

### MATH430

#### fall 09 review problems

1. True or False? If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_m]$  is an  $n \times m$  matrix and  $\forall \mathbf{x} \in \mathbf{R}^m$  one has  $A\mathbf{x} = \mathbf{0}$ , then  $A = \mathbf{0}$ .
2. True or False? If  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  is a linearly dependent vector set, then one of the vectors is a linear combination of the others.
3. True or False? If  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  is a linearly independent vector set, then one of the vectors is a linear combination of the others.
4. Let  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  be a linearly independent vector set. If the entries  $b_{ij}$  of the matrix  $B$  are defined by  $b_{ij} = \mathbf{a}_i^T \mathbf{a}_j$ , then  $\det B \neq 0$  (i.e.  $\text{rank } B = n$ , and  $n$  columns of  $B$  are linearly independent).
5. Let  $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  be a linearly independent vector set in a vector space  $\mathbf{R}^m$ . If  $\mathbf{x} \in \mathbf{R}^m$ , and  $\mathbf{x} \notin \text{span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ , then  $\mathbf{x} = \mathbf{a} + \mathbf{b}$  such that  $\mathbf{a} \in \text{span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ , and  $\mathbf{b}^T \mathbf{a}_i = 0$ ,  $i = 1, \dots, n$ .
6. If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_m]$  is an  $n \times m$  matrix and  $\mathbf{x} \in \mathbf{R}^m$ , then  $A\mathbf{x} = x_1\mathbf{a}_1 + \dots + x_m\mathbf{a}_m$ .
7. If  $A$  is an  $n \times m$  matrix and  $B = [\mathbf{b}_1, \dots, \mathbf{b}_k]$ ,  $\mathbf{b}_i \in \mathbf{R}^m$ , then  $AB = [A\mathbf{b}_1, \dots, A\mathbf{b}_k]$ .
8. True or False? If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_m]$ ,  $B = [\mathbf{b}_1, \dots, \mathbf{b}_n]$ , and the vector sets  $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$  and  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  are linearly independent, then the set  $\{A\mathbf{b}_1, \dots, A\mathbf{b}_n\}$  is linearly independent.
9. True or False? If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_m]$ ,  $B = [\mathbf{b}_1, \dots, \mathbf{b}_n]$ , and the vector set  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is linearly dependent, then the set  $\{A\mathbf{b}_1, \dots, A\mathbf{b}_n\}$  is linearly dependent.
10. If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_m]$ , then  $AA^T = \mathbf{a}_1\mathbf{a}_1^T + \dots + \mathbf{a}_m\mathbf{a}_m^T$ .
11. True or False? If  $A$  and  $P$  are  $n \times n$  matrices so that  $AP = A$ , then  $P = I$ .
12. Let  $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$  so that  $\mathbf{v}^T \mathbf{u} \neq 1$ . For the matrices  $I - \mathbf{u}\mathbf{v}^T$  and  $I - \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T \mathbf{u} - 1}$  compute the product 
$$\left(I - \mathbf{u}\mathbf{v}^T\right) \left(I - \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T \mathbf{u} - 1}\right).$$
13. Let  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix}$ ,  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .
  - For  $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_2$  compute  $E_1 = I - \mathbf{u}\mathbf{u}^T$  and  $(I - \mathbf{u}\mathbf{u}^T)A = E_1A$ .
  - Compute  $E_2 = I - (1 - \alpha)\mathbf{e}_2\mathbf{e}_2^T$ , and  $E_2A$ .
  - Compute  $E_3 = I - \alpha\mathbf{e}_2\mathbf{e}_1^T$ , and  $E_3A$ .
14. Elementary row operations:
  - multiplication of a row by a nonzero number,
  - switching two different rows,

- addition of a multiple of row  $i$  to row  $j$ .

- Reduced row echelon form of  $A$ .
- If  $B$  is obtained from  $A$  by a single elementary row operation, and  $k$  rows of  $A$  are linearly independent, then there are  $k$  linearly independent rows of  $B$ .
- If  $B$  is obtained from  $A$  by a single elementary row operation, then  $R(A^T) = R(B^T)$ .
- If  $B$  is obtained from  $A$  by a single elementary row operation, then there is an invertible matrix  $E$  such that  $EA = B$  (the matrix  $E$  is called an elementary matrix).
- For each  $A$  there is a finite set of elementary matrices  $E_1, E_2, \dots, E_n$  so that  $E_n E_{n-1} \dots E_2 E_1 A$  is in a row echelon form.
- If  $B$  is obtained from  $A$  by a single elementary row operation, then  $N(B) = N(A)$ .
- Apply elementary row operations to an  $n \times m$  matrix  $A$  and reduce it to

$$\begin{bmatrix} I_k & B \\ 0_{(n-k) \times k} & 0_{(n-k) \times (m-k)} \end{bmatrix}$$

$B = [\mathbf{b}_1, \dots, \mathbf{b}_{m-k}]$  is  $k \times (m - k)$ . Let  $\hat{B} = [\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_{m-k}]$ , where  $\hat{\mathbf{b}}_i = (\mathbf{b}_i^T, \mathbf{e}_i^T)^T$ , and  $\mathbf{e}_i$  is the  $i^{\text{th}}$  column of  $I_{m-k}$ . True or False? The set  $\{\hat{\mathbf{b}}_1, \dots, \hat{\mathbf{b}}_{m-k}\}$  is a basis for  $N(A)$ .

- If  $B$  is obtained from  $A$  by a single elementary row operation, and  $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}\}$  are linearly independent, then  $\{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_k}\}$  are linearly independent.
- The number of linearly independent columns of a matrix  $A$  is the same as the number of linearly independent rows of  $A$  (rank  $A$ ).
- If  $B$  is nonsingular, then  $N(BA) = N(A)$  (in particular  $N(A)$  and the null space of the reduced row echelon form of  $A$  are identical).

15.  $\text{tr}(AB) = \text{tr}(BA)$ .

16. If  $A = A^T$ , then the eigenvalues  $\lambda_i$  are all real, and  $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$ .

17. If  $A = A^T$ , then  $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$ .

18. If  $A^{-1}$  exists, then  $A\mathbf{x} = \lambda\mathbf{x}$  yields  $\lambda \neq 0$ .

19. If  $A^{-1}$  exists,  $A = A^T$ , and  $A\mathbf{x} = \lambda\mathbf{x}$ , then  $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$ .

20. If  $A = A^T$ , and  $A^{-1}$  exists, then  $A^{-1}$  is symmetric.

21. If  $A$  is an  $n \times n$  matrix such that  $I + A + A^2 + \dots + A^n + \dots$  converges, then

$$(I - A)^{-1} = I + A + A^2 + \dots + A^n + \dots$$

22. If  $S = -S^T$  (is skew-symmetric), then  $I + S$  is nonsingular.

23. If  $A$  and  $B$  are nonsingular, and  $AB = BA$ , then  $AB^{-1} = B^{-1}A$ .
24. If  $A$  is a matrix,  $A\mathbf{x}_i = \mathbf{y}_i$ ,  $i = 1, \dots, n$  and  $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$  is a linearly independent set, then  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is a linearly independent set.
25. True or False? If  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is a linearly independent set, and  $X = [\mathbf{x}_1, \dots, \mathbf{x}_n]$ , then  $XX^T$  is nonsingular.
26. True or False? If  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is a linearly independent set, and  $X = [\mathbf{x}_1, \dots, \mathbf{x}_n]$ , then  $X^T X$  is nonsingular.
27. Let  $A$  and  $B$  be  $n \times m$  matrices. True or False? If there are matrices  $P$  and  $Q$  such that  $AP = B$  and  $A = BQ$ , then  $P$  and  $Q$  are nonsingular.
28. If  $E$  is an elementary matrix, then  $\text{rank}(EA) = \text{rank } A$ . If  $B, C$  are invertible, then  $\text{rank}(BA) = \text{rank}(AC) = \text{rank } A$ .
29. True or False? If  $A, B$  are  $n \times n$  matrices, then  $\text{rank}(AB) = \text{rank}(BA)$ .
30. Let  $A$  be an  $n \times m$  matrix. If  $A = [\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{a}_{p+1}, \dots, \mathbf{a}_m]$ , and  $\dim R(A) = p$ , then  $\dim N(A^T) = m - p$ .