Ch. 2 First-Order ODEs

2.1 Linear

2.2 Separable only concrete solution methods covered

A few solution methods for certain types of ODEs in the exercises

LRC tutoring is available in 225

2.7 Direction fields

HW 4 posted in P36 P

What is the meaning of \( y' = f(x, y) \), \( y(x_0) = y_0 \)?

Initial condition \( y(0) > 2 \)

Quick last time:

\[
y' = -\frac{x}{y}, \quad y(0) = 2
\]
\[
y(x) = \sqrt{4 - x^2}
\]
\[
\Rightarrow y' = \frac{1}{2\sqrt{4 - x^2}} = -\frac{x}{y} \text{ dh}
\]

\( y' = f \) mean that at a point \((x, y)\) in the \(x-y\)-plane, the solution \( y(x) \) has slope \( y'(x) = f(x, y) \).
The plot above is a direction field, that is, a collection of slopes for a family of sample points \((x, y)\) throughout the \(x-y\)-plane.

\(\dot{y} = f(x, y)\) for \(x = 0, 2\) \(\Rightarrow\) \(y' = f(x, y) = -\frac{x}{y} = 0\)
\((x, y) = (1, 2) \Rightarrow y' = f(x, y) = -\frac{x}{y} = -\frac{1}{2}\)
\((x, y) = (1, 1) \Rightarrow y' = -\frac{1}{y} = -1\)

**Initial Conditions**

**IC\( y(0) = 0, \quad y_0 < 0 \text{ would pick } y(x) = -\sqrt{4-x^2} \to \text{ we quit!}**

Since we have \(y(x)\) explicitly, we can easily determine that its domain is all \(x\) with \(4 - x^2 \geq 0 \iff 4 \geq x^2 \iff x^2 \leq 4 \iff |x| = \frac{1}{2}\).
(1) "Initial condition" is just the usual phrase for it; you often can actually extend the solution also to the left for $x < x_0$, i.e., backwards in time.

(2) The direction field can tell you the domain of the solution, even if you do not have an explicit solution: The domain of $y(x)$ needs to include the initial point. The domain will go from there to some point in $x$ where $y' = f$ becomes undefined, e.g., $f = \frac{1}{x}$ in this example.

The direction field, in Figure 2.7 at beginning of Sec. 2.7 is on the one hand the actual meaning of ODE $y' = f(x, y)$ and on the other hand the connection to the general theory (Lioard's theorem) in Sec. 1.2 and numerical methods.
as in Sec. 2.7, 2.8.

Example \[ y' = \frac{3x^2 + 4x + 2}{2(y-1)} \]

\[ y(0) = -1 \]

from last time

with \[ y(x) = \frac{1 - \sqrt{x^3 + 2x^2 + 2x + 4}}{1 - \sqrt{x^3 + 2x^2 + 2x + 4}} \]

The direction field seems to tell us that

\[ y(x) \text{ is valid for all } x \geq x', \text{ where } x' \text{ is some number } \approx -2. \]

We also see that \( y(x) \to -\infty \) as \( x \to \infty \).

We can conclude this just from the direction field, but we the known \( y(x) \) to check:

\[ y(x) = 1 - \sqrt{\cdots} \text{ is defined for all } x \text{ such that } x^3 + 2x^2 + 2x + 4 \geq 0. \]

Notice that this is in fact 0 for \( x = -2 \).

\[ y(-2) = 0 \]

Do polynomial division to find that

\[ 13, 0, 0, -1, 2, -2. \]
\[ x^3 + 2x^2 + 2x + 4 = (x+2)(x^2 + 2) \]

which is \( \geq 0 \) for all \( x \geq -2 \).

Since \( x^2 + 2 \geq 2 > 0 \) for any \( x \).

Also notice that \( x^3 \to +\infty \) as \( x \to \infty \)

since the cubic term will dominate over any \( x^2, x, \) etc. terms.

\[ y(x) = 1 - \sqrt{x^3 + 2} \to -\infty \]

**Polynomial division:**

\[
\begin{align*}
(x^3 + 2x^2 + 2x + 4) : (x+2) &= x^2 + 2 \\
- (x^3 + 2x^2) &
\end{align*}
\]

\[
\begin{align*}
2x + 4 \\
- (2x + 4)
\end{align*}
\]

\( 0 \) remainder \( 0 \) means exactly that \( x^3 + 2x^2 + 2x + 4 = 0 \) for \( x = -2 \).

This division means that

\[
\frac{x^3 + 2x^2 + 2x + 4}{x + 2} = x^2 + 2
\]

\( \Rightarrow \quad x^3 + 2x^2 + 2x + 4 = (x+2)(x^2 + 2) \)