There are n zeros of each function near the finite curve extending from $z=-n$ to $z=n$; the asymptotic expansions of these zeros for large n are given by the right side of **9.5.22** or **9.5.24** with $\nu=n$ and $\zeta=e^{-2\pi i/3}n^{-2/3}a_s$ or $\zeta=e^{-2\pi i/3}n^{-2/3}a'_s$.

Zeros of Cross-Products

If ν is real and λ is positive, the zeros of the function

9.5.27 $$J_\nu(z)Y_\nu(\lambda z)-J_\nu(\lambda z)Y_\nu(z)$$

are real and simple. If $\lambda>1$, the asymptotic expansion of the sth zero is

9.5.28 $$\beta+\frac{p}{\beta}+\frac{q-p^2}{\beta^3}+\frac{r-4pq+2p^3}{\beta^5}+\ldots$$

where with $4\nu^2$ denoted by μ,

9.5.29
$$\beta=s\pi/(\lambda-1)$$
$$p=\frac{\mu-1}{8\lambda},\qquad q=\frac{(\mu-1)(\mu-25)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu-1)(\mu^2-114\mu+1073)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros (not necessarily the sth) of the function

9.5.30 $$J'_\nu(z)Y'_\nu(\lambda z)-J'_\nu(\lambda z)Y'_\nu(z)\qquad(\lambda>1)$$

is given by **9.5.28** with the same value of β, but instead of **9.5.29** we have

9.5.31
$$p=\frac{\mu+3}{8\lambda},\qquad q=\frac{(\mu^2+46\mu-63)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu^3+185\mu^2-2053\mu+1899)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros of the function

9.5.32 $$J'_\nu(z)Y_\nu(\lambda z)-Y'_\nu(z)J_\nu(\lambda z)$$

is given by **9.5.28** with

9.5.33
$$\beta=(s-\tfrac{1}{2})\pi/(\lambda-1)$$
$$p=\frac{(\mu+3)\lambda-(\mu-1)}{8\lambda(\lambda-1)}$$
$$q=\frac{(\mu^2+46\mu-63)\lambda^3-(\mu-1)(\mu-25)}{6(4\lambda)^3(\lambda-1)}$$
$$5(4\lambda)^5(\lambda-1)r=(\mu^3+185\mu^2-2053\mu+1899)\lambda^5-(\mu-1)(\mu^2-114\mu+1073)$$

Modified Bessel Functions *I* and *K*

9.6. Definitions and Properties

Differential Equation

9.6.1 $$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)w=0$$

Solutions are $I_{\pm\nu}(z)$ and $K_\nu(z)$. Each is a regular function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq0)$ each is an entire function of ν. When $\nu=\pm n$, $I_\nu(z)$ is an entire function of z.

$I_\nu(z)$ $(\mathscr{R}\nu\geq0)$ is bounded as $z\to0$ in any bounded range of arg z. $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent except when ν is an integer. $K_\nu(z)$ tends to zero as $|z|\to\infty$ in the sector $|\arg z|<\frac{1}{2}\pi$, and for all values of ν, $I_\nu(z)$ and $K_\nu(z)$ are linearly independent. $I_\nu(z)$, $K_\nu(z)$ are real and positive when $\nu>-1$ and $z>0$.

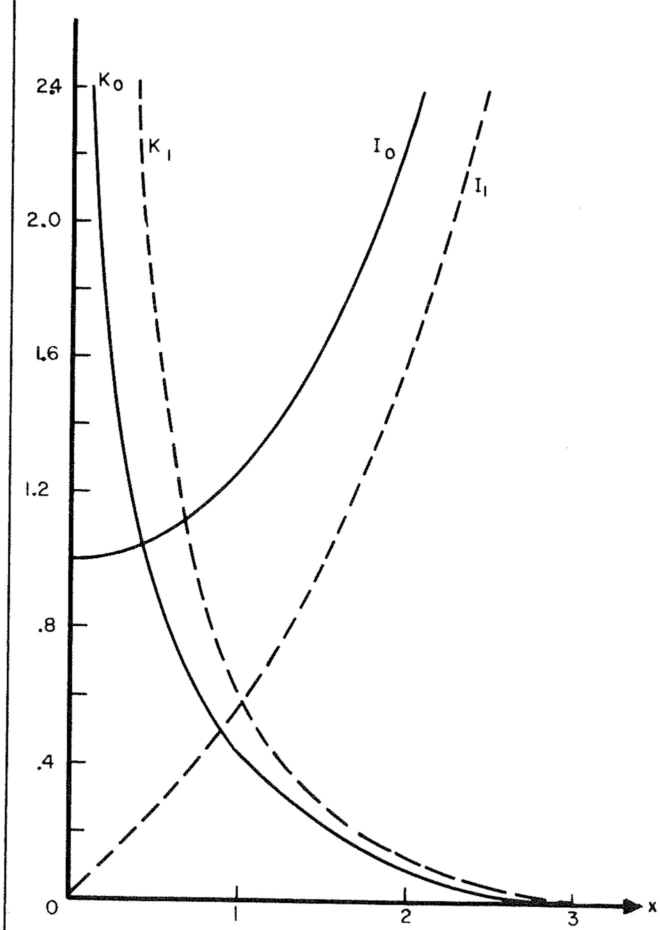

FIGURE 9.7. $I_0(x)$, $K_0(x)$, $I_1(x)$ *and* $K_1(x)$.

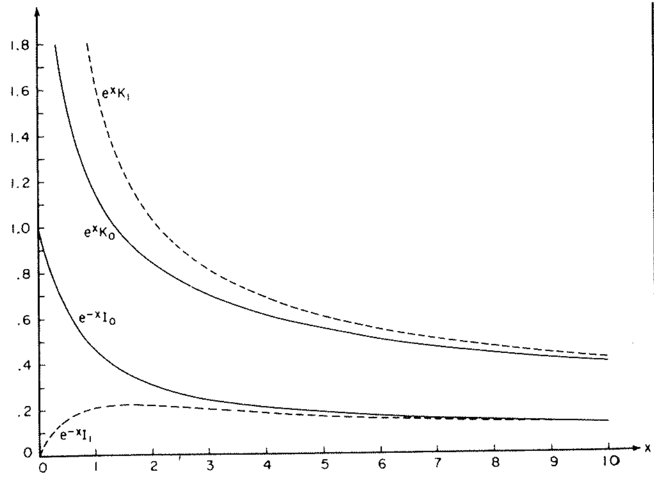

FIGURE 9.8. $e^{-x}I_0(x), e^{-x}I_1(x), e^xK_0(x)$ *and* $e^xK_1(x)$.

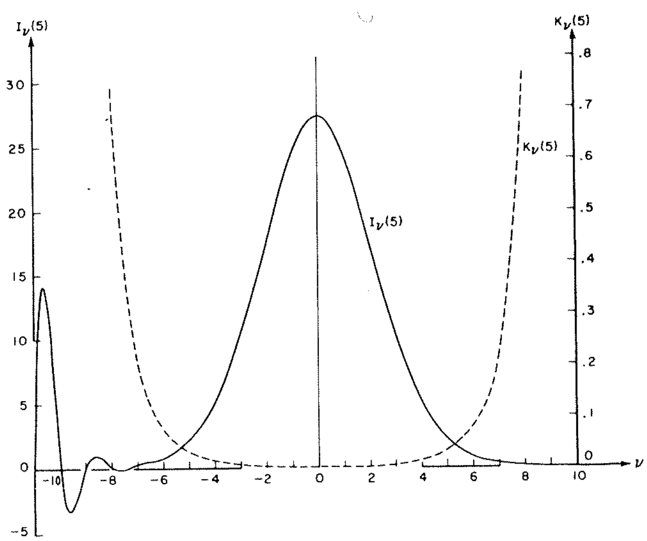

FIGURE 9.9. $I_\nu(5)$ *and* $K_\nu(5)$.

Relations Between Solutions

9.6.2 $$K_\nu(z)=\tfrac{1}{2}\pi\,\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.6.3

$$I_\nu(z)=e^{-\frac{1}{2}\nu\pi i}J_\nu(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\le\tfrac{1}{2}\pi)$$

$$I_\nu(z)=e^{3\nu\pi i/2}J_\nu(ze^{-3\pi i/2})\qquad(\tfrac{1}{2}\pi<\arg z\le\pi)$$

9.6.4

$$K_\nu(z)=\tfrac{1}{2}\pi ie^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\le\tfrac{1}{2}\pi)$$

$$K_\nu(z)=-\tfrac{1}{2}\pi ie^{-\frac{1}{2}\nu\pi i}H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i})\qquad(-\tfrac{1}{2}\pi<\arg z\le\pi)$$

9.6.5

$$Y_\nu(ze^{\frac{1}{2}\pi i})=e^{\frac{1}{2}(\nu+1)\pi i}I_\nu(z)-(2/\pi)e^{-\frac{1}{2}\nu\pi i}K_\nu(z)\qquad(-\pi<\arg z\le\tfrac{1}{2}\pi)$$

9.6.6 $$I_{-n}(z)=I_n(z),\ K_{-\nu}(z)=K_\nu(z)$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

Limiting Forms for Small Arguments

When ν is fixed and $z\to0$

9.6.7 $$I_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1)\qquad(\nu\neq-1,-2,\ldots)$$

9.6.8 $$K_0(z)\sim-\ln z$$

9.6.9 $$K_\nu(z)\sim\tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}\qquad(\mathscr{R}\nu>0)$$

Ascending Series

9.6.10 $$I_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}$$

9.6.11

$$K_n(z)=\tfrac{1}{2}(\tfrac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac{1}{4}z^2)^k+(-)^{n+1}\ln(\tfrac{1}{2}z)I_n(z)$$
$$+(-)^n\tfrac{1}{2}(\tfrac{1}{2}z)^n\sum_{k=0}^{\infty}\{\psi(k+1)+\psi(n+k+1)\}\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by **6.3.2.**

9.6.12 $$I_0(z)=1+\frac{\frac{1}{4}z^2}{(1!)^2}+\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

9.6.13

$$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+\frac{\frac{1}{4}z^2}{(1!)^2}+(1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

Wronskians

9.6.14

$$W\{I_\nu(z),I_{-\nu}(z)\}=I_\nu(z)I_{-(\nu+1)}(z)-I_{\nu+1}(z)I_{-\nu}(z)=-2\sin(\nu\pi)/(\pi z)$$

9.6.15

$$W\{K_\nu(z),I_\nu(z)\}=I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=1/z$$

Integral Representations

9.6.16
$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}d\theta=\frac{1}{\pi}\int_0^\pi \cosh(z\cos\theta)d\theta$$

9.6.17 $$K_0(z)=-\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}\{\gamma+\ln(2z\sin^2\theta)\}d\theta$$

9.6.18
$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta$$
$$=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}e^{\pm zt}dt \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

9.6.19 $$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)d\theta$$

9.6.20
$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)d\theta$$
$$-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.21
$$K_0(x)=\int_0^\infty \cos(x\sinh t)dt=\int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}dt \qquad (x>0)$$

9.6.22
$$K_\nu(x)=\sec(\tfrac{1}{2}\nu\pi)\int_0^\infty \cos(x\sinh t)\cosh(\nu t)dt$$
$$=\csc(\tfrac{1}{2}\nu\pi)\int_0^\infty \sin(x\sinh t)\sinh(\nu t)dt \qquad (|\mathscr{R}\nu|<1,\ x>0)$$

9.6.23
$$K_\nu(z)=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu}t\,dt$$
$$=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}dt \qquad (\mathscr{R}\nu>-\tfrac{1}{2},\ |\arg z|<\tfrac{1}{2}\pi)$$

9.6.24 $$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt \quad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.25
$$K_\nu(xz)=\frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\pi^{\frac{1}{2}}x^\nu}\int_0^\infty \frac{\cos(xt)dt}{(t^2+z^2)^{\nu+\frac{1}{2}}} \qquad (\mathscr{R}\nu>-\tfrac{1}{2},\ x>0,\ |\arg z|<\tfrac{1}{2}\pi)^*$$

Recurrence Relations

9.6.26
$$\mathscr{Z}_{\nu-1}(z)-\mathscr{Z}_{\nu+1}(z)=\frac{2\nu}{z}\mathscr{Z}_\nu(z)$$
$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu-1}(z)-\frac{\nu}{z}\mathscr{Z}_\nu(z)$$
$$\mathscr{Z}_{\nu-1}(z)+\mathscr{Z}_{\nu+1}(z)=2\mathscr{Z}'_\nu(z)$$
$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu+1}(z)+\frac{\nu}{z}\mathscr{Z}_\nu(z)$$

$\mathscr{Z}_\nu$ denotes I_ν, $e^{\nu\pi i}K_\nu$ or any linear combination of these functions, the coefficients in which are independent of z and ν.

9.6.27 $$I_0'(z)=I_1(z), \qquad K_0'(z)=-K_1(z)$$

Formulas for Derivatives

9.6.28
$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^\nu\mathscr{Z}_\nu(z)\}=z^{\nu-k}\mathscr{Z}_{\nu-k}(z)$$
$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^{-\nu}\mathscr{Z}_\nu(z)\}=z^{-\nu-k}\mathscr{Z}_{\nu+k}(z) \qquad (k=0,1,2,\ldots)$$

9.6.29
$$\mathscr{Z}_\nu^{(k)}(z)=\frac{1}{2^k}\left\{\mathscr{Z}_{\nu-k}(z)+\binom{k}{1}\mathscr{Z}_{\nu-k+2}(z)+\binom{k}{2}\mathscr{Z}_{\nu-k+4}(z)+\ldots+\mathscr{Z}_{\nu+k}(z)\right\} \qquad (k=0,1,2,\ldots)$$

Analytic Continuation

9.6.30 $$I_\nu(ze^{m\pi i})=e^{m\nu\pi i}I_\nu(z) \qquad (m \text{ an integer})$$

9.6.31
$$K_\nu(ze^{m\pi i})=e^{-m\nu\pi i}K_\nu(z)-\pi i\sin(m\nu\pi)\csc(\nu\pi)I_\nu(z) \qquad (m \text{ an integer})$$

9.6.32 $$I_\nu(\bar{z})=\overline{I_\nu(z)}, \qquad K_\nu(\bar{z})=\overline{K_\nu(z)} \qquad (\nu \text{ real})$$

Generating Function and Associated Series

9.6.33 $$e^{\frac{1}{2}z(t+1/t)}=\sum_{k=-\infty}^{\infty} t^k I_k(z) \qquad (t\neq 0)$$

9.6.34 $$e^{z\cos\theta}=I_0(z)+2\sum_{k=1}^{\infty} I_k(z)\cos(k\theta)$$

9.6.35
$$e^{z\sin\theta}=I_0(z)+2\sum_{k=0}^{\infty}(-)^k I_{2k+1}(z)\sin\{(2k+1)\theta\}+2\sum_{k=1}^{\infty}(-)^k I_{2k}(z)\cos(2k\theta)$$

9.6.36 $$1=I_0(z)-2I_2(z)+2I_4(z)-2I_6(z)+\ldots$$

9.6.37 $$e^z=I_0(z)+2I_1(z)+2I_2(z)+2I_3(z)+\ldots$$

9.6.38 $$e^{-z}=I_0(z)-2I_1(z)+2I_2(z)-2I_3(z)+\ldots$$

9.6.39
$$\cosh z=I_0(z)+2I_2(z)+2I_4(z)+2I_6(z)+\ldots$$

9.6.40 $$\sinh z=2I_1(z)+2I_3(z)+2I_5(z)+\ldots$$

*See page II.

Other Differential Equations

The quantity λ^2 in equations **9.1.49** to **9.1.54** and **9.1.56** can be replaced by $-\lambda^2$ if at the same time the symbol $\mathscr{C}$ in the given solutions is replaced by $\mathscr{Z}$.

9.6.41

$$z^2w''+z(1\pm 2z)w'+(\pm z-\nu^2)w=0, \qquad w=e^{\mp z}\mathscr{Z}_\nu(z)$$

Differential equations for products may be obtained from **9.1.57** to **9.1.59** by replacing z by iz.

Derivatives With Respect to Order

9.6.42

$$\frac{\partial}{\partial\nu}I_\nu(z)=I_\nu(z)\ln(\tfrac{1}{2}z)-(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!}$$

9.6.43

$$\frac{\partial}{\partial\nu}K_\nu(z)=\tfrac{1}{2}\pi\csc(\nu\pi)\left\{\frac{\partial}{\partial\nu}I_{-\nu}(z)-\frac{\partial}{\partial\nu}I_\nu(z)\right\}$$
$$-\pi\cot(\nu\pi)K_\nu(z) \qquad (\nu\neq 0,\pm 1,\pm 2,\ldots)$$

9.6.44

$$(-)^n\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=n}=$$
$$-K_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}(-)^k\frac{(\frac{1}{2}z)^kI_k(z)}{(n-k)k!}$$

9.6.45

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=n}=\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kK_k(z)}{(n-k)k!}$$

9.6.46

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=0}=-K_0(z), \qquad \left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=0}=0$$

Expressions in Terms of Hypergeometric Functions

9.6.47

$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1;\tfrac{1}{4}z^2)$$
$$=\frac{(\frac{1}{2}z)^\nu e^{-z}}{\Gamma(\nu+1)}M(\nu+\tfrac{1}{2},2\nu+1,2z)=\frac{z^{-\frac{1}{2}}M_{0,\nu}(2z)}{2^{2\nu+\frac{1}{2}}\Gamma(\nu+1)}$$

9.6.48 $$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}W_{0,\nu}(2z)$$

(${}_0F_1$ is the generalized hypergeometric function. For $M(a,b,z)$, $M_{0,\nu}(z)$ and $W_{0,\nu}(z)$ see chapter **13**.)

Connection With Legendre Functions

If μ and z are fixed, $\mathscr{R}z>0$, and $\nu\to\infty$ through real positive values

9.6.49 $$\lim\left\{\nu^\mu P_\nu^{-\mu}\left(\cosh\frac{z}{\nu}\right)\right\}=I_\mu(z)$$

9.6.50 $$\lim\left\{\nu^{-\mu}e^{-\mu\pi i}Q_\nu^\mu\left(\cosh\frac{z}{\nu}\right)\right\}=K_\mu(z)$$

For the definition of $P_\nu^{-\mu}$ and Q_ν^μ, see chapter **8.**

Multiplication Theorems

9.6.51

$$\mathscr{Z}_\nu(\lambda z)=\lambda^{\pm\nu}\sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k(\frac{1}{2}z)^k}{k!}\mathscr{Z}_{\nu\pm k}(z) \qquad (|\lambda^2-1|<1)$$

If $\mathscr{Z}=I$ and the upper signs are taken, the restriction on λ is unnecessary.

9.6.52

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}J_{\nu+k}(z), \qquad J_\nu(z)=\sum_{k=0}^{\infty}(-)^k\frac{z^k}{k!}I_{\nu+k}(z)$$

Neumann Series for $K_n(z)$

9.6.53

$$K_n(z)=(-)^{n-1}\{\ln(\tfrac{1}{2}z)-\psi(n+1)\}I_n(z)$$
$$+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}(-)^k\frac{(\frac{1}{2}z)^kI_k(z)}{(n-k)k!}$$
$$+(-)^n\sum_{k=1}^{\infty}\frac{(n+2k)I_{n+2k}(z)}{k(n+k)}$$

9.6.54 $$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+2\sum_{k=1}^{\infty}\frac{I_{2k}(z)}{k}$$

Zeros

Properties of the zeros of $I_\nu(z)$ and $K_\nu(z)$ may be deduced from those of $J_\nu(z)$ and $H_\nu^{(1)}(z)$ respectively, by application of the transformations **9.6.3** and **9.6.4.**

For example, if ν is real the zeros of $I_\nu(z)$ are all complex unless $-2k<\nu<-(2k-1)$ for some positive integer k, in which event $I_\nu(z)$ has two real zeros.

The approximate distribution of the zeros of $K_n(z)$ in the region $-\frac{3}{2}\pi\leq\arg z\leq\frac{1}{2}\pi$ is obtained on rotating **Figure 9.6** through an angle $-\frac{1}{2}\pi$ so that the cut lies along the positive imaginary axis. The zeros in the region $-\frac{1}{2}\pi\leq\arg z\leq\frac{3}{2}\pi$ are their conjugates. $K_n(z)$ has no zeros in the region $|\arg z|\leq\frac{1}{2}\pi$; this result remains true when n is replaced by any real number ν.

9.7. Asymptotic Expansions

Asymptotic Expansions for Large Arguments

When ν is fixed, $|z|$ is large and $\mu=4\nu^2$

9.7.1

$$I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left\{1-\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}\right.$$
$$\left.-\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3}+\ldots\right\} \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.7.2

$$K_\nu(z)\sim\sqrt{\frac{\pi}{2z}}\,e^{-z}\{1+\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}+\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3}+\ldots\}\qquad(|\arg z|<\tfrac{3}{2}\pi)$$

9.7.3

$$I'_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\{1-\frac{\mu+3}{8z}+\frac{(\mu-1)(\mu+15)}{2!(8z)^2}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots\}\qquad(|\arg z|<\tfrac{1}{2}\pi)$$

9.7.4

$$K'_\nu(z)\sim-\sqrt{\frac{\pi}{2z}}e^{-z}\{1+\frac{\mu+3}{8z}+\frac{(\mu-1)(\mu+15)}{2!(8z)^2}+\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots\}\qquad(|\arg z|<\tfrac{3}{2}\pi)$$

The general terms in the last two expansions can be written down by inspection of **9.2.15** and **9.2.16.**

If ν is real and non-negative and z is positive the remainder after k terms in the expansion **9.7.2** does not exceed the $(k+1)$th term in absolute value and is of the same sign, provided that $k\geq\nu-\frac{1}{2}$.

9.7.5

$$I_\nu(z)K_\nu(z)\sim\frac{1}{2z}\{1-\frac{1}{2}\frac{\mu-1}{(2z)^2}+\frac{1\cdot3}{2\cdot4}\frac{(\mu-1)(\mu-9)}{(2z)^4}-\ldots\}\qquad(|\arg z|<\tfrac{1}{2}\pi)$$

9.7.6

$$I'_\nu(z)K'_\nu(z)\sim-\frac{1}{2z}\{1+\frac{1}{2}\frac{\mu-3}{(2z)^2}-\frac{1\cdot1}{2\cdot4}\frac{(\mu-1)(\mu-45)}{(2z)^4}+\ldots\}\qquad(|\arg z|<\tfrac{1}{2}\pi)$$

The general terms can be written down by inspection of **9.2.28** and **9.2.30.**

Uniform Asymptotic Expansions for Large Orders

9.7.7
$$I_\nu(\nu z)\sim\frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta}}{(1+z^2)^{1/4}}\{1+\sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\}$$

9.7.8

$$K_\nu(\nu z)\sim\sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{1/4}}\{1+\sum_{k=1}^{\infty}(-)^k\frac{u_k(t)}{\nu^k}\}$$

9.7.9
$$I'_\nu(\nu z)\sim\frac{1}{\sqrt{2\pi\nu}}\frac{(1+z^2)^{1/4}}{z}e^{\nu\eta}\{1+\sum_{k=1}^{\infty}\frac{v_k(t)}{\nu^k}\}$$

9.7.10

$$K'_\nu(\nu z)\sim-\sqrt{\frac{\pi}{2\nu}}\frac{(1+z^2)^{1/4}}{z}e^{-\nu\eta}\{1+\sum_{k=1}^{\infty}(-)^k\frac{v_k(t)}{\nu^k}\}$$

When $\nu\to+\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z|\leq\frac{1}{2}\pi-\epsilon$, where ϵ is an arbitrary positive number. Here

9.7.11
$$t=1/\sqrt{1+z^2},\qquad \eta=\sqrt{1+z^2}+\ln\frac{z}{1+\sqrt{1+z^2}}$$

and $u_k(t)$, $v_k(t)$ are given by **9.3.9, 9.3.10, 9.3.13** and **9.3.14**. See [9.38] for tables of η, $u_k(t)$, $v_k(t)$, and also for bounds on the remainder terms in **9.7.7** to **9.7.10**.

9.8. Polynomial Approximations [4]

In equations **9.8.1** to **9.8.4**, $t=x/3.75$.

9.8.1 $\qquad -3.75\leq x\leq3.75$

$$I_0(x)=1+3.51562\,29t^2+3.08994\,24t^4+1.20674\,92t^6+.26597\,32t^8+.03607\,68t^{10}+.00458\,13t^{12}+\epsilon$$
$$|\epsilon|<1.6\times10^{-7}$$

9.8.2 $\qquad 3.75\leq x<\infty$

$$x^{\frac{1}{2}}e^{-x}I_0(x)=.39894\,228+.01328\,592t^{-1}+.00225\,319t^{-2}-.00157\,565t^{-3}+.00916\,281t^{-4}-.02057\,706t^{-5}+.02635\,537t^{-6}-.01647\,633t^{-7}+.00392\,377t^{-8}+\epsilon$$
$$|\epsilon|<1.9\times10^{-7}$$

9.8.3 $\qquad -3.75\leq x\leq3.75$

$$x^{-1}I_1(x)=\tfrac{1}{2}+.87890\,594t^2+.51498\,869t^4+.15084\,934t^6+.02658\,733t^8+.00301\,532t^{10}+.00032\,411t^{12}+\epsilon$$
$$|\epsilon|<8\times10^{-9}$$

9.8.4 $\qquad 3.75\leq x<\infty$

$$x^{\frac{1}{2}}e^{-x}I_1(x)=.39894\,228-.03988\,024t^{-1}-.00362\,018t^{-2}+.00163\,801t^{-3}-.01031\,555t^{-4}+.02282\,967t^{-5}-.02895\,312t^{-6}+.01787\,654t^{-7}-.00420\,059t^{-8}+\epsilon$$
$$|\epsilon|<2.2\times10^{-7}$$

[4] See footnote 2, section **9.4**.