IS 709/809: Computational Methods in IS Research

Math Review: Algorithm Analysis

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Topics

- Proof techniques
  - Proof by induction
  - Proof by counterexample
  - Proof by contradiction
- Recursion
- Summary
What do we want to prove?

- Properties of a data structure always hold for all operations
- Algorithm running time/memory usage will never exceed some limit
- Algorithm will always be correct
- Algorithm will always terminate
Proof by Induction

- **Goal**: Prove some hypothesis is true
- **Three-step process**
  - Prove the Base case:
    - Show hypothesis is true for some initial conditions
    - This step is almost always trivial
  - **Inductive hypothesis**: Assume hypothesis is true for all values $\leq k$
  - Using the inductive hypothesis, show that the theorem is true for the next value, typically $k + 1$
Induction Example

- Prove arithmetic series \( \sum_{i=1}^{N} i = \frac{N(N+1)}{2} \)

- (Step 1) Base case: Show true for N=1

\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}
\]
Induction Example (cont’d)

- (Step 2) Assume true for N=k
- (Step 3) Show true for N=k+1

\[
\sum_{i=1}^{k+1} i = (k + 1) + \sum_{i=1}^{k} i
\]

\[
= (k + 1) + \frac{k(k + 1)}{2}
\]

\[
= \frac{2(k + 1) + k(k + 1)}{2}
\]

\[
= \frac{(k + 1)(k + 2)}{2}
\]
More Induction Examples

- Prove the geometric series \( \sum_{i=0}^{N} A^i = \frac{A^{N+1} - 1}{A - 1} \)

- Prove that the number of nodes \( N \) in a complete binary tree of depth \( D \) is \( 2^{D+1} - 1 \)

- Prove that \( \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6} \)
Proof by **Counterexample**

- Prove hypothesis is not true by giving an example that doesn’t work
  - Example: $2^N > N^2$?
  - Example: Prove or disprove “all prime numbers are odd numbers”

- Proof by example?
- Proof by lots of examples?
- Proof by all possible examples?
  - Empirical proof
  - Hard when input size and contents can vary arbitrarily
Another Example

- Traveling salesman problem
  - Given N cities and costs for traveling between each pair of cities, find the least-cost tour to visit each city exactly once

- Hypothesis
  - Given a least-cost tour for N cities, the same tour will be least-cost for (N-1) cities
  - e.g., if A→B→C→D is the least-cost tour for cities {A,B,C,D}, then A→B→C will be the least-cost tour for cities {A,B,C}
Another Example (cont’d)

* Counterexample
  - Cost \((A \rightarrow B \rightarrow C \rightarrow D)\) = 40 (optimal)
  - Cost \((A \rightarrow B \rightarrow C)\) = 30
  - Cost \((A \rightarrow C \rightarrow B)\) = 20
Proof by **Contradiction**

- Assume hypothesis is false
- Show this assumption leads to a contradiction (i.e., some known property is violated)
  - Can’t use special cases or specific examples
- Therefore, hypothesis must be true
Contradiction Example

- Variant of traveling salesman problem
  - Given $N$ cities and costs for traveling between each pair of cities, find the least-cost path to go from city $X$ to city $Y$

- Hypothesis
  - A least-cost path from $X$ to $Y$ contains least-cost paths from $X$ to every city on the path
  - E.g., if $X \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow Y$ is the least-cost path from $X$ to $Y$, then
    - $X \rightarrow C_1 \rightarrow C_2 \rightarrow C_3$ is the least-cost path from $X$ to $C_3$
    - $X \rightarrow C_1 \rightarrow C_2$ is the least-cost path from $X$ to $C_2$
    - $X \rightarrow C_1$ is the least-cost path from $X$ to $C_1$
Contradiction Example (cont’d)

- Assume hypothesis is false
  - i.e., Given a least-cost path $P$ from $X$ to $Y$ that goes is a better path $P'$ from $X$ to $C$ than the one in $P$

- Show a contradiction
  - But we could replace the subpath from $X$ to $C$ in $P$ with this lesser-cost path $P'$
  - The path cost from $C$ to $Y$ is the same
  - Thus we now have a better path from $X$ to $Y$
  - But this violates the assumption that $P$ is the least-cost path from $X$ to $Y$

- Therefore, the original hypothesis must be true
More Contradiction Example

- Example: Prove that the square root of 2 is irrational (a number that cannot be expressed as a fraction $a/b$, where $a$ and $b$ are integers, $b \neq 0$)

- Proof: will be derived in the class
  - assume root of 2 is a rational number
  - assume $a/b$ is simplified to the lowest terms
    - can be done with any fraction
    - in order for $a/b$ to be in its simplest terms, both $a$ and $b$ must not be even. One or both must be odd. Otherwise, you could simplify
Recursion

- A recursive function is defined in terms of itself
- Examples of recursive functions
  - Factorial
  - Fibonacci

\[
\begin{align*}
  n! &= \begin{cases} 
    1 & \text{if } n = 0 \\
    n \times (n-1)! & \text{if } n > 0
  \end{cases}
\end{align*}
\]

Factorial (n)
if n = 0
then return 1
else return (n * Factorial (n-1))
Example

- Fibonacci numbers
  - F(0) = 0
  - F(1) = 1
  - F(2) = 1
  - F(3) = 2
  - F(4) = 3
  - F(5) = 5
- F(n) = F(n-1) + F(n-2)

Fibonacci (n)
if (n ≤ 1)
  then return 1
else return (Fibonacci (n-1) + Fibonacci (n-2))
Basic Rules of Recursion

- **Base cases**
  - Must always have some base cases, which can be solved without recursion

- **Making progress**
  - Recursive calls must always make progress toward a base case

- **Design rule**
  - Assume that all the recursive calls work

- **Compound interest rule**
  - Never duplicate work by solving the same instance of a problem in separate recursive calls
Example (cont’d)

- **Fibonacci (5)**

  ![Fibonacci Tree Diagram]

  The Fibonacci numbers are the numbers in the following integer sequence:
  
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144............
Summary

- Proofs by mathematical induction, counterexample and contradiction
- Recursion
- Tools to help us analyze the performance of our data structures and algorithms

Next:
- Floors, ceilings, exponents, logarithms, series, and modular arithmetic
Questions