Hwk 1 Solutions:

1. (This question was not graded. This proof is due to Geoffrey Clapp)

We will show that the n columns of B are linearly independent by contradiction. Assume not. That is, assume that $\exists$ $\{y_1, \ldots, y_n\}$ with at least one non-zero element, such that

$$y_1 b_1 + \cdots + y_n b_n = \vec{0},$$

where $B = [b_1, \ldots, b_n]$. This can be reexpressed as $B\vec{y}$, where

$$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Now define $A = [a_1, \ldots, a_n]$. By construction of B, $A^T A = B$. By substituting $A^T A$ into $B\vec{y} = \vec{0}$, we get $A^T A\vec{y} = \vec{0}$. Also $\vec{y}^T \left(A^T A\vec{y}\right) = \vec{y} \cdot (\vec{0}) = \vec{0}$. If we reposition parentheses we get

$$\left(\vec{y}^T A^T\right)(A\vec{y}) = (A\vec{y})^T (A\vec{y}) = \vec{0}.$$

Anything multiplied by its transpose can only be zero if it is zero itself. Therefore, $A\vec{y} = \vec{0}$. But this contradicts that $\{a_1, \ldots, a_n\}$are linearly independent since $A\vec{y} = \vec{0}$ implies

$$y_1 a_1 + y_2 a_2 + \cdots + y_n a_n = 0.$$

Therefore the n columns of B are linearly independent, and we know that B is a square matrix, so $\det B \neq 0$.

2. write $A = [A_{11}]$, block form with just one block $A_{11} = A$. Write

$$B = [B_{11}, B_{12}, \ldots, B_{1k}]$$

where the $(1, j)$ -block is $B_{1j} = b_j$. Then according to the block multiplication rules on pg. 111, the (i, j)-block is

$$A_{11} B_{1j} = A_{11} b_j.$$

Therefore

$$AB = [Ab_1, \ldots, Ab_k].$$

3. True. Linear independence of $a_1, \ldots, a_m$ implies that $Av \neq 0$, for all $v \neq 0$. Suppose that $\{Ab_1, \ldots, Ab_n\}$ is not linearly independent. Then there must exists n scalars $c_1, \ldots, c_n$ not all zero such that

$$c_1 Ab_1 + \cdots + c_n Ab_n = 0.$$

But this equals

$$A(c_1 b_1) + \cdots + A(c_n b_n)$$

$$= A\left(c_1 b_1 + \cdots + c_n b_n\right) \Rightarrow c_1 b_1 + \cdots + c_n b_n = 0.$$

But this is not possible since $\{b_1, \ldots, b_n\}$ are linearly independent.

4. Suppose that the $\mathbf{a}_i$'s are $n \times 1$ column vectors. Block decompose A as $[\mathbf{a}_1, \ldots, \mathbf{a}_m]$ where each $\mathbf{a}_i$ is a $n \times 1$ block. Similarly block decompose A^T as

$$\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}.$$

Then according to block matrix multiplication $AA^T =$

$$[\mathbf{a}_1, \ldots, \mathbf{a}_m] \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} = \mathbf{a}_1 \mathbf{a}_1^T + \cdots + \mathbf{a}_m \mathbf{a}_m^T.$$

5.

$$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad E_2 A = \begin{bmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}, \qquad E_3 A = \begin{bmatrix} a_{11} & a_{12} \\ (1-\alpha) a_{11} & (1-\alpha) a_{12} \end{bmatrix}.$$

6. Assume $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times m$ matrix.

$$\operatorname{tr}(AB) = \sum_{k=1}^{m} \left(\sum_{l=1}^{n} a_{kl} b_{lk} \right).$$

And

$$\operatorname{tr}(BA) = \sum_{k=1}^{n} \left(\sum_{l=1}^{m} b_{kl} a_{lk} \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{l=1}^{m} a_{lk} b_{kl} \right) = \sum_{l=1}^{m} \left(\sum_{k=1}^{n} a_{lk} b_{kl} \right).$$

where in the last step we just switched the order of summation. Now just relabel k as l, and l as k.

7. True. Assume that $\{x_1, \ldots, x_n\}$ is linearly dependent. Then there exist $c_1, \ldots, c_n$ not all zero such that $c_1 x_1 + \cdots + c_n x_n = 0$. Then

$$0 = A\left(c_1 x_1 + \cdots + c_n x_n\right) = c_1 A x_1 + \cdots + c_n A x_n$$

$$= c_1 y_1 + \cdots + c_n y_n.$$

But the c_i's are not all zero and this contradicts the linear independence of the y_i's. Therefore $\{x_1, \ldots, x_n\}$ is a linearly independent set.

8. False. Here is one counterexample. Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Clearly $\mathbf{x}_1$and $\mathbf{x}_2$ are linearly independent. If $X = [\mathbf{x}_1, \mathbf{x}_2]$, then

$$XX^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly this matrix is singular.