MATH 221, Fall 2016 - Homework 6 Solutions

Due Tuesday, October 18

Section 2.3

Page 115, Problem 2:

$$A = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$$

Notice that $\mathbf{a}_2 = -\frac{1}{2}\mathbf{a}_1$ where $\mathbf{a}_i$ is the column vector of the matrix A. Thus, the columns are linearly dependent. By Theorem 8 of this section, the **matrix is singular (nonivertible).** Also, notice that the determinant is equal to 0. So, by Theorem 4 of the previous section, the matrix is singular.

Page 115, Problem 4:

$$A = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix} \; A^T = \begin{bmatrix} -5 & 0 & 1 \\ 1 & 0 & 4 \\ 4 & 0 & 9 \end{bmatrix}$$

Notice that the columns of A^T are linearly dependent because the zero vector is a member of the set.

Thus, A^T is singular (noninvertible). Hence A is singular (nonivertible), by Theorem 8.

Also, because A contains a row of zeros, it cannot be reduced to the identity matrix.

Therefore, by Theorem 8, it is signular (noninvertible).

Page 115, Problem 8:

$$A = \begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the matrix is in echelon form, it is clear that there is a pivot in every row.

Hence, the matrix is invertible by Theorem 8.

Page 115, Problem 11a:

True or False: If the equation A$\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.

TRUE: Because (d) of Theorem 8 is true, (b) must also be true.

Page 115, Problem 11d:

True or False: If the equation A$\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

TRUE: Because (d) of Theorem 8 is false, (c) must also be false. An $n \times n$ matrix can never have more than n pivot positions, so it must have fewer than n.

Page 115, Problem 11e:

True or False: If A^T is not invertible, then A is not invertible.

TRUE: Because (l) of Theorem 8 is false, (a) must also be false.

Page 115, Problem 12a:

True or False: If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.

TRUE: Because (k) of Theorem 8 is true, (j) is also true. Because $AD = I$, $D = A^{-1}$, so $DA = A^{-1}A = I$.

Page 115, Problem 12b:

True or False: If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^{\mathrm{n}}$ into $\mathbb{R}^{\mathrm{n}}$, then the row reduced echelon form of A is I.

FALSE: In order for this to follow from Theorem 8, $\mathbf{x} \mapsto A\mathbf{x}$ must map $\mathbb{R}^{\mathrm{n}}$ **onto** $\mathbb{R}^{\mathrm{n}}$, not into.

Page 115, Problem 12c:

True or False: If the columns of A are linearly independent, then the columns of A span $\mathbb{R}^{\mathrm{n}}$.

TRUE: Because (e) of Theorem 8 is true, (h) must also be true.

Page 115, Problem 21:

Notice that on page 112, in the paragraph at the end of the page, it says (g) in Theorem 8 could be rewritten as

"The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^{\mathrm{n}}$."

In problem 21, this statement is false, thus (h) of Theorem 8 must also be false, so the columns of C **do not span** $\mathbb{R}^{\mathrm{n}}$.

Page 115, Problem 27:

Assume AB is invertible. Then, by Theorem 8(k) of this section, there exists an $n \times n$ matrix W such that $ABW = I$.

By properties of matrices (and because the order is defined), $ABW = A(BW) = I$.

Because A is square, let $BW = D$. Thus, by Theorem 8(k), A is invertible.

Page 115, Problem 33:

Let $T(\mathbf{x}) = A\mathbf{x}$. $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Because $\det A = 35 - 36 = -1$, A is invertible.

Thus, by Theorem 9 of this section, T is invertible.

$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. $A^{-1} = \frac{1}{-1}\begin{bmatrix} -7 & -9 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. Thus, $T^{-1}(\mathbf{x}) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$.

Page 115, Problem 39:

Because T maps $\mathbb{R}^{\mathrm{n}}$ onto $\mathbb{R}^{\mathrm{n}}$, then the standard matrix A is invertible, by Theorem 8 of this section.

Hence, by Theorem 9 of this section, T is invertible and A^{-1} is the standard matrix of T^{-1}.

Thus, by Theorem 8 of this section, the columns of A^{-1} are linearly independent and span $\mathbb{R}^{\mathrm{n}}$.

By Theorem 12 in Section 1.9, this shows that T^{-1} is a one-to-one mapping of $\mathbb{R}^{\mathrm{n}}$ onto $\mathbb{R}^{\mathrm{n}}$.