MATH 221, Spring 2016 - Homework 7 Solutions

Due Tuesday, April 12

Section 4.4

Page 222, Problem 3:

Let $\mathcal{B}$=[$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$]. Then, $\mathbf{x} = 1\mathbf{b}_1 + 0\mathbf{b}_2 + -2\mathbf{b}_3 = 1\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} + -2\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$.

Page 222, Problem 7:

In this problem, we are solving the equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for the coordinates c_1, c_2, and c_3. In this problem, this equation is represented by $\begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, which amounts to solving the augmented system $\begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix}$. Row-reducing yields $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

So, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

Page 223, Problem 10:

As stated in this section (on page 219), the matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ is the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Therefore, $P_{\mathcal{B}} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ 6 & -4 & 3 \end{bmatrix}$.

Page 223, Problem 14:

Any polynomial $a + bt + ct^2$ in $\mathbb{P}_2$ can be written in vector form as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Therefore, the set $\mathcal{B}$ as a set of vectors is $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$ and the vector $\mathbf{p}$ is $\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$. Solve the augmented system $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{bmatrix}$.

The solution in reduced-echelon form is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, so $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

Page 223, Problem 22:

Let $P_{\mathcal{B}} = \left[\begin{array}{ccc} \mathbf{b}_1 & \dots & \mathbf{b}_n \end{array}\right]$ (which is an $n \times n$ matrix because its columns form a basis for $\mathbb{R}^{\mathrm{n}}$). By definition, $\mathbf{x} = P_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$ which is a transformation of $[\mathbf{x}]_{\mathcal{B}}$ to $\mathbf{x}$. Because the columns of $P_{\mathcal{B}}$ are linearly independent (the form a basis for $\mathbb{R}^{\mathrm{n}}$), $P_{\mathcal{B}}$ is invertible. Thus, left-side multiplication of $P_{\mathcal{B}}^{-1}$ results in $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, which is a transformation of $\mathbf{x}$ to $[\mathbf{x}]_{\mathcal{B}}$ ($\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$). Therefore, take $A = P_{\mathcal{B}}^{-1}$.

Section 4.4

Page 242, Problem 1:

- To find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$, we need to solve the systems $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1$ and $y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2$ and form the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$. In this problem, it is clear from the description that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$.
- Because $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. So, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

Page 243, Problem 6:

- Using the same method as problem 1 (in more dimensions), it is clear that $\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.
- As before, $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{F}}{P}[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

Page 243, Problem 9:

- To find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, use the method described on page 241, solving two systems simultaneously: $\left[\begin{array}{cc:cc} 2 & -2 & 4 & 8 \\ 2 & 2 & 4 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & -1 & 2 & 4 \\ 1 & 1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right] \Rightarrow \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$.
- To find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$, use the same method, but change the order: $\left[\begin{array}{cc:cc} 4 & 8 & 2 & -2 \\ 4 & 4 & 2 & 2 \end{array}\right] \rightarrow$ $\left[\begin{array}{cc:cc} 2 & 4 & 1 & -1 \\ 2 & 2 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 0.5 & 1.5 \\ 0 & 1 & 0 & -1 \end{array}\right] \Rightarrow \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 0.5 & 1.5 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$.

Section 3.2

Page 175, Problem 20:

Transforming the matrix $\begin{bmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{bmatrix}$ by $-R_2 + R_1 \rightarrow R_1$ yields $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Because $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$ and the only row-operations on the matrix were adding a multiple of a row, $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = 7$.

Page 175, Problem 21:

Use row-operations to reduce the matrix:

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix} : R_3 \leftrightarrow R_1 : \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & 0 \end{bmatrix} : -R_1 + R_2 \to R_2 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} : -2R_1 + R_3 \to R_3 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} :$$

$R_2 + R_3 \to R_3 : \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, the determinant is -1 (because of the row-interchange at the beginning).

Because the determinant does not equal 0, the matrix is invertible.

Page 175, Problem 22:

Use row-operations to reduce the matrix:

$$\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix} : R_1 \leftrightarrow R_2 : \begin{bmatrix} 1 & -3 & -2 \\ 5 & 0 & -1 \\ 0 & 5 & 3 \end{bmatrix} : -5R_1 + R_2 \to R_2 : \begin{bmatrix} 1 & -3 & -2 \\ 0 & 15 & 9 \\ 0 & 5 & 3 \end{bmatrix} :$$

$-\frac{1}{3}R_2 + R_3 \to R_3 : \begin{bmatrix} 1 & 3 & -2 \\ 0 & 15 & 9 \\ 0 & 0 & 0 \end{bmatrix}$. Because the determinant is equal to 0, the matrix is not invertible.

Page 176, Problem 37:

Straightforward calculation of $\det A = 3(1) - 0(6) = 3$ and $\det B = 2(4) - 0(5) = 8$ shows that

$(\det A)(\det B) = 3(8) = 24$. The determinant of the matrix product $AB = \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix}$ is

$\det AB = 6(4) - 0(17) = 24$. Thus, $24 = \det AB = (\det A)(\det B) = 24$.

Page 176, Problem 41:

Calculate each determinant: $\det A = (a+e)(d) - (b+f)(c) = ad + de - bc - cf = ad - bc + ed - fc$,

$\det B = ad - bc$, $\det C = ed - fc$. It is clear then that $\det A = \det B + \det C$