Ch. 2 First-order ODEs

2.1 Linear
2.2 Separable

} only concrete solution methods covered

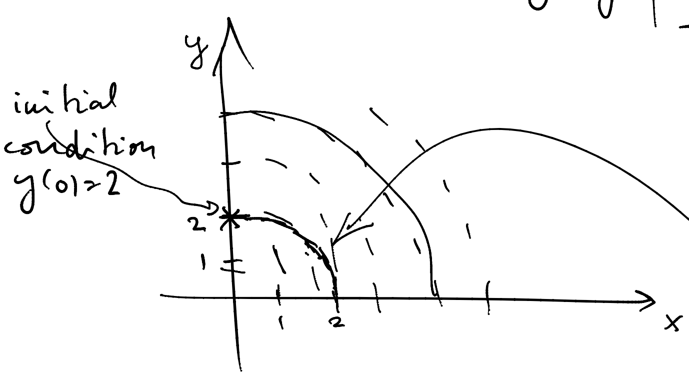
▷ More solution methods for certain types of ODEs in the exercises ▷

LRC tutoring is available for 225

2.7 Directionfields

HW 4 posted in Bb!

What is the meaning of $\boxed{y' = f(x,y)}$, $y(x_0) = y_0$?



Quiz last time:

$y' = -\frac{x}{y}$, $y(0) = 2$

$y(x) = \sqrt{4 - x^2}$

$\Rightarrow y' = \frac{1}{2}\frac{(-2x)}{\sqrt{4-x^2}} = -\frac{x}{y}$ ok.

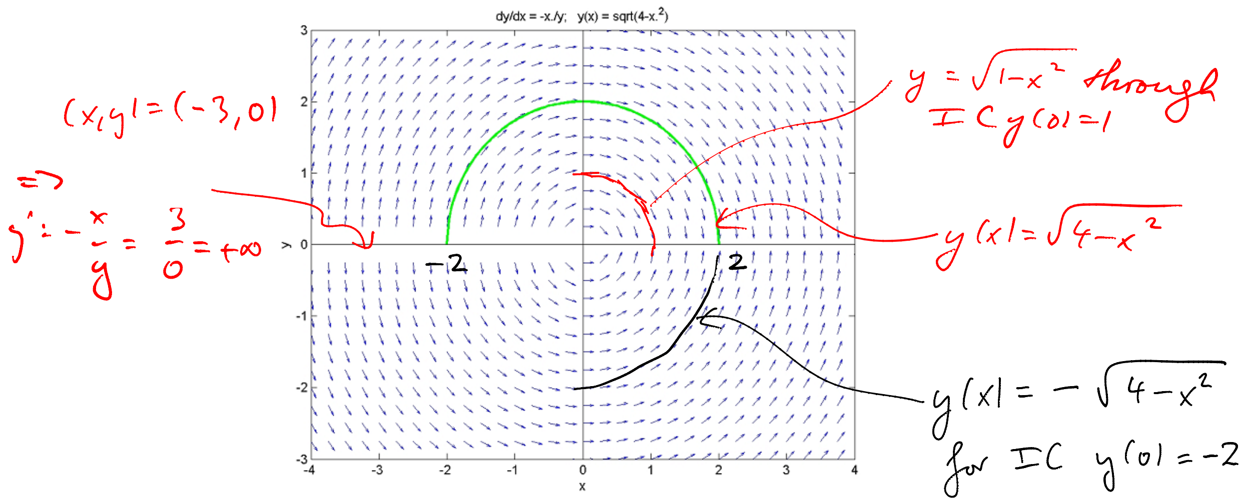
$y' = f$ mean that at a point (x,y) in the x-y-plane, the solution $y(x)$ has slope $y'(x) = f(x,y)$

Ex.: $(x,y) = (0,2) \Rightarrow y' = f(x,y) = -\frac{x}{y} = 0$

$(x,y) = (1,2) \Rightarrow y' = f(x,y) = -\frac{x}{y} = -\frac{1}{2}$

$(x,y) = (1,1) \Rightarrow y' = -\frac{1}{1} = -1$

The plot above is a direction field, that is, a collection of slopes for a number sample points (x,y) throughout the x-y-plane.



IC $y(0) = y_0$ with $y_0 < 0$ would pick $y(x) = -\sqrt{C - x^2} \rightarrow$ see quiz!

Since we have $y(x)$ explicitly, we can easily determine that its domain is all x with $4 - x^2 \geq 0 \iff 4 \geq x^2 \iff x^2 \leq 4$

$\iff |x| \leq 2 \iff -2 \leq x \leq 2$

Notice:

Notes:

① "Initial condition" is just the usual phrase for it; you often can actually extend the solution also to the left for $x < x_0$, i.e., backwards in time.

② The direction field can tell you the domain of the solution, even if you do not have an explicit solution: The domain of $y(x)$ needs to include the initial point!

The domain will go from there to some point in x where $y' = f$ becomes undefined, e.g., $f = \frac{1}{0}$ in this example.

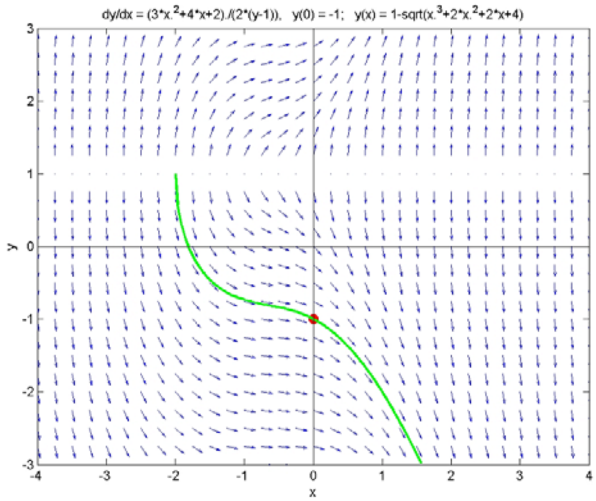
The direction field, in Farlow at beginning of Sec. 2.7 is on the one hand the actual meaning of ODE $y' = f(x,y)$ and on the other hand the connection to the general theory (Picard's theorem) in Sec. 1.2 and numerical methods as in Sec. 2.7, 2.8.

Example $y' = \dfrac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$

from last time

with $y(x) =$

$1 - \sqrt{x^3+2x^2+2x+4}$



the direction field seems to tell us that

$y(x)$ is valid for all $x \geq x'$, where x' is some number ≈ -2. We also see that $y(x) \to -\infty$ as $x \to \infty$.

We can conclude this just from the direction field, but use the known $y(x)$ to check:

$y(x) = 1 - \sqrt{\cdots}$ is defined for all x such that $x^3+2x^2+2x+4 \geq 0$.

Notice that this is in fact 0 for $x = -2$

$\Rightarrow$ Do polynomial division to find that

... to find that

$$x^3+2x^2+2x+4 = (x+2)(x^2+2)$$

Which is ≥ 0 for all $x \geq -2$

since $x^2+2 \geq 2 > 0$ for any x.

Also, notice that $x^3+\cdots \longrightarrow +\infty$ as $x \to \infty$

since the cubic term will dominate

over any x^2, x, etc. terms

$$\Rightarrow y(x) = 1 - \sqrt{x^3+\dots} \longrightarrow -\infty$$

Polynomial division:

$$\begin{array}{l} \quad (x^3+2x^2+2x+4) : (x+2) = x^2+2 \\ \underline{-(x^3+2x^2 \qquad\quad)} \\ \qquad\qquad 2x+4 \\ \qquad\quad \underline{-(2x+4)} \\ \qquad\qquad\quad 0 \end{array}$$

$0 \Leftrightarrow$ remainder 0 means exactly that $x^3+2x^2+2x+4=0$ for $x=-2$

This division means that

$$\frac{x^3+2x^2+2x+4}{x+2} = x^2+2$$

$$\Leftrightarrow x^3+2x^2+2x+4 = (x+2)(x^2+2)$$