Homework 4 solutions:
1. Interchanging rows, matrix should look something like this

$$E_1 = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & 0 & 1 & & \\ & & 0 & 1 & 0 & & \\ & & 1 & 0 & 0 & & \\ & & & & & 1 & \end{bmatrix}.$$

Of course we could have something like this

$$E_1 = \begin{bmatrix} 0 & & & & 1 & & \\ & \ddots & & & & & \\ & & 1 & 0 & & & \\ & & 0 & 1 & & & \\ 1 & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

A general description of the matrix that interchanges rows $n$ and $m$ would be

$$E_{ij} = \begin{cases} 1 & i = j \text{ and } i, j \neq n, m \\ 0 & i \neq j \text{ and } i, j \neq n, m \\ 1 & i = m \; j = n \text{ or } i = n, \; j = m \\ 0 & \text{else} \end{cases}.$$

Multiplying row $i$ by a scalar $c$

$$E_2 = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

A general description of this matrix is (multiplying row $n$ by the scalar),

$$E_{ij} = \begin{cases} 1 & i = j \text{ and } i, j \neq n \\ c & i = j = n \\ 0 & \text{else} \end{cases}.$$

Multiplying row $n$ by $c$ and adding it to row $m$:

$$E_3 = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & & \\ & & c & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

A general description of this matrix is

$$E_{ij} = \begin{cases} 1 & i = j \\ c & i = m \text{ and } j = n \\ 0 & \text{else} \end{cases}.$$

2.
If $A$ and $B$ are non-singular then $A^{-1}$ and $B^{-1}$ both exist.

$$AB = BA \Rightarrow B^{-1}AB = B^{-1}BA$$
$$\Rightarrow B^{-1}AB = A \Rightarrow B^{-1}ABB^{-1} = AB^{-1}$$
$$\Rightarrow B^{-1}A = AB^{-1}$$

Which is exactly what we want.
3.
False. Let

$$A = \begin{bmatrix} c & 0 \\ c & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & 0 \end{bmatrix},\ Q = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $AP = B$ and $BQ = A$**.** This counterexample is only possible because we not require $A$ and $B$ to have non-zero dimensional Nullspace (there may exist a vector which gets mapped to 0).

4. Use the Rank of a Product theorem on page 210:

If $\text{rank}(BA) = \text{rank}(A) - \dim N(B) \cap R(A)$, together with the fact that $N(B)$ is the zero subspace and has dimension 0 (since $B$ is invertible/non-singular), imply that $\text{rank}(BA) = \text{rank}(A)$.

By using 4.4.9 page 199 we have that $\dim R(A^T) = \text{rank}(A)$. Now using the Rank of a Product theorem,

$$\text{rank}(AC) =$$
$$\text{rank}(C^T A^T) = \text{rank}(A^T) - \dim N(C^T) \cap R(A^T).$$

Then $\dim N(C^T) = 0$ since $C^T$ is a non-singular matrix (same reason as above). Therefore $\text{rank}(AC) = \text{rank}(A^T) = \text{rank}(A)$.