Name:________________________

MATH225

quiz 2

Sections 4.1–4.4, 4.6, 4.7

03/13/08

Total 100

Solutions

Show all work legibly.

1. (20) Determine the general solution of $y'' + 2y' = 3 + 4\sin 2x$.

   The characteristic equations is $0 = r^2 + 2r = r(r+2)$, with the roots $r_1 = 0$ and $r_2 = -2$. The two linearly independent solutions of the homogenious equation are $y_1(x) = 1$, and $y_2(x) = e^{-2x}$.

   A particular solution $y_{p1}(x)$ for the equation $y'' + 2y' = 3$ has the form $y_{p1}(x) = Ax$, with $A = \frac{3}{2}$, so that $y_{p1}(x) = \frac{3}{2}x$.

   A particular solution $y_{p21}(x)$ for the equation $y'' + 2y' = 4\sin 2x$ has the form $y_{p2}(x) = A\sin 2x + B\cos 2x$ with $A = -\frac{1}{2}$, and $B = -\frac{1}{2}$, i.e. $y_{p2}(x) = -\frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x$.

   The general solution is $c_1 y_1(x) + c_2 y_2(x) + y_{p1}(x) + y_{p21}(x)$.

   the solution is: $y(x) = c_1 + c_2 e^{-2x} + \frac{3}{2}x - \frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x$.

2. (20) Solve the initial value problem: $y'' - 6y' + 9y = 0,\ y(0) = 0,\ y'(0) = 2.$

The characteristic equation is $0 = r^2 - 6r + 9 = (r-3)^2$, and the fundamental set is $y_1(x) = e^{3x}$, $y_2(x) = xe^{3x}$.

the solution is: $y(x) = 2xe^{3x}$

3. (20) Solve the differential equation: $y'' + 2y' + 2y = 0$.

The characteristic equation is $r^2 + 2r + 2 = 0$, and the roots are $r_1 = -1 - i$, $r_2 = -1 + i$. The fundamental set is $y_1(x) = e^{-x} \sin x$ and $y_2(x) = e^{-x} \cos x$.

the solution is: $y(x) = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x$

4. (20) Find the general solution $y(t)$ of the differential equation: $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$.

The characteristic equation is $0 = r^2 - 2r + 1 = (r-1)^2$, and a fundamental set is $y_1(x) = e^x$, $y_2(x) = xe^x$. We are looking for a particular solution

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$$

with $u_1'(x) = \dfrac{W_1(x)}{W(x)}$ and $u_2'(x) = \dfrac{W_2(x)}{W(x)}$ where

$$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x},\ W_1(x) = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & e^x + xe^x \end{vmatrix} = -\frac{xe^{2x}}{1+x^2},\ W_2(x) = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix} = \frac{e^{2x}}{1+x^2}.$$

So

$$u_1'(x) = -\frac{x}{1+x^2}, \text{ and } u_1(x) = -\frac{1}{2}\ln(1+x^2)$$

and

$$u_2'(x) = \frac{1}{1+x^2}, \text{ and } u_2(x) = \tan^{-1} x.$$

the solution is: $y(x) = c_1 e^x + c_2 xe^x - \dfrac{1}{2}\ln(1+x^2)e^x + xe^x \tan^{-1} x$.

5. (20) Use the substitution $x = e^t$ to transform the Euler equation $x^2y'' - 3xy' + 13y = 4 + 3x$ to a differential equation with constant coefficients and solve the original equation.

With $x(t) = e^t$ one has $\frac{dx}{dt} = e^t = x$, and

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad \text{and } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

The differential equation becomes

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 4 + 3e^t.$$

The characteristic equation is $r^2 - 4r + 13 = 0$, and a pair of linearly independent solutions is given by

$$y_1(t) = e^{2t}\cos 3t, \text{ and } y_2(t) = e^{2t}\sin 3t.$$

A particular solution for this equation is $y_p(t) = \frac{4}{13} + \frac{3}{10}e^t$, and the general solution for the Euler equation is

$$c_1x^2\cos(3\ln|x|) + c_2x^2\sin(3\ln|x|) + \frac{4}{13} + \frac{3}{10}x.$$

the solution is: $c_1x^2\cos(3\ln|x|) + c_2x^2\sin(3\ln|x|) + \frac{4}{13} + \frac{3}{10}x.$

6. (**extra credit**) (20) Find a particular solution $y_p(t)$ of the differential equation: $2y'' + 3y' + y = x^2 + 3\sin x$.

The characteristic equation is $0 = r^2 + \frac{3}{2}r + \frac{1}{2} = \left(r + \frac{1}{2}\right)(r+1)$. $y_p(x) = y_{p1}(x) + y_{p2}(x)$, where $y_{p1}(x)$ is a particular solution of the equation $2y'' + 3y' + y = x^2$, and $y_{p2}(x)$ is a particular solution of the equation $2y'' + 3y' + y = 3\sin x$.

$y_{p1}(x) = A_1x^2 + A_2x + A_3$, and substitution to the equation leads to $y_{p1}(x) = 2x^2 - 12x + 28$.

$y_{p2}(x) = A\sin x + B\cos x$, and substitution to the equation leads to $y_{p2}(x) = -\frac{3}{10}\sin x - \frac{9}{10}\cos x$.

the solution is: $y_p(t) = 2x^2 - 12x + 28 - \frac{3}{10}\sin x - \frac{9}{10}\cos x$.