3.5.8A. $A = [a_{ij}]$, $B = [b_{ij}]$ such that if $i > j$ then $a_{ij} = b_{ij} = 0$, $C = AB$, where the entries of C are c_{ij}. We want to see if $c_{ij} = 0$ when $i > j$. By definition of matrix multiplication:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{n} a_{ik} b_{kj}$$

Now we use the fact that for $i > k$, $a_{ik} = 0$ to knock off the first sum. Use the fact that for $k \geq i > j$, $b_{kj} = 0$, to knock off the second sum.

B. Diagonal entries are

$$c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}.$$

If $i > k$ then $a_{ik} = 0$. If $i < k$ then $b_{ki} = 0$. Therefore the only non zero term in this sum is $a_{kk} b_{kk}$.

C. If A, B are lower triangular matrices then A^T and B^T are upper triangular matrices. Consider $B^T A^T$. This matrix by part A is upper triangular, and therefore the transpose of this matrix is lower triangular. But the transpose of this matrix is nothing but AB.

3.5.9. $c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t) b_{kj}(t)$ implies that (by the ordinary product rule)

$$\frac{d}{dt} c_{ij}(t) = \sum_{k=1}^{n} \frac{d}{dt} \left(a_{ik}(t) b_{kj}(t) \right)$$

$$= \sum_{k=1}^{n} \left\{ \left(\frac{d}{dt} a_{ik}(t) \right) b_{kj}(t) + a_{ik}(t) \left(\frac{d}{dt} b_{kj}(t) \right) \right\}$$

$$= \sum_{k=1}^{n} \left(\frac{d}{dt} a_{ik}(t) \right) b_{kj}(t) + \sum_{k=1}^{n} a_{ik}(t) \left(\frac{d}{dt} b_{kj}(t) \right)$$

$$= \frac{dA}{dt} B + A \frac{dB}{dt},$$

since $\left(\frac{d}{dt} a_{ik}(t)\right)$ is the i, kth compononent of $\frac{dA}{dt}$ and $\left(\frac{d}{dt} b_{kj}(t)\right)$ is the k, jth component of $\frac{dB}{dt}$.

3.6.2 see attached sheet.

3.6.3 Let

$$A = \begin{bmatrix} I & B \\ 0 & B \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Since $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{3}$ we have that $B^2 = B$. Now

$$A^2 = \begin{bmatrix} I \cdot I + B \cdot 0 & I \cdot B + B^2 \\ 0 \cdot I + B \cdot 0 & B^2 \end{bmatrix} = \begin{bmatrix} I & 2B \\ 0 & B \end{bmatrix}.$$

Assume that

$$A^n = \begin{bmatrix} I & nB \\ 0 & B \end{bmatrix}.$$

Then

$$A^{n+1} = A^n A = \begin{bmatrix} I & nB \\ 0 & B \end{bmatrix} \begin{bmatrix} I & B \\ 0 & B \end{bmatrix}$$

$$= \begin{bmatrix} I \cdot I + B \cdot 0 & I \cdot B + nB^2 \\ 0 \cdot I + B \cdot 0 & B^2 \end{bmatrix} = \begin{bmatrix} I & (n+1)B \\ 0 & B \end{bmatrix}.$$

Therefore by induction, for all $n = 1, 2, \ldots$

$$A^n = \begin{bmatrix} I & nB \\ 0 & B \end{bmatrix}.$$

Therefore

$$A^{300} = \begin{bmatrix} I & 300B \\ 0 & B \end{bmatrix}.$$

3.6.5 A is a symmetric matrix if and only if $A - A^T = 0$. Let A and B be symmetric matrices such that $AB = BA$. Then

$$AB - (AB)^T = AB - B^T A^T = AB - BA = 0.$$

Now if we did not have commutativity we would not have that $AB - (AB)^T = 0$, and then AB would not be symmetric.

3.6.7 Assume that X does exist. Since we know that $\mathrm{tr}(AX) = \mathrm{tr}(XA)$, and that $\mathrm{tr}(AX + XA) = \mathrm{tr}(AX) + \mathrm{tr}(XA)$, we have that

$$\mathrm{tr}(I) = \mathrm{tr}(AX - XA) = 0,$$

which is a contradiction since we know that the trace of the identity matrix is never 0 .

3.7.5 We have $AA^{-1} = I$, by the definition of matrix inverse. Now $I = I^T$ so

$$\left(AA^{-1}\right)^T = I \Rightarrow \left(A^{-1}\right)^T A^T = I.$$

But $A = A^T$ so

$$\left(A^{-1}\right)^T A^T = I \Rightarrow \left(A^{-1}\right)^T A = I.$$

Now we can see that $\left(A^{-1}\right)^T$ is the matrix inverse of A. Now using the fact that the matrix inverse is unique we have that $\left(A^{-1}\right)^T = A^{-1}$, or that A^{-1} is symmetric.

3.7.6 We know that $(I - A)^{-1}(I - A) = I$. Multiply both sides from the right by A,

$$(I - A)^{-1}(I - A)A = A.$$

Now $(I - A)A = A - A^2 = A(I - A)$, so we have

$$(I - A)^{-1}A(I - A) = A.$$

Then multiply from the right by $(I-A)^{-1}$,

$$(I-A)^{-1}A\left\{(I-A)(I-A)^{-1}\right\} = A(I-A)^{-1}$$

$$\Rightarrow (I-A)^{-1}A = A(I-A)^{-1}.$$

3.7.9a We assume that there is a vector solution x to the equation $(I-S)x = 0$, and then show that such a solution has to be the zero vector, thus showing that $I-S$ is nonsingular. If

$$(I-S)\,x = 0$$

then

$$x^T\,(I-S)\,x = 0 \Rightarrow x^Tx - x^TSx = 0$$

$$\Rightarrow x^Tx = x^TSx. \tag{1}$$

Now

$$x^T\,(I-S)\,x = 0 \Rightarrow \left(x^T\,(I-S)\,x\right)^T = 0$$

$$\Rightarrow x^T(I-S)^Tx = 0 \Rightarrow x^Tx - x^TS^Tx = 0.$$

Now by the skew symmetry of S (that $S^T = -S$) we have that

$$x^Tx + x^TSx = 0.$$

And by (1) we have that this is equivalent to

$$x^Tx + x^Tx = 0 \Rightarrow x^Tx = 0.$$

But this only occurs when x is the zero vector.

b.

$$A^{-1} = \left[(I+S)(I-S)^{-1}\right]^{-1}$$

$$= (I-S)(I+S)^{-1}$$

$$= (I+S)^{-1} - S(I+S)^{-1}$$

and by 3.7.6 (and that $-S$ is also skew symmetric)

$$= (I+S)^{-1} - (I+S)^{-1}S$$

$$= (I+S)^{-1}I - (I+S)^{-1}S$$

$$= (I+S)^{-1}(I-S)$$

$$= (I-S^T)^{-1}(I+S^T)$$

$$= \left((I-S)^T\right)^{-1}(I+S)^T$$

$$= \left((I-S)^{-1}\right)^T(I+S)^T$$

$$= \left((I+S)(I-S)^{-1}\right)^T = A^T.$$