous from the weak topology of $\boldsymbol{\mathcal{V}}$.

We note, first, that essentially the same argument used above to show compactness of $\overline{\mathcal{R}}$ shows precompactness of $\{x_f(t;\xi,\varphi)\}$ for φ bounded in $\boldsymbol{\mathcal{V}}$, uniformly for $0 < \tau \leq t \leq T$. Since we also have equicontinuity of such $\{x_f(\cdot;\xi,\varphi)\}$, it follows that we can extract a subsequence converging, uniformly on each $[\tau, T]$, to some $\bar{x} \in C((0,T] \to \mathcal{X})$. This also, of course, gives pointwise convergence on $(0,T]$ of $\mathbf{F}(\varphi_j)$ to $f(\bar{x})$ and using that in (6.7) shows that $\bar{x} = x_f(\cdot;\xi,\bar{\varphi})$ by the uniqueness of the solution. Since $\boldsymbol{\gamma}$ is closed so η_f is well-defined, this gives

$$\eta_f(\bar{\varphi}) = \boldsymbol{\gamma}\,\bar{x}(T) = \boldsymbol{\gamma}\,\lim x_f(T;) = \lim \eta_f(\varphi_j)$$

as desired. ■

EXAMPLE 6:

A somewhat artificial but easy example, related to Example 5, shows that in the context of **R**-approximate controllability one cannot generally permit perturbations of linear growth as, e.g., in [27], [28]. The assumption in Theorem 6 that f is uniformly bounded seems rather strong, but is apparently unavoidable without requiring deeper information regarding the linear control problem (2.1) than may conveniently be available here. For example, in the context of **O**-approximate controllability, the other results of [22] require more detailed knowledge about the growth rate of $\|\varphi\|$ needed to approximate to within ε, the results of Khapalov [13] require knowledge of the growth rate for nullcontrol as $T \to 0$, and the results of Zuazua [27], [28] require uniform control results for the family $\{\mathbf{A} \hookleftarrow [\mathbf{A} + a\cdot] : a(\cdot) \text{ bounded}\}$ to permit linear growth of f.

For this example we take our original problem to be the 1-dimensional heat equation (5.2) with vanishing Neumann data and the perturbation to be $f(u) = a(x)u$, not just of linear growth, but actually linear.

Note, first, that we can choose $a(x)$ so the operator: $u \mapsto u_{xx} + au$ has an eigenfunction b which is *not* one of the eigenfunctions e_k of $u \mapsto$

u_{xx}: we use this pair a, b to specify the unperturbed control problem (5.2) and the perturbation $f(u) = a(x)u$. As will always be possible, we then fix k, ℓ so β_k, $\beta_\ell \neq 0$ and take γ_k, $\gamma_\ell \neq 0$ so $\beta_k\gamma_k + \beta_\ell\gamma_\ell = 0$; finally, set $c(x) = \gamma_k e_k(x) + \gamma_\ell e_\ell(x)$ and $\boldsymbol{\gamma} : u \mapsto \langle c, u\rangle$. For any φ the perturbed problem $u_t = u_{xx} + au + b\varphi$ (say, with initial data $u\big|_{t=0} = 0$) always has the form $u(t,x) = \omega(t)b(x)$, giving $\eta_f(\varphi) \equiv 0$, certainly a complete failure of **R**-approximate controllability. On the other hand, for the unperturbed problem (5.2) with the same $b, \boldsymbol{\gamma}$, we get

$$\eta_0(\varphi) = \beta_k\gamma_k \int_0^T e^{-k^2\pi^2(T-s)}\varphi(s)\,ds + \beta_\ell\gamma_\ell \int_0^T e^{-\ell^2\pi^2(T-s)}\varphi(s)\,ds$$

and it is easy to choose $\varphi(\cdot)$ to give this any arbitrary value $\eta \in \mathbb{R} =: \mathcal{Y}$. Thus the unperturbed problem is actually **R**-completely controllable in this example while the range of the perturbed problem is trivial.

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