MATH 221, Spring 2016 - Homework 8 Solutions

Due Tuesday, April 19

Section 5.1

Page 271, Problem 7:

4 is an eigenvalue if and only if the equation $A\mathbf{x} = 4\mathbf{x}$ has a nontrivial solution, which is equivalent to solving the system

$(A - 4I)\mathbf{x} = \mathbf{0}$: $(A - 4I) = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$. Because the columns of this matrix are linearly dependent, the system must have a nontrivial solution, so 4 is an eigenvalue. To find the eigenvector corresponding to $\lambda = 4$, solve the system by row-reducing:

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Each vector of this form with $x_3 \neq 0$ is an eigenvector corresponding to $\lambda = 4$.

Page 271, Problem 9:

To find a basis for the eigenspace of each eigenvalue, find the vectors that span the eigenspace and are linearly independent (i.e. the vectors that form the general solution of $(A - \lambda I)\mathbf{x} = \mathbf{0}$):

- When $\lambda = 1$: $A - I = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$. So, $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.
- When$\lambda = 3$: $A - 3I = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$. So, $\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace.

Page 272, Problem 17:

Because the matrix is upper-triangular (every element below the diagnoal is 0), the eigenvalues are the entries of the diagnoal. Thus, $\lambda = 0$, $\lambda = 3$, $\lambda = -2$.

Page 272, Problem 24:

Because the diagonal entries of an upper-triangular matrix are its eigenvalues, let $A = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$ where λ, $a \in \mathbb{R}$.

Thus, the diagonal entries are the eigenvalues, but because they are the same value, the matrix has one distinct eigenvalue.

Section 5.2

Page 279, Problem 2:

- $A-\lambda I = \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix}$ and the characteristic polynomial is $\det(A-\lambda I) = (-4-\lambda)(1-\lambda)-(-1)(6) = \lambda^2+3\lambda+2$
- The solutions to the equation $\lambda^2+3\lambda+2=0$ are $\lambda=-1$, $\lambda=-2$.

Page 279, Problem 4:

- $A-\lambda I = \begin{bmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix}$ and the characteristic polynomial is $\det(A-\lambda I) = (8-\lambda)(3-\lambda)-(3)(2) = \lambda^2-11\lambda+18$
- The solutions to $\lambda^2-11\lambda+18=0$ are $\lambda=9$, $\lambda=2$.

Page 272, Problem 7:

- $A-\lambda I = \begin{bmatrix} 5-\lambda & 3 \\ -4 & 4-\lambda \end{bmatrix}$ and the characteristic polynomial is $\det(A-\lambda I) = (5-\lambda)(4-\lambda)-(3)(-4) = \lambda^2-9\lambda+32$
- The solutions to $\lambda^2-9\lambda+32=0$ are found using the quadratic formula $\lambda = \frac{9\pm\sqrt{9^2-4(1)(32)}}{2(1)} \Rightarrow \lambda = \frac{9}{2} \pm \frac{\sqrt{81-128}}{2}$. Because expression involves complex roots, **there are no REAL eigenvalues**.

Page 279, Problem 8:

- $A-\lambda I = \begin{bmatrix} -4-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix}$ and the characteristic polynomial is $\det(A-\lambda I) = (-4-\lambda)(1-\lambda)-(3)(2) = \lambda^2+3\lambda-10$
- The solutions to $\lambda^2+3\lambda-10=0$ are $\lambda=-5$, $\lambda=2$.

Page 280, Problem 25a:

- Because we know that $\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ is an eigenvector, compute $A\mathbf{v}_1 = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$. So, $\lambda=1$ must be the eigenvalue corresponding to $\mathbf{v}_1$.
- To find the other eigenvector, find the eignevalues of the matrix: $A-\lambda I = \begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix}$, so the characteristic polynomial is $\lambda^2-1.3\lambda+0.3$ and the solutions to $\lambda^2-1.3\lambda+0.3=0$ are $\lambda=1$ and $\lambda=.3$. Thus, the other eigenvector must correspond to $\lambda=.3$.
- To find the other eigenvector, solve $(A-.3I)\mathbf{x}=\mathbf{0}$ for the general solution: $\begin{bmatrix} .3 & .3 & 0 \\ .4 & .4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, an eigenvector corresponding to $\lambda=.3$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- Because eigenvectors corresponding to different eigenvalues are linearly independent (and two non-zero linearly independent vectors in $\mathbb{R}^2$ must also span $\mathbb{R}^2$), the set $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

Page 280, Problem 25b:

- Solve for c: $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2 \Rightarrow \mathbf{x}_0 - \mathbf{v}_1 = c\mathbf{v}_2$. So, $\begin{bmatrix} .5 \\ .5 \end{bmatrix} - \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} 1/14 \\ -1/14 \end{bmatrix} = -\frac{1}{14}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\frac{1}{14}\mathbf{v}_2$. So, $c = -\frac{1}{14}$ and $\mathbf{x}_0 = \mathbf{v}_1 - \frac{1}{14}\mathbf{v}_2$.

Page 280, Problem 25c:

- To begin, realize that $\mathbf{x}_k = A^k\mathbf{x}_0 = A^k(\mathbf{v}_1 - \frac{1}{14}\mathbf{v}_2) = A^k\mathbf{v}_1 - A^k\frac{1}{14}\mathbf{v}_2 = A^k\mathbf{v}_1 - \frac{1}{14}A^k\mathbf{v}_2$.
- Then, $\mathbf{x}_1 = A\mathbf{v}_1 - \frac{1}{14}A\mathbf{v}_2$. Remember the definition of an eigenvector: if $\mathbf{v}$ is an eigenvector corresponding to λ, then $A\mathbf{v} = \lambda\mathbf{v}$.
- Because $\mathbf{v}_1$ is an eigenvector corresponding to $\lambda = 1$ and $\mathbf{v}_2$ is an eigenvector corresponding to $\lambda = .3$, this equation can be rewritten as $\mathbf{x}_1 = 1\mathbf{v}_1 - \frac{1}{14}(0.3\mathbf{v}_2) = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + \begin{bmatrix} 3/140 \\ -3/140 \end{bmatrix} = \begin{bmatrix} 9/20 \\ 11/20 \end{bmatrix}$.
- Similarly, $\mathbf{x}_2 = A^2\mathbf{v}_1 - \frac{1}{14}A^2\mathbf{v}_2 = A(A\mathbf{v}_1) - \frac{1}{14}A(A\mathbf{v}_2) = A(1\mathbf{v}_1) - \frac{1}{14}A(.3\mathbf{v}_2) = A\mathbf{v}_1 - \frac{.3}{14}A\mathbf{v}_2 = 1\mathbf{v}_1 - \frac{.3}{14}(.3\mathbf{v}_2) = \mathbf{v}_1 - \frac{1}{14}(0.3)^2\mathbf{v}_2$. This is equal to $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + \begin{bmatrix} 9/1400 \\ -9/1400 \end{bmatrix} = \begin{bmatrix} 87/200 \\ 113/200 \end{bmatrix}$.
- It is clear to see that that the formula for $\mathbf{x}_k = \mathbf{v}_1 - \frac{1}{14}(0.3)^k\mathbf{v}_2$.
- As k gets larger (tends to infinity), $(0.3)^k$ tends to 0. Therefore, as $k \to \infty$, $\mathbf{x}_k \to \mathbf{v}_1$.

Section 5.3

Page 286, Problem 6:

A matrix A of the form $A = PDP^{-1}$ where D is a diagonal matrix consisting of the eigenvalues of A has vectors that form a basis for the eigenspace in the column of P that correspond to the eigenvalue in D. Therefore, the eigenvalues of A are 3 and 4. The vectors corresponding to $\lambda = 3$ that forms a basis for the eigenspace are columns 1 and 3 of the matrix P: $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} \right\}$. The vector corresponding to $\lambda = 4$ that forms a basis for the eigenspace is column 2 of the matrix P: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Page 286, Problem 7:

- To diagonalize the matrix, first find the eigenvalues: $\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 6(0) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$. Then, find a basis for each eigenspace.
- When $\lambda = 1$, $(A - I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 6 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a basis.
- When $\lambda = -1$, $(A + I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis.
- Then, these bases form the columns of P with the **associated eigenvalue in the corresponding column of D (this is very important!):** $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Page 287, Problem 12:

- Because the eigenvalues are given, we just need to find a basis for each eigenspace. Note: Because there are only 2 distinct eigenvalues, the sum of the dimensions of the eigenspaces must equal 3 in order for A to be diagonalizable.

- When $\lambda = 2$, $(A - 2I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

- When $\lambda = 5$, $(A-5I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis.

- Then, these bases form the columns of P with the **associated eigenvalue in the corresponding column of** D **(this is very important!):** $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Page 287, Problem 20:

- Because the matrix is triangular, the eigenvalues are the entries on the diagonal: $\lambda = 2$, $\lambda = 3$ (each with multiplicity 2). Note: Because there are only 2 distinct eigenvalues, the sum of the dimensions of the eigenspaces must equal 4 in order for A to be diagonalizable.

- When $\lambda = 2$, $(A - 2I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis.

- When $\lambda = 3$, $(A - 3I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So, $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

- Because the dimension of the basis corresponding to $\lambda = 3$ is 1 and the basis corresponding to $\lambda = 2$ is 2 and $1 + 2 = 3 \neq 4$, the matrix is not diagonalizable.

Page 287, Problem 21a:

True or False: A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

FALSE: The matrix D needs to be a diagonal matrix (the notation D does not automatically denote a diagonal matrix).

Page 287, Problem 21b:

True or False: If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diaognalizable.

TRUE: Because A is an $n \times n$ matrix (stated in the directions), this statement is true and follows from the Diagonalization Theorem on page 282.