

IS 709/809: Computational Methods in IS Research

Math Review: Algorithm Analysis

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Topics

- Proof techniques
 - Proof by induction
 - Proof by counterexample
 - Proof by contradiction
- Recursion
- Summary

Proof Techniques

- What do we want to prove?
 - Properties of a data structure always hold for all operations
 - Algorithm running time/memory usage will never exceed some limit
 - Algorithm will always be correct
 - Algorithm will always terminate

Proof by Induction

- Goal: Prove some hypothesis is true
- Three-step process
 - Prove the Base case:
 - Show hypothesis is true for some initial conditions
 - This step is almost always trivial
 - **Inductive hypothesis:** Assume hypothesis is true for all values $\leq k$
 - Using the inductive hypothesis, show that the theorem is true for the next value, typically $k + 1$

Induction Example

- Prove arithmetic series $\sum_{i=1}^N i = \frac{N(N+1)}{2}$

- (Step 1) Base case: Show true for $N=1$

$$\sum_{i=1}^1 i = 1 = \frac{1(1+1)}{2}$$

Induction Example (cont'd)

- (Step 2) Assume true for $N=k$
- (Step 3) Show true for $N=k+1$

$$\begin{aligned}\sum_{i=1}^{k+1} i &= (k+1) + \sum_{i=1}^k i \\ &= (k+1) + \frac{k(k+1)}{2} \\ &= \frac{2(k+1) + k(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

More Induction Examples

- Prove the geometric series $\sum_{i=0}^N A^i = \frac{A^{N+1} - 1}{A - 1}$
- Prove that the number of nodes N in a complete binary tree of depth D is $2^{D+1} - 1$
- Prove that $\sum_{i=1}^N i^2 = \frac{N(N+1)(2N+1)}{6}$

Proof by Counterexample

- Prove hypothesis is not true by giving an example that doesn't work
 - Example: $2^N > N^2$?
 - Example: Prove or disprove “all prime numbers are odd numbers”
- Proof by example?
- Proof by lots of examples?
- Proof by all possible examples?
 - Empirical proof
 - Hard when input size and contents can vary arbitrarily

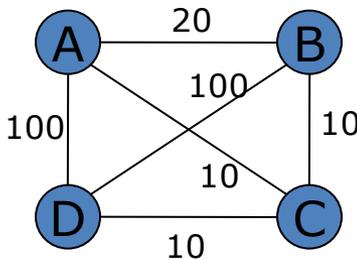
Another Example

■ Traveling salesman problem

- Given N cities and costs for traveling between each pair of cities, find the least-cost tour to visit *each* city *exactly once*

■ Hypothesis

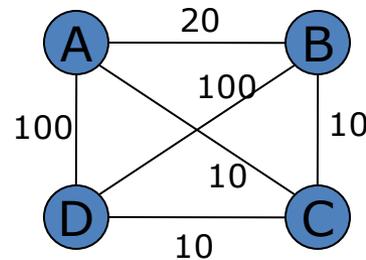
- Given a least-cost tour for N cities, the same tour will be least-cost for $(N-1)$ cities
- e.g., if $A \rightarrow B \rightarrow C \rightarrow D$ is the least-cost tour for cities $\{A, B, C, D\}$, then $A \rightarrow B \rightarrow C$ will be the least-cost tour for cities $\{A, B, C\}$



Another Example (cont'd)

- Counterexample

- Cost $(A \rightarrow B \rightarrow C \rightarrow D) = 40$ (optimal)
- Cost $(A \rightarrow B \rightarrow C) = 30$
- Cost $(A \rightarrow C \rightarrow B) = 20$

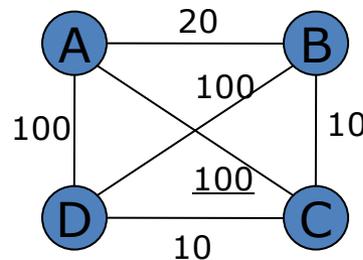


Proof by Contradiction

- Assume hypothesis is false
- Show this assumption leads to a contradiction (i.e., some known property is violated)
 - Can't use special cases or specific examples
- Therefore, hypothesis must be true

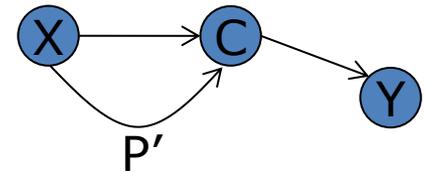
Contradiction Example

- Variant of traveling salesman problem
 - Given N cities and costs for traveling between each pair of cities, find the least-cost path to go from city X to city Y
- Hypothesis
 - A least-cost path from X to Y contains least-cost paths from X to every city on the path
 - E.g., if $X \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow Y$ is the least-cost path from X to Y , then
 - $X \rightarrow C_1 \rightarrow C_2 \rightarrow C_3$ is the least-cost path from X to C_3
 - $X \rightarrow C_1 \rightarrow C_2$ is the least-cost path from X to C_2
 - $X \rightarrow C_1$ is the least-cost path from X to C_1



Contradiction Example (cont'd)

- Assume hypothesis is false
 - i.e., Given a least-cost path P from X to Y that goes through C , there is a better path P' from X to C than the one in P
- Show a contradiction
 - But we could replace the subpath from X to C in P with this lesser-cost path P'
 - The path cost from C to Y is the same
 - Thus we now have a better path from X to Y
 - But this violates the assumption that P is the least-cost path from X to Y
- Therefore, the original hypothesis must be true



More Contradiction Example

- Example: Prove that the square root of 2 is irrational (a number that cannot be expressed as a fraction a/b , where a and b are integers, $b \neq 0$)
- Proof: will be derived in the class
 - assume root of 2 is a rational number
 - assume a/b is simplified to the lowest terms
 - can be done with any fraction
 - in order for a/b to be in its simplest terms, both a and b must not be even. One or both must be odd. Otherwise, you could simplify

Recursion

- A recursive function is defined in terms of itself
- Examples of recursive functions
 - Factorial
 - Fibonacci

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n * (n-1)! & \text{if } n > 0 \end{cases}$$

```
Factorial (n)
  if n = 0
  then return 1
  else return (n * Factorial (n-1))
```

Example

■ Fibonacci numbers

- $F(0) = 0$
- $F(1) = 1$
- $F(2) = 1$
- $F(3) = 2$
- $F(4) = 3$
- $F(5) = 5$
- -----
- $F(n) = F(n-1) + F(n-2)$

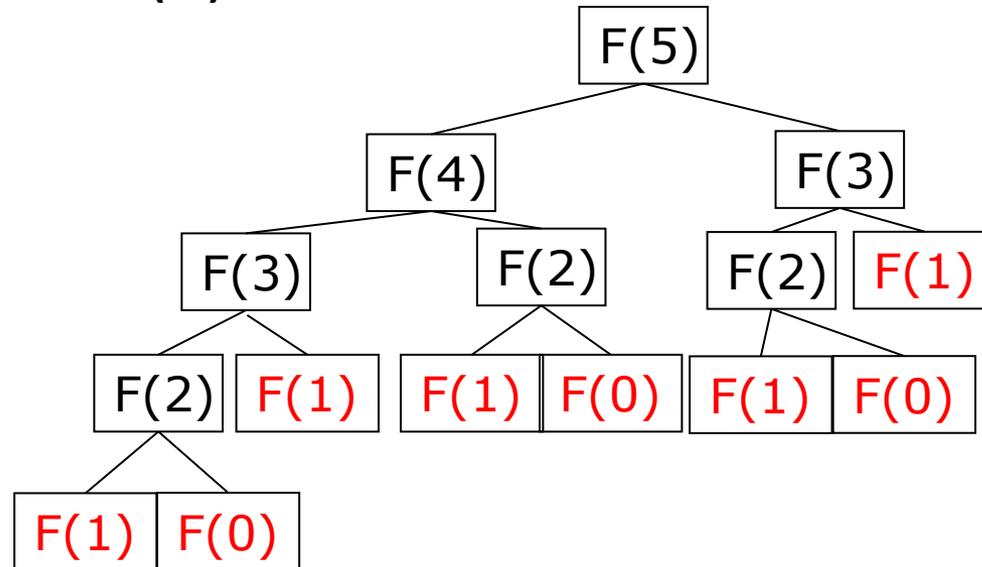
```
Fibonacci (n)
  if (n ≤ 1)
    then return 1
  else return (Fibonacci (n-1) + Fibonacci (n-2))
```

Basic Rules of Recursion

- Base cases
 - Must always have some base cases, which can be solved without recursion
- Making progress
 - Recursive calls must always make progress toward a base case
- Design rule
 - Assume that all the recursive calls work
- Compound interest rule
 - Never duplicate work by solving the same instance of a problem in separate recursive calls

Example (cont'd)

- Fibonacci (5)



- The Fibonacci numbers are the numbers in the following integer sequence:

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.....

Summary

- Proofs by mathematical induction, counterexample and contradiction
- Recursion
- Tools to help us analyze the performance of our data structures and algorithms

- Next:
 - Floors, ceilings, exponents, logarithms, series, and modular arithmetic

Questions

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