MATH221-04
final examination 12/13/18
Total 200
Solutions

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Show all work legibly. **Name**:______________________

1. (20) Consider the linear system of equations

$$\begin{array}{rcrcrcr} x & - & y & & & = & a \\ 2x & + & y & - & 3z & = & 8 \\ x & - & 2y & + & 3z & = & -5 \end{array}$$

Find the value of a if $y = 0$.

Solution. Since $y = 0$ the system becomes

$$\begin{array}{rcrcrcr} x & + & & & & = & a \\ 2x & + & & - & 3z & = & 8 \\ x & + & & + & 3z & = & -5 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} x & - & a & & & = & 0 \\ 2x & + & & - & 3z & = & 8 \\ x & + & & + & 3z & = & -5 \end{array}$$

The augmented matrix of the systems is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -3 & 8 \\ 1 & 0 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -3 & 8 \\ 0 & 1 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & -5 \\ 0 & 2 & -3 & 8 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & -9 & 18 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$a = 1$

2. (40) Let $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be the eigenvectors of a 2×2 matrix A with the corresponding eigenvalues 2 and 3.

(a) (20) Compute A^4.

Solution. $A = U \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} U^{-1}$, where $U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $U^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

$$A^4 = U \begin{bmatrix} 2^4 & 0 \\ 0 & 3^4 \end{bmatrix} U^{-1} = \begin{bmatrix} 2^3 + 3^4/2 & 2^3 - 3^4/2 \\ 2^3 - 3^4/2 & 2^3 + 3^4/2 \end{bmatrix}.$$

(b) (20) Compute det (A).

Solution. det $(A) =$ det U det $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ det $U^{-1} = 6$.

3. (20) Let T be a linear transformation that reflects a vector in $\mathbf{R}^2$ with respect to the line $y = -x$.

(a) (10) Find A the standard matrix for the linear transformation T.

Solution. $A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = [-\mathbf{e}_2, -\mathbf{e}_1]$.

(b) (10) True or False? T is one–to–one.

Solution. $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$. If $T(\mathbf{x}) = 0$, then $\mathbf{x} = 0$, and T is one–to–one.

4. (20) Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(a) (10) Find A^{-1} if exists.

Solution.

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$A^{-1} = A$.

(b) (10) If B is a 2×3 matrix so that $AB = C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find B.

Solution. $B = A^{-1}C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -5 & -6 \\ -1 & -2 & -3 \end{bmatrix}$.

5. (60) Let $\mathbf{u}_1,\ \mathbf{u}_2$ be vectors of magnitude 1 (i.e., $\mathbf{u}_1^T\mathbf{u}_1 = \mathbf{u}_2^T\mathbf{u}_2 = 1$).

(a) (20) True or False? $\mathbf{u}_1^T\mathbf{u}_2 \le 1$

Solution. Note that for each real number t

$$0 \le (\mathbf{u}_1 - t\mathbf{u}_2)^T(\mathbf{u}_1 - t\mathbf{u}_2) = t^2\mathbf{u}_2^T\mathbf{u}_2 - 2t\mathbf{u}_2^T\mathbf{u}_1 + \mathbf{u}_1^T\mathbf{u}_1 = t^2 - t\left(2\mathbf{u}_2^T\mathbf{u}_1\right) + 1.$$

This yields

$$\left(2\mathbf{u}_2^T\mathbf{u}_1\right)^2 - 4 \le 0, \text{ and } \mathbf{u}_1^T\mathbf{u}_2 \le \left|\mathbf{u}_1^T\mathbf{u}_2\right| \le 1.$$

(b) (20) True or False? If $\mathbf{u}_1^T\mathbf{u}_2 = 1$, then $\mathbf{u}_1 = \mathbf{u}_2$

Solution. When $\mathbf{u}_1^T\mathbf{u}_2 = 1$ one has $|\mathbf{u}_1 - t\mathbf{u}_2|^2 = (t-1)^2$. When $t = 1$ one has $|\mathbf{u}_1 - \mathbf{u}_2|^2 = 0$, and $\mathbf{u}_1 = \mathbf{u}_2$.

(c) (20) Let $\mathbf{v}_1,\ \mathbf{v}_2$ be linearly independent vectors of magnitude 1. True or False? If $\mathbf{u}_i^T\mathbf{v}_j = 0$ for each i, j then the set $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_1,\ \mathbf{v}_2\}$ is linearly independent.

Solution. If $\mathbf{u}_1 = \pm\mathbf{u}_2$ the four vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_1,\ \mathbf{v}_2\}$ are linearly dependent.

6. (40) Let $A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & 4 & 8 \\ 7 & 3 & 9 & 27 \\ 7 & 4 & 16 & 64 \\ 7 & 5 & 25 & 125 \end{bmatrix}$.

(a) (20) Compute $\det A$.

Solution.

$$\det\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 5 & 5^2 & 5^3 \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & 2 & 2^2 & 2^3 \\ 0 & 3-2 & 3^2-2^2 & 3^3-2^3 \\ 0 & 4-2 & 4^2-2^2 & 4^3-2^3 \\ 0 & 5-2 & 5^2-2^2 & 5^3-2^3 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)\det\begin{bmatrix} 1 & 3+2 & 3^2+3\times 2+2^2 \\ 1 & 4+2 & 4^2+4\times 2+2^2 \\ 1 & 5+2 & 5^2+5\times 2+2^2 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)\det\begin{bmatrix} 1 & 3+2 & 3^2+3\times 2+2^2 \\ 0 & 4-3 & 4^2+4\times 2-3^2-3\times 2 \\ 0 & 5-3 & 5^2+5\times 2-3^2-3\times 2 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)\det\begin{bmatrix} 4-3 & 4^2+4\times 2-3^2-3\times 2 \\ 5-3 & 5^2+5\times 2-3^2-3\times 2 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)\det\begin{bmatrix} 4-3 & 4^2-3^2+4\times 2-3\times 2 \\ 5-3 & 5^2-3^2+5\times 2-3\times 2 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)\det\begin{bmatrix} 4-3 & (4-3)(4+3)+(4-3)\times 2 \\ 5-3 & (5-3)(5+3)+(5-3)\times 2 \end{bmatrix} =$$

$$(3-2)(4-2)(5-2)(4-3)(5-3)\det\begin{bmatrix} 1 & 4+3+2 \\ 1 & 5+3+2 \end{bmatrix} =$$

$$(3-2)(4-2)(4-3)(5-2)(5-3)(5-4) = 12.$$

(b) (20) Compute $\det\ B$.

Solution. $\det\ B = 7\det\ A = 84$.