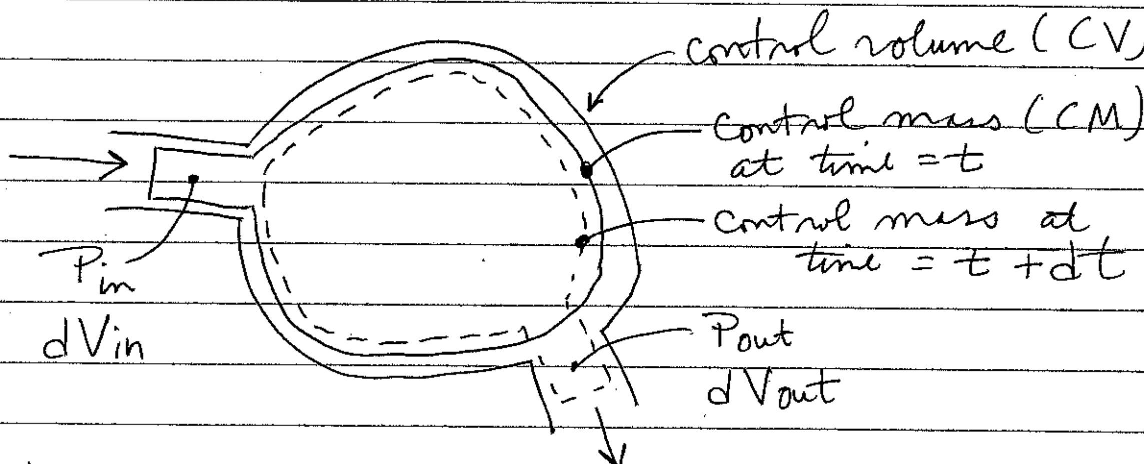
ENCH 630
The Reynolds Transport Theorem

The Reynold's transport theorem permits the conversion of a conservation law for a control mass (a closed system) to the equivalent law for a control volume (an open system).

Consider the following system and ignore conduction through the walls of the control volume:



P^t = any extensive property
P = corresponding intensive property
Therefore:

$$P^t_{CV} = \int_{CV} P\,dV \;;\; P^t_{CM} = \int_{CM} P\,dV$$

We have, from the figure, at time = t and at time = t+dt the following:

$$P^t_{CM}\Big|_t = P^t_{CV}\Big|_t + P_{in}\,dV_{in}$$

$$P^t_{CM}\Big|_{t+dt} = P^t_{CV}\Big|_{t+dt} + P_{out}\,dV_{out}$$

The Reynolds' Transport Theorem (cont.)

Subtract the two previous equations and divide by dt:

$$\frac{d\,P_{CM}^{t}}{dt} = \underbrace{\frac{d\,P_{CV}^{t}}{dt}}_{\substack{\text{accumulation}\\ \text{in control}\\ \text{volume}}} \underbrace{- P_{in}\left(\frac{dV}{dt}\right)_{in}}_{\substack{\text{inflow to}\\ \text{control}\\ \text{volume}}} \underbrace{+ P_{out}\left(\frac{dV}{dt}\right)_{out}}_{\substack{\text{outflow to}\\ \text{control}\\ \text{volume}}}$$

The right side of above equation can be interpreted as the production of P in the control volume since

production = accumulation + outflow − inflow

In general, if $\frac{dP_{CM}^{t}}{dt} = B$ for a control mass, then for a control volume this conservation law can be written as

Production of P in CV $= B$

Examples:

mass: $P_{CM}^{t} = m^{t}$; $B = 0$

momentum: $P_{CM}^{t} = (mv)^{t}$; $B = \Sigma$ Forces

When conduction through CV walls is included then theorem takes following form:

$$\text{If} \quad dP_{CM}^{t}/dt = B + \left(dP_{CM}^{t}/dt\right)_{\text{conduction}}$$

the "apparent" B that comes from conduction

then: Production of P including from conduction in CV $= B$

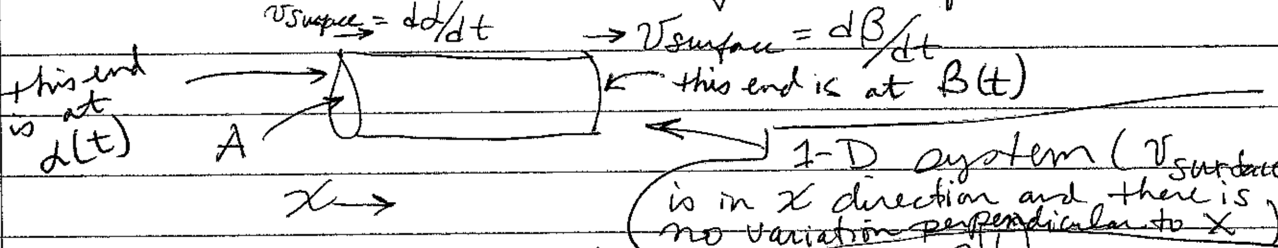
The Reynolds transport theorem (Cont.) – 3 –

Let us take a microscopic view, and start with the Leibniz formula for differentiating a volume integral:

$$\frac{d}{dt}\int_V \rho S \, dV = \int_V \frac{\partial \rho S}{\partial t} + \int_{S'} \rho S (\vec{v}_f \cdot \vec{n}) \, dS' \qquad (1)$$

where the surface of the volume under consideration moves at the local fluid velocity ($v_{surface} = v$) and S is any property per unit ~~volume~~ mass (an intensive property).

First, it is useful to derive the one dimensional Leibniz formula from the above equation. Let $\rho S = f(x,t)$ and we consider the following volume



This yields from Eq. 1:

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t) \cancel{A} \, dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x,t) \cancel{A} \, dx$$

velocity of upper limit of integration

$$+ f(\beta(t),t)\frac{d\beta}{dt}\cancel{A}$$

$$- f(\alpha(t),t)\frac{d\alpha}{dt}\cancel{A}$$

velocity of lower limit of integration

The above is the 1-D Leibniz formula

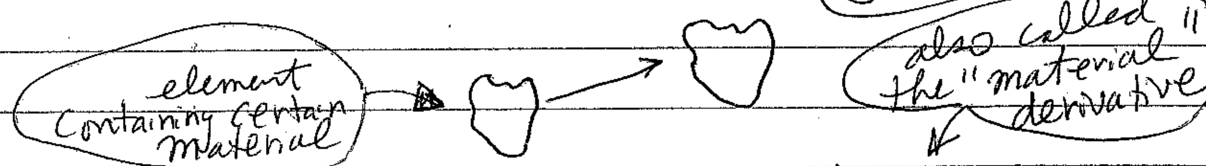
The Reynolds transport theorem (cont.)

Starting from Eq. 1, if you apply the product rule to the first term on the right side and Gauss's theorem to the second term on the right side (i.e., $\int_S \rho s (\vec{v}_f \cdot \vec{n}) dS = \int_V \vec{\nabla} \cdot \rho s \vec{v}_f dV$), and then apply the continuity equation in the form $\partial \rho / \partial t + \vec{\nabla} \cdot \rho \vec{v}_f = 0$, the right side of equation 1 can be written

$$\int_V \rho \frac{Ds}{Dt} dV$$

Note first that when the time derivatives are all inside the integral, it does not matter if the surface of the volume being considered is moving or not. Only the instantaneous position of the surface matters. Thus the above integral can be considered to apply to a stationary control volume whose surface does not move (with respect to the coordinates).

Now consider Ds/Dt



Ds/Dt is the substantial (i.e., Lagrangian) derivative, i.e., the derivative following fluid motion. It is the change in

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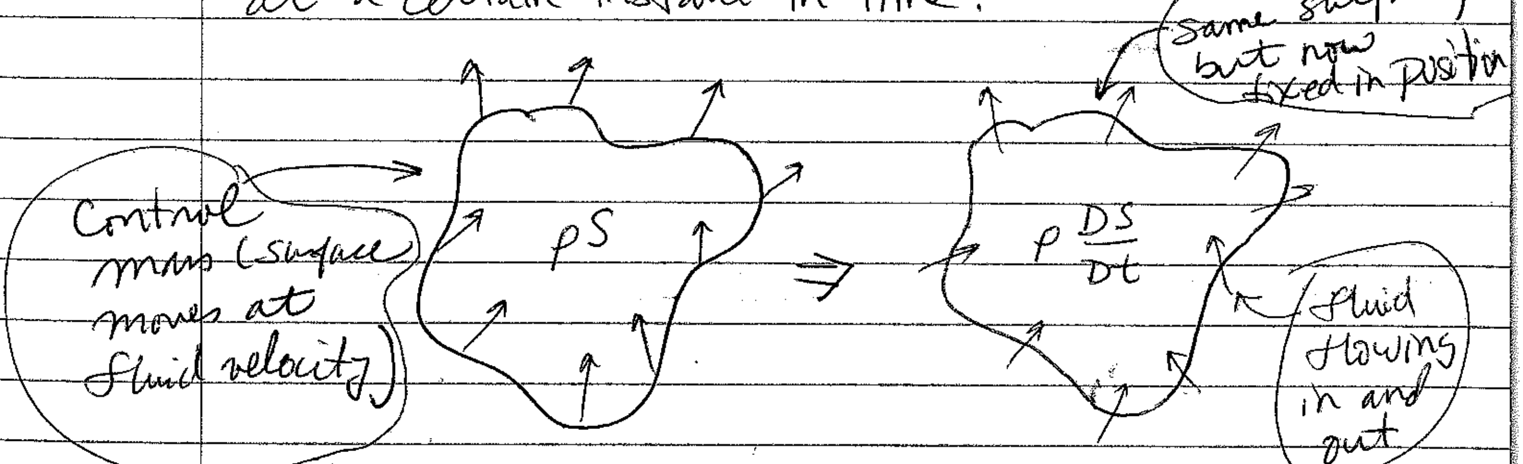
S (amount per unit mass) for a small mass element, the surface of which travels at the fluid velocity. Therefore, even if the element changes its volume (so the property becomes thereby diluted or concentrated by the volume change) S will not change. Only true "production" will cause S to change. Thus DS/Dt is the rate of production of S per unit mass so that

$$\int_V \rho \underbrace{\frac{DS}{Dt}}\, dV = \text{rate of production of } S \text{ for the volume being considered}$$

mass / volume

production of S per unit mass

We can then make the following correspondence at a certain instant in time.



$$\underbrace{\frac{d}{dt}\int_V \rho S\, dV}_{\text{control mass}} = \underbrace{\int_V \rho \frac{DS}{Dt}\, dV}_{\text{control volume, fixed in space (location)}}$$

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The left side of the previous equation is the change in a property for a control mass. If this is again equal to a parameter "B" by a conservation law, then B is also equal (according to the right side) to the production of the property S in the fixed control volume. According to ~~the~~ the definition of "production", B is also equal to the accumulation + outflow − inflow of B for the fixed position control volume.

Note finally that a correction for conduction must also be made here, similar to ~~that~~ the same correction for the macroscopic case. More specifically, for the control mass, an "apparent" B that comes from conduction needs to be subtracted ~~the~~ from the term $\frac{d}{dt}\int_{V} \rho S \, dV$ to yield a corrected B. ~~Then~~ Then, this corrected B equals the production of S in the control volume, where outflow − inflow by conduction is included in the determination of production.