UMBC

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Computational Methods in IS Research
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Graph Algorithms
Minimum Spanning Tree

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Minimum Spanning Tree Problem

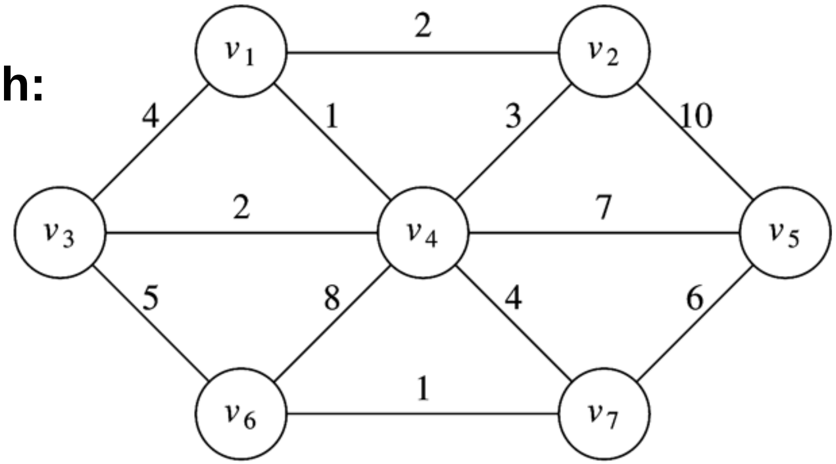
- Find a minimum-cost set of edges that *connect all vertices* of a graph at *lowest total cost*
- Applications
  - Connecting "nodes" with a minimum of "wire"
    - Networking
    - Circuit design
  - Collecting nearby nodes
    - Clustering, taxonomy construction
  - Approximating graphs
    - Most graph algorithms are faster on trees

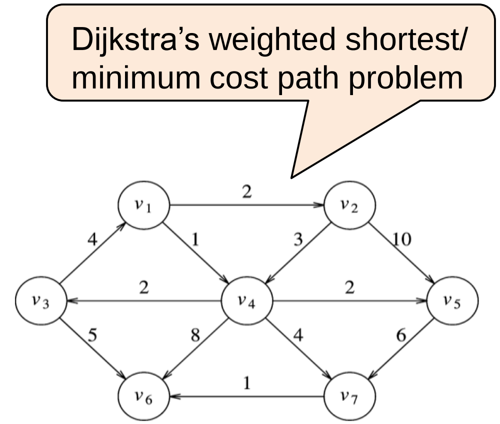
Minimum Spanning Tree

- A tree is an acyclic, undirected, connected graph
- A spanning tree of a graph is a tree containing all vertices from the graph
- A minimum spanning tree is a spanning tree, where the sum of the weights on the tree's edges are minimal

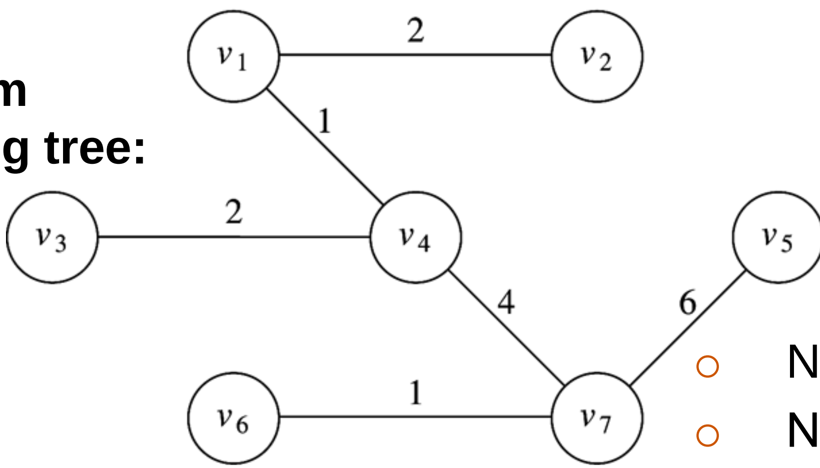
Minimum Spanning Tree (cont'd)

Graph:





Minimum spanning tree:



- Number of edges in MST is |V| - 1
- No fixed start vertex like Dijkstra's
- Undirected graph
- Minimizing summation of total edge cost instead of finding distinct shortest path

Minimum Spanning Tree (cont'd)

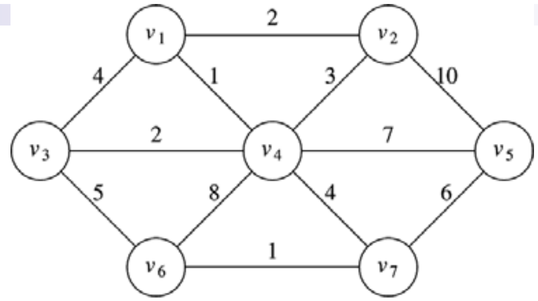
- Problem
  - Given an undirected, weighted graph $G = (V, E)$ with weights $w(u, v)$ for each edge $(u, v) \in E$
  - Find an acyclic, connected graph $G' = (V', E')$, $E' \subseteq E$, that minimizes $\sum_{(u, v) \in E'} w(u, v)$
  - G' is a minimum spanning tree of G
    - There can be more than one minimum spanning tree of a graph G
  - Two algorithms
    - Prim's algorithm
    - Kruskal's algorithm
    - Differ in how a minimum edge is selected

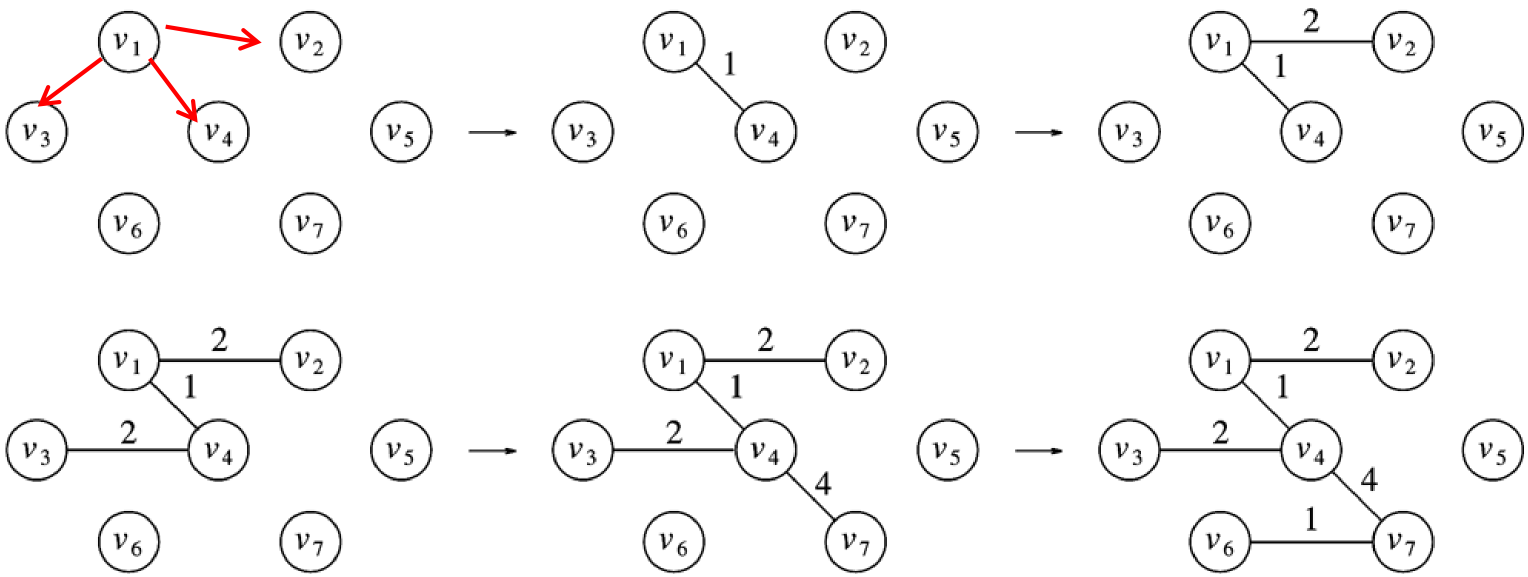
Minimum Spanning Tree

- Solution #1 (Prim's Algorithm (1957))
  - Start with an empty tree T
  - T = {x}, where x is an arbitrary node in the input graph
  - While T is not a spanning tree
    - Find the **lowest-weight edge** that connects a vertex in T to a vertex not in T
    - Add this edge to T
  - T will be a minimum spanning tree

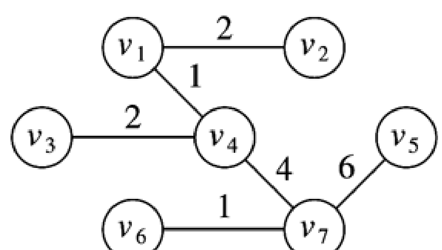
Prim's Algorithm: Example

Input graph:





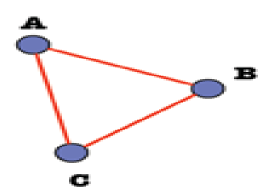
Minimum spanning tree:

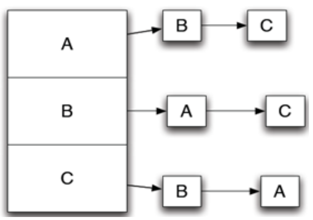


Prim’s Algorithm

- Similar to Dijkstra’s shortest-path algorithm
- Except
  - `v.known` = v in T
  - `v.dist` = weight of lowest-weight edge connecting v to a `known` vertex in T
  - `v.path` = last neighboring vertex changing (lowering) v’s dist value (same as before)
  - Undirected graph, so two entries for every edge in the adjacency list

```
struct Vertex
{
    List      adj;
    bool      known;
    DistType  dist;
    Vertex    path;
        // Other data
};
```



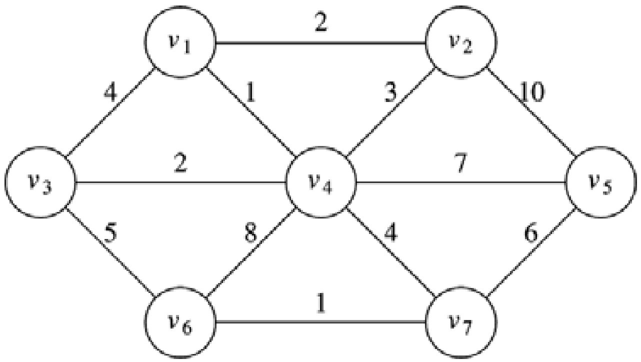


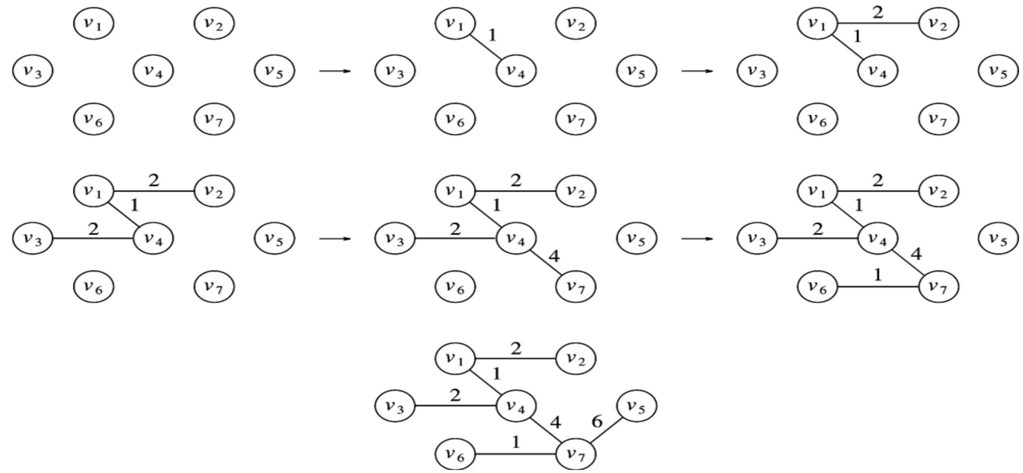
Prim’s Algorithm (cont’d)

```
void Graph::dijkstra( Vertex s )
{
    for each Vertex v
    {
        v.dist = INFINITY;
        v.known = false;
    }

    s.dist = 0;

    for( ; ; )
    {
        Vertex v = smallest unknown distance vertex;
        if( v == NOT_A_VERTEX )
            break;
        v.known = true;

        for each Vertex w adjacent to v
            if( !w.known )
                if( v.dist + cvw < w.dist )
                {
                    // Update w
                    decrease( w.dist to v.dist + cvw );
                    w.path = v;
                }
    }
}
```

Running time same as Dijkstra: O(|E| log |V|) using binary heaps

Prim's Algorithm: Example



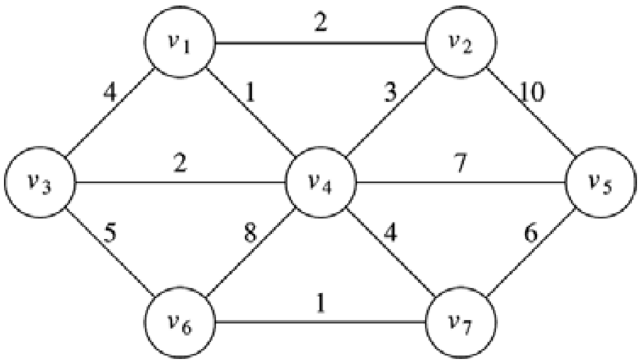


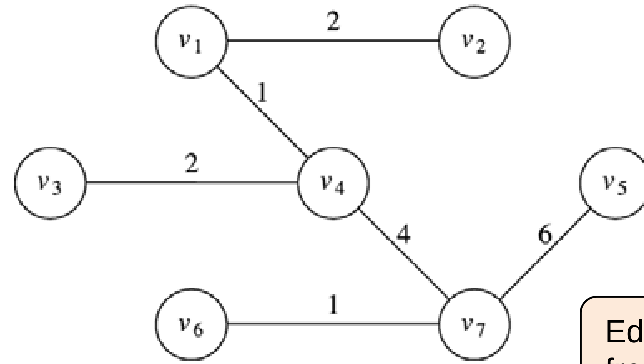
v	*known*	d_v	p_v
v_1	F	0	0
v_2	F	∞	0
v_3	F	∞	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

v	*known*	d_v	p_v
v_1	T	0	0
v_2	F	2	v_1
v_3	F	4	v_1
v_4	F	1	v_1
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

v	*known*	d_v	p_v
v_1	T	0	0
v_2	F	2	v_1
v_3	F	2	v_4
v_4	T	1	v_1
v_5	F	7	v_4
v_6	F	8	v_4
v_7	F	4	v_4

Prim's Algorithm: Example (cont'd)





Edges can be read from this final table

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	2	v_4
v_4	T	1	v_1
v_5	F	7	v_4
v_6	F	5	v_3
v_7	F	4	v_4

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	2	v_4
v_4	T	1	v_1
v_5	F	6	v_7
v_6	F	1	v_7
v_7	T	4	v_4

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	2	v_4
v_4	T	1	v_1
v_5	T	6	v_7
v_6	T	1	v_7
v_7	T	4	v_4

Prim's Algorithm: Analysis

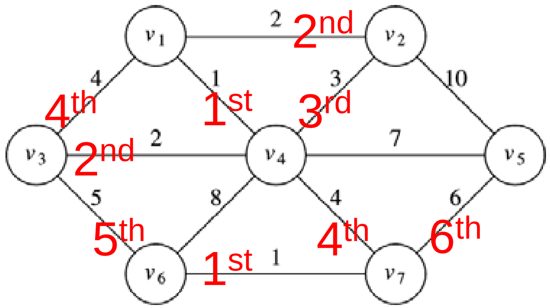
- Running time = $O(|V|^2)$ without heap
  - Optimal for dense graph
- $O(|E| \log |V|)$ using binary heaps
  - Good for sparse graph

Minimum Spanning Tree

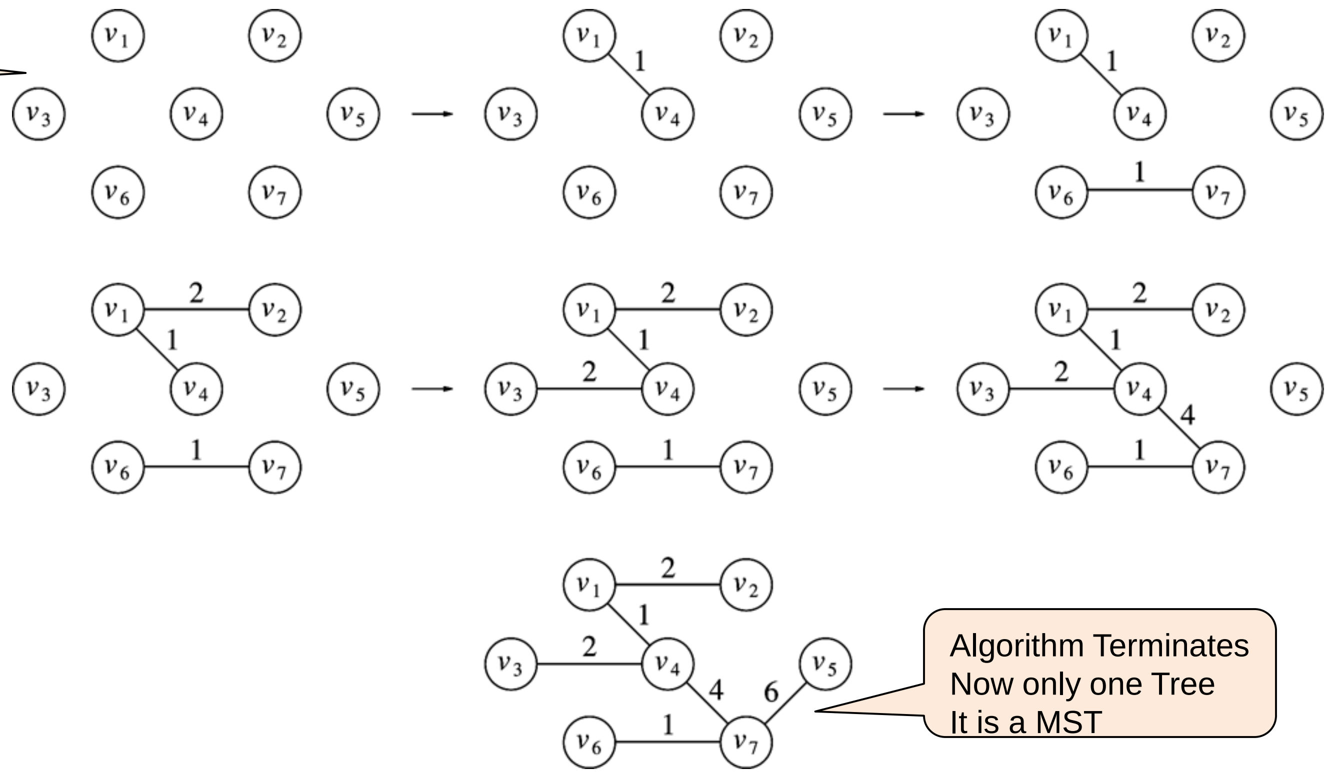
- Solution #2 (Kruskal's algorithm (1956))
  - Start with T = V (with no edges)
  - For each edge in increasing order by weight
    - If adding edge to T does not create a cycle
    - Then add edge to T
  - T will be a minimum spanning tree

Kruskal’s Algorithm: Example

Input graph:



Collection of Trees
|V| single-node Trees



Edge	Weight	Action
(v_1, v_4)	1	Accepted
(v_6, v_7)	1	Accepted
(v_1, v_2)	2	Accepted
(v_3, v_4)	2	Accepted
(v_2, v_4)	3	Rejected
(v_1, v_3)	4	Rejected
(v_4, v_7)	4	Accepted
(v_3, v_6)	5	Rejected
(v_5, v_7)	6	Accepted

Algorithm Terminates
Now only one Tree
It is a MST

Kruskal's Algorithm

```
void Graph::kruskal( )
{
    int edgesAccepted = 0;
    DisjSet ds( NUM_VERTICES );
    PriorityQueue<Edge> pq( getEdges( ) );
    Edge e;
    Vertex u, v;

    while( edgesAccepted < NUM_VERTICES - 1 )
    {
        pq.deleteMin( e );      // Edge e = (u. v)
        SetType uset = ds.find( u );
        SetType vset = ds.find( v );
        if( uset != vset )
        {
            // Accept the edge
            edgesAccepted++;
            ds.unionSets( uset, vset );
        }
    }
}
```

Uses Disjoint Set and Priority Queue data structures

The edges can be sorted, but building a heap in linear time is a better option

deleteMins give the edge to be tested in order

deleteMin: O(|V| log |V|)

find: O(|E| log |V|)

Kruskal's Algorithm: Analysis

- Worst-case: $O(|E| \log |E|)$
- Since $|E| = O(|V|^2)$, worst-case also $O(|E| \log |V|)$
  - Running time dominated by heap operations
- Typically terminates before considering all edges, so faster in practice

Summary

- Finding set of edges that minimally connect all vertices in a graph
- Fast algorithm with many important applications
- Utilizes advanced data structures to achieve fast performance