# 6. Gamma Function and Related Functions

## Mathematical Properties

### 6.1. Gamma (Factorial) Function

**Euler's Integral**

**6.1.1** $$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt \qquad (\mathscr{R}z>0)$$

$$=k^z\int_0^\infty t^{z-1}e^{-kt}\,dt \qquad (\mathscr{R}z>0,\ \mathscr{R}k>0)$$

**Euler's Formula**

**6.1.2**
$$\Gamma(z)=\lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)} \qquad (z\neq 0,-1,-2,\ldots)$$

**Euler's Infinite Product**

**6.1.3** $$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right] \qquad (|z|<\infty)$$

$$\gamma=\lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{m}-\ln m\right]$$

$$=.57721\ 56649\ldots$$

$\gamma$ is known as Euler's constant and is given to 25 decimal places in chapter 1. $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z=-n(n=0,\ 1,\ 2,\ \ldots)$ where it possesses simple poles with residue $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z=-n(n=0,\ 1,\ 2,\ \ldots)$.

**Hankel's Contour Integral**

**6.1.4** $$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_C (-t)^{-z}e^{-t}\,dt \qquad (|z|<\infty)$$

The path of integration $C$ starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

**Factorial and Π Notations**

**6.1.5** $$\Pi(z)=z!=\Gamma(z+1)$$

**Integer Values**

**6.1.6** $$\Gamma(n+1)=1\cdot2\cdot3\ldots(n-1)n=n!$$

**6.1.7**
$$\lim_{z\to n}\frac{1}{\Gamma(-z)}=0=\frac{1}{(-n-1)!} \qquad (n=0,\ 1,\ 2,\ \ldots)$$

**Fractional Values**

**6.1.8**
$$\Gamma(\tfrac{1}{2})=2\int_0^\infty e^{-t^2}\,dt=\pi^{\frac{1}{2}}=1.77245\ 38509\ldots=(-\tfrac{1}{2})!$$

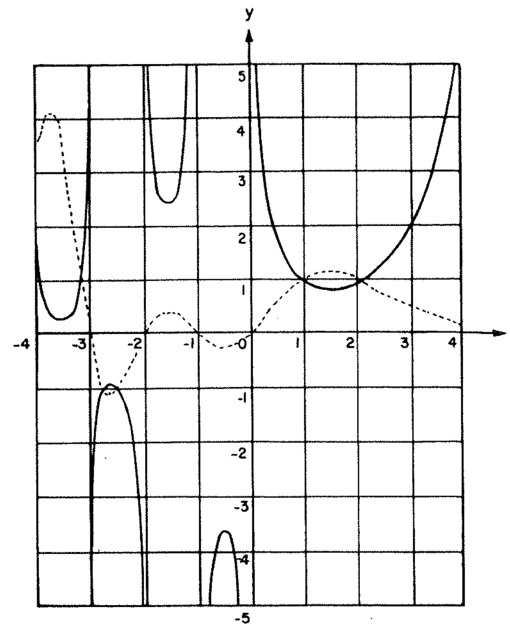


FIGURE 6.1. *Gamma function.* *

———, $y=\Gamma(x)$, – – – –, $y=1/\Gamma(x)$

**6.1.9** $$\Gamma(3/2)=\tfrac{1}{2}\pi^{\frac{1}{2}}=.88622\ 69254\ldots=(\tfrac{1}{2})!$$

**6.1.10** $$\Gamma(n+\tfrac{1}{4})=\frac{1\cdot5\cdot9\cdot13\ldots(4n-3)}{4^n}\Gamma(\tfrac{1}{4})$$

$$\Gamma(\tfrac{1}{4})=3.62560\ 99082\ldots$$

**6.1.11** $$\Gamma(n+\tfrac{1}{3})=\frac{1\cdot4\cdot7\cdot10\ldots(3n-2)}{3^n}\Gamma(\tfrac{1}{3})$$

$$\Gamma(\tfrac{1}{3})=2.67893\ 85347\ldots$$

**6.1.12** $$\Gamma(n+\tfrac{1}{2})=\frac{1\cdot3\cdot5\cdot7\ldots(2n-1)}{2^n}\Gamma(\tfrac{1}{2})$$

**6.1.13** $$\Gamma(n+\tfrac{2}{3})=\frac{2\cdot5\cdot8\cdot11\ldots(3n-1)}{3^n}\Gamma(\tfrac{2}{3})$$

$$\Gamma(\tfrac{2}{3})=1.35411\ 79394\ldots$$

**6.1.14** $$\Gamma(n+\tfrac{3}{4})=\frac{3\cdot7\cdot11\cdot15\ldots(4n-1)}{4^n}\Gamma(\tfrac{3}{4})$$

$$\Gamma(\tfrac{3}{4})=1.22541\ 67024\ldots$$

*See page II.

**Recurrence Formulas**

**6.1.15** $\Gamma(z+1)=z\Gamma(z)=z!=z(z-1)!$

**6.1.16**

$$\Gamma(n+z)=(n-1+z)(n-2+z)\ldots(1+z)\Gamma(1+z)$$
$$=(n-1+z)!$$
$$=(n-1+z)(n-2+z)\ldots(1+z)z!$$

**Reflection Formula**

**6.1.17** $\Gamma(z)\Gamma(1-z)=-z\Gamma(-z)\Gamma(z)=\pi\csc\pi z$

$$=\int_0^\infty \frac{t^{z-1}}{1+t}\,dt \qquad (0<\mathscr{R}z<1)$$

**Duplication Formula**

**6.1.18** $\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$

**Triplication Formula**

**6.1.19** $\Gamma(3z)=(2\pi)^{-1}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{3})\Gamma(z+\tfrac{2}{3})$

**Gauss' Multiplication Formula**

**6.1.20** $\Gamma(nz)=(2\pi)^{\frac{1}{2}(1-n)}n^{nz-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)$

**Binomial Coefficient**

**6.1.21** $\binom{z}{w}=\frac{z!}{w!(z-w)!}=\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$

**Pochhammer's Symbol**

**6.1.22**

$(z)_0=1,$

$(z)_n=z(z+1)(z+2)\ldots(z+n-1)=\frac{\Gamma(z+n)}{\Gamma(z)}$

**Gamma Function in the Complex Plane**

**6.1.23** $\Gamma(\bar{z})=\overline{\Gamma(z)};\ \ln\Gamma(\bar{z})=\overline{\ln\Gamma(z)}$

**6.1.24** $\arg\Gamma(z+1)=\arg\Gamma(z)+\arctan\frac{y}{x}$

**6.1.25** $\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2=\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(x+n)^2}\right]^{-1}$

**6.1.26** $|\Gamma(x+iy)|\leq|\Gamma(x)|$

**6.1.27**

$$\arg\Gamma(x+iy)=y\psi(x)+\sum_{n=0}^{\infty}\left(\frac{y}{x+n}-\arctan\frac{y}{x+n}\right)$$
$$(x+iy\neq 0,-1,-2,\ldots)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$

**6.1.28** $\Gamma(1+iy)=iy\,\Gamma(iy)$

**6.1.29** $\Gamma(iy)\Gamma(-iy)=|\Gamma(iy)|^2=\frac{\pi}{y\sinh\pi y}$

**6.1.30** $\Gamma(\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}-iy)=|\Gamma(\tfrac{1}{2}+iy)|^2=\frac{\pi}{\cosh\pi y}$

**6.1.31** $\Gamma(1+iy)\Gamma(1-iy)=|\Gamma(1+iy)|^2=\frac{\pi y}{\sinh\pi y}$

**6.1.32** $\Gamma(\tfrac{1}{4}+iy)\Gamma(\tfrac{3}{4}-iy)=\frac{\pi\sqrt{2}}{\cosh\pi y+i\sinh\pi y}$

**Power Series**

**6.1.33**

$$\ln\Gamma(1+z)=-\ln(1+z)+z(1-\gamma)$$
$$+\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]z^n/n \qquad (|z|<2)$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter 23).

**Series Expansion[2] for $1/\Gamma(z)$**

**6.1.34** $\frac{1}{\Gamma(z)}=\sum_{k=1}^{\infty}c_k z^k \qquad (|z|<\infty)$

| $k$ | $c_k$ |
|---|---|
| 1 | 1.00000 00000 000000 |
| 2 | 0.57721 56649 015329 |
| 3 | −0.65587 80715 202538 |
| 4 | −0.04200 26350 340952 |
| 5 | 0.16653 86113 822915 |
| 6 | −0.04219 77345 555443 |
| 7 | −0.00962 19715 278770 |
| 8 | 0.00721 89432 466630 |
| 9 | −0.00116 51675 918591 |
| 10 | −0.00021 52416 741149 |
| 11 | 0.00012 80502 823882 |
| 12 | −0.00002 01348 547807 |
| 13 | −0.00000 12504 934821 |
| 14 | 0.00000 11330 272320 |
| 15 | −0.00000 02056 338417 |
| 16 | 0.00000 00061 160950 |
| 17 | 0.00000 00050 020075 |
| 18 | −0.00000 00011 812746 |
| 19 | 0.00000 00001 043427 |
| 20 | 0.00000 00000 077823 |
| 21 | −0.00000 00000 036968 |
| 22 | 0.00000 00000 005100 |
| 23 | −0.00000 00000 000206 |
| 24 | −0.00000 00000 000054 |
| 25 | 0.00000 00000 000014 |
| 26 | 0.00000 00000 000001 |

[2] The coefficients $c_k$ are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.

**Polynomial Approximations**[3]

**6.1.35** $0\leq x\leq 1$

$$\Gamma(x+1)=x!=1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\epsilon(x)$$

$$|\epsilon(x)|\leq 5\times 10^{-5}$$

$a_1=-.57486\ 46$ $\quad a_4=\ .42455\ 49$
$a_2=\ .95123\ 63$ $\quad a_5=-.10106\ 78$
$a_3=-.69985\ 88$

**6.1.36** $0\leq x\leq 1$

$$\Gamma(x+1)=x!=1+b_1x+b_2x^2+\ .\ .\ .\ +b_8x^8+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-7}$$

$b_1=-.57719\ 1652$ $\quad b_5=-.75670\ 4078$
$b_2=\ .98820\ 5891$ $\quad b_6=\ .48219\ 9394$
$b_3=-.89705\ 6937$ $\quad b_7=-.19352\ 7818$
$b_4=\ .91820\ 6857$ $\quad b_8=\ .03586\ 8343$

**Stirling's Formula**

**6.1.37**

$$\Gamma(z)\sim e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left[1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+\cdots\right]\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

**6.1.38**

$$x!=\sqrt{2\pi}\,x^{x+\frac{1}{2}}\exp\left(-x+\frac{\theta}{12x}\right)\qquad (x>0,\ 0<\theta<1)$$

**Asymptotic Formulas**

**6.1.39**

$$\Gamma(az+b)\sim\sqrt{2\pi}\,e^{-az}(az)^{az+b-\frac{1}{2}}\qquad (|\arg z|<\pi,\ a>0)$$

**6.1.40**

$$\ln\Gamma(z)\sim(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)z^{2m-1}}\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

For $B_n$ see chapter **23**

**6.1.41**

$$\ln\Gamma(z)\sim(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}-\frac{1}{1680z^7}+\cdots\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

[3] From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

**Error Term for Asymptotic Expansion**

**6.1.42**

If

$$R_n(z)=\ln\Gamma(z)-(z-\tfrac{1}{2})\ln z+z-\tfrac{1}{2}\ln(2\pi)-\sum_{m=1}^{n}\frac{B_{2m}}{2m(2m-1)z^{2m-1}}$$

then

$$|R_n(z)|\leq\frac{|B_{2n+2}|K(z)}{(2n+1)(2n+2)|z|^{2n+1}}$$

where

$$K(z)=\underset{u\geq 0}{\text{upper bound}}|z^2/(u^2+z^2)|$$

For $z$ real and positive, $R_n$ is less in absolute value than the first term neglected and has the same sign.

**6.1.43**

$$\mathscr{R}\ln\Gamma(iy)=\mathscr{R}\ln\Gamma(-iy)=\tfrac{1}{2}\ln\left(\frac{\pi}{y\sinh\pi y}\right)\sim\tfrac{1}{2}\ln(2\pi)-\tfrac{1}{2}\pi y-\tfrac{1}{2}\ln y,\qquad (y\to+\infty)$$

**6.1.44**

$$\mathscr{I}\ln\Gamma(iy)=\arg\Gamma(iy)=-\arg\Gamma(-iy)=-\mathscr{I}\ln\Gamma(-iy)\sim y\ln y-y-\tfrac{1}{4}\pi-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}}{(2n-1)(2n)y^{2n-1}}\qquad (y\to+\infty)$$

**6.1.45** $$\lim_{|y|\to\infty}(2\pi)^{-\frac{1}{2}}|\Gamma(x+iy)|e^{\frac{1}{2}\pi|y|}|y|^{\frac{1}{2}-x}=1$$

**6.1.46** $$\lim_{n\to\infty}n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)}=1$$

**6.1.47**

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\frac{(a-b)(a+b-1)}{2z}+\frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2-a+b-1\right)\frac{1}{z^2}+\ .\ .\ .$$

as $z\to\infty$ along any curve joining $z=0$ and $z=\infty$, providing $z\neq -a,\ -a-1,\ .\ .\ .\ ;\ z\neq -b,\ -b-1,\ .\ .\ .\ .$

**Continued Fraction**

**6.1.48**

$$\ln \Gamma(z)+z-(z-\tfrac{1}{2})\ln z-\tfrac{1}{2}\ln(2\pi)$$

$$=\frac{a_0}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\,\frac{a_3}{z+}\,\frac{a_4}{z+}\,\frac{a_5}{z+}\cdots \qquad (\mathscr{R}z>0)$$

$$a_0=\frac{1}{12},\ a_1=\frac{1}{30},\ a_2=\frac{53}{210},\ a_3=\frac{195}{371},$$

$$a_4=\frac{22999}{22737},\ a_5=\frac{29944523}{19733142},\ a_6=\frac{109535241009}{48264275462}$$

**Wallis' Formula**[4]

**6.1.49**

$$\frac{2}{\pi}\int_0^{\pi/2}\binom{\sin}{\cos}^{2n} x\,dx=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}$$

$$=\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1}{2^{2n}}\binom{2n}{n}=\frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+1)}$$

$$\sim\frac{1}{\pi^{\frac{1}{2}}n^{\frac{1}{2}}}\left[1-\frac{1}{8n}+\frac{1}{128n^2}-\cdots\right] \qquad (n\to\infty)$$

**Some Definite Integrals**

**6.1.50**

$$\ln\Gamma(z)=\int_0^\infty\left[(z-1)e^{-t}-\frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right]\frac{dt}{t} \qquad (\mathscr{R}z>0)$$

$$=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln 2\pi$$

$$+2\int_0^\infty\frac{\arctan(t/z)}{e^{2\pi t}-1}\,dt \qquad (\mathscr{R}z>0)$$

## 6.2. Beta Function

**6.2.1**

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt=\int_0^\infty\frac{t^{z-1}}{(1+t)^{z+w}}\,dt$$

$$=2\int_0^{\pi/2}(\sin t)^{2z-1}(\cos t)^{2w-1}\,dt$$

$$(\mathscr{R}z>0,\ \mathscr{R}w>0)$$

**6.2.2** $$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}=B(w,z)$$

## 6.3. Psi (Digamma) Function[5]

**6.3.1** $\psi(z)=d[\ln\Gamma(z)]/dz=\Gamma'(z)/\Gamma(z)$

[4] Some authors employ the special double factorial notation as follows:

$(2n)!!=2\cdot 4\cdot 6\ldots(2n)=2^n n!$

$(2n-1)!!=1\cdot 3\cdot 5\ldots(2n-1)=\pi^{-\frac{1}{2}}2^n\,\Gamma(n+\frac{1}{2})$

[5] Some authors write $\psi(z)=\frac{d}{dz}\ln\Gamma(z+1)$ and similarly for the polygamma functions.

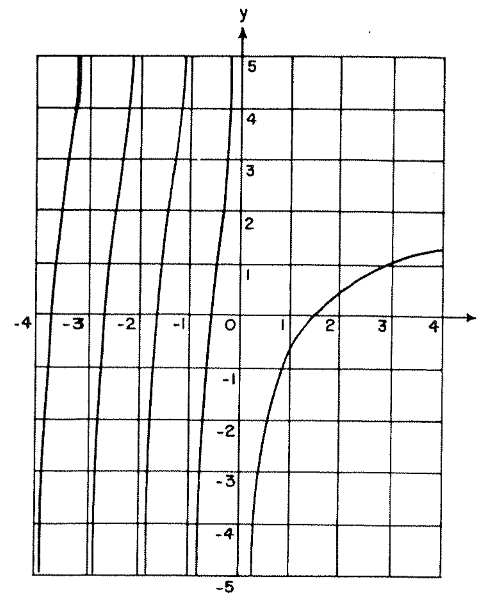


FIGURE 6.2. *Psi function.*

$y=\psi(x)=d\ln\Gamma(x)/dx$

**Integer Values**

**6.3.2** $$\psi(1)=-\gamma,\ \psi(n)=-\gamma+\sum_{k=1}^{n-1}k^{-1} \qquad (n\geq 2)$$

**Fractional Values**

**6.3.3**

$$\psi(\tfrac{1}{2})=-\gamma-2\ln 2=-1.96351\ 00260\ 21423\ldots$$

**6.3.4**

$$\psi(n+\tfrac{1}{2})=-\gamma-2\ln 2+2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) \qquad (n\geq 1)$$

**Recurrence Formulas**

**6.3.5** $$\psi(z+1)=\psi(z)+\frac{1}{z}$$

**6.3.6**

$$\psi(n+z)=\frac{1}{(n-1)+z}+\frac{1}{(n-2)+z}+\cdots$$

$$+\frac{1}{2+z}+\frac{1}{1+z}+\psi(1+z)$$

**Reflection Formula**

**6.3.7** $\psi(1-z)=\psi(z)+\pi\cot\pi z$

**Duplication Formula**

**6.3.8** $\psi(2z)=\frac{1}{2}\psi(z)+\frac{1}{2}\psi(z+\frac{1}{2})+\ln 2$

**Psi Function in the Complex Plane**

**6.3.9** $\psi(\bar{z})=\overline{\psi(z)}$

**6.3.10**

$$\mathscr{R}\psi(iy)=\mathscr{R}\psi(-iy)=\mathscr{R}\psi(1+iy)=\mathscr{R}\psi(1-iy)$$

**6.3.11** $\mathscr{I}\psi(iy)=\frac{1}{2}y^{-1}+\frac{1}{2}\pi\coth\pi y$

**6.3.12** $\mathscr{I}\psi(\frac{1}{2}+iy)=\frac{1}{2}\pi\tanh\pi y$

**6.3.13**

$$\mathscr{I}\psi(1+iy)=-\frac{1}{2y}+\frac{1}{2}\pi\coth\pi y$$

$$=y\sum_{n=1}^{\infty}(n^2+y^2)^{-1}$$

**Series Expansions**

**6.3.14** $\psi(1+z)=-\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1}\qquad(|z|<1)$

**6.3.15**

$$\psi(1+z)=\frac{1}{2}z^{-1}-\frac{1}{2}\pi\cot\pi z-(1-z^2)^{-1}+1-\gamma$$

$$-\sum_{n=1}^{\infty}[\zeta(2n+1)-1]z^{2n}\qquad(|z|<2)$$

**6.3.16**

$$\psi(1+z)=-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)}\qquad(z\neq-1,-2,-3,\ldots)$$

**6.3.17**

$$\mathscr{R}\psi(1+iy)=1-\gamma-\frac{1}{1+y^2}$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}[\zeta(2n+1)-1]y^{2n}\qquad(|y|<2)$$

$$=-\gamma+y^2\sum_{n=1}^{\infty}n^{-1}(n^2+y^2)^{-1}\qquad(-\infty<y<\infty)$$

**Asymptotic Formulas**

**6.3.18**

$$\psi(z)\sim\ln z-\frac{1}{2z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}$$

$$=\ln z-\frac{1}{2z}-\frac{1}{12z^2}+\frac{1}{120z^4}-\frac{1}{252z^6}+\ldots\qquad(z\to\infty\text{ in }|\arg z|<\pi)$$

**6.3.19**

$$\mathscr{R}\psi(1+iy)\sim\ln y+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}}{2ny^{2n}}$$

$$=\ln y+\frac{1}{12y^2}+\frac{1}{120y^4}+\frac{1}{252y^6}+\ldots\qquad(y\to\infty)$$

**Extrema[6] of $\Gamma(x)$ — Zeros of $\psi(x)$**

$$\Gamma'(x_n)=\psi(x_n)=0$$

| $n$ | $x_n$ | $\Gamma(x_n)$ |
|---|---|---|
| 0 | +1.462 | +0.886 |
| 1 | −0.504 | −3.545 |
| 2 | −1.573 | +2.302 |
| 3 | −2.611 | −0.888 |
| 4 | −3.635 | +0.245 |
| 5 | −4.653 | −0.053 |
| 6 | −5.667 | +0.009 |
| 7 | −6.678 | −0.001 |

$x_0=1.46163\ 21449\ 68362$

$\Gamma(x_0)=.88560\ 31944\ 10889$

**6.3.20** $x_n=-n+(\ln n)^{-1}+o[(\ln n)^{-2}]$

**Definite Integrals**

**6.3.21**

$$\psi(z)=\int_0^{\infty}\left[\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right]dt\qquad(\mathscr{R}z>0)$$

$$=\int_0^{\infty}\left[e^{-t}-\frac{1}{(1+t)^z}\right]\frac{dt}{t}$$

$$=\ln z-\frac{1}{2z}-2\int_0^{\infty}\frac{tdt}{(t^2+z^2)(e^{2\pi t}-1)}\qquad\left(|\arg z|<\frac{\pi}{2}\right)$$

**6.3.22**

$$\psi(z)+\gamma=\int_0^{\infty}\frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt=\int_0^1\frac{1-t^{z-1}}{1-t}dt$$

$$\gamma=\int_0^{\infty}\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)dt$$

$$=\int_0^{\infty}\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}$$

[6] From W. Sibagaki, Theory and applications of the gamma function, Iwanami Syoten, Tokyo, Japan, 1952 (with permission).

## 6.4. Polygamma Functions[7]

**6.4.1**

$$\psi^{(n)}(z)=\frac{d^n}{dz^n}\psi(z)=\frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z) \qquad (n=1,2,3,\ldots)$$

$$* \quad =(-1)^{n+1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}dt \qquad (\mathscr{R}z>0)$$

$\psi^{(n)}(z),(n=0,1,\ldots)$, is a single valued analytic function over the entire complex plane save at the points $z=-m(m=0,1,2,\ldots)$ where it possesses poles of order $(n+1)$.

### Integer Values

**6.4.2**

$$\psi^{(n)}(1)=(-1)^{n+1}n!\zeta(n+1) \qquad (n=1,2,3,\ldots)$$

**6.4.3**

$$\psi^{(m)}(n+1)=(-1)^m m!\left[-\zeta(m+1)+1+\frac{1}{2^{m+1}}+\ldots+\frac{1}{n^{m+1}}\right]$$

### Fractional Values

**6.4.4**

$$\psi^{(n)}(\tfrac{1}{2})=(-1)^{n+1}n!(2^{n+1}-1)\zeta(n+1) \qquad (n=1,2,\ldots)$$

**6.4.5** $$\psi'(n+\tfrac{1}{2})=\tfrac{1}{2}\pi^2-4\sum_{k=1}^{n}(2k-1)^{-2}$$

### Recurrence Formula

**6.4.6** $$\psi^{(n)}(z+1)=\psi^{(n)}(z)+(-1)^n n!z^{-n-1}$$

### Reflection Formula

**6.4.7**

$$\psi^{(n)}(1-z)+(-1)^{n+1}\psi^{(n)}(z)=(-1)^n\pi\frac{d^n}{dz^n}\cot\pi z$$

### Multiplication Formula

**6.4.8**

$$* \quad \psi^{(n)}(mz)=\delta\ln m+\frac{1}{m^{n+1}}\sum_{k=0}^{m-1}\psi^{(n)}\left(z+\frac{k}{m}\right)$$

$$\delta=1,\quad n=0$$
$$\delta=0,\quad n>0$$

[7] $\psi'$ is known as the trigamma function. $\psi''$, $\psi^{(3)}$, $\psi^{(4)}$ are the tetra-, penta-, and hexagamma functions respectively. Some authors write $\psi(z)=d[\ln\Gamma(z+1)]/dz$, and similarly for the polygamma functions.

*See page II.

### Series Expansions

**6.4.9**

$$\psi^{(n)}(1+z)=(-1)^{n+1}\left[n!\zeta(n+1)-\frac{(n+1)!}{1!}\zeta(n+2)z+\frac{(n+2)!}{2!}\zeta(n+3)z^2-\ldots\right] \qquad (|z|<1)$$

**6.4.10**

$$\psi^{(n)}(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}(z+k)^{-n-1} \qquad (z\neq 0,-1,-2,\ldots)$$

### Asymptotic Formulas

**6.4.11**

$$\psi^{(n)}(z)\sim(-1)^{n-1}\left[\frac{(n-1)!}{z^n}+\frac{n!}{2z^{n+1}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+n-1)!}{(2k)!z^{2k+n}}\right] \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

**6.4.12**

$$\psi'(z)\sim\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\frac{1}{42z^7}-\frac{1}{30z^9}+\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

**6.4.13**

$$\psi''(z)\sim-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{2z^4}+\frac{1}{6z^6}-\frac{1}{6z^8}+\frac{3}{10z^{10}}-\frac{5}{6z^{12}}+\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

**6.4.14**

$$\psi^{(3)}(z)\sim\frac{2}{z^3}+\frac{3}{z^4}+\frac{2}{z^5}-\frac{1}{z^7}+\frac{4}{3z^9}-\frac{3}{z^{11}}+\frac{10}{z^{13}}-\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

## 6.5. Incomplete Gamma Function

(see also **26.4**)

**6.5.1**

$$P(a,x)=\frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}\,dt \qquad (\mathscr{R}a>0)$$

**6.5.2**

$$\gamma(a,x)=P(a,x)\Gamma(a)=\int_0^x e^{-t}t^{a-1}\,dt \qquad (\mathscr{R}a>0)$$

**6.5.3**

$$\Gamma(a,x)=\Gamma(a)-\gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt$$

**6.5.4**

$$\gamma^*(a,x)=x^{-a}P(a,x)=\frac{x^{-a}}{\Gamma(a)}\gamma(a,x)$$

$\gamma^*$ is a single valued analytic function of $a$ and $x$ possessing no finite singularities.

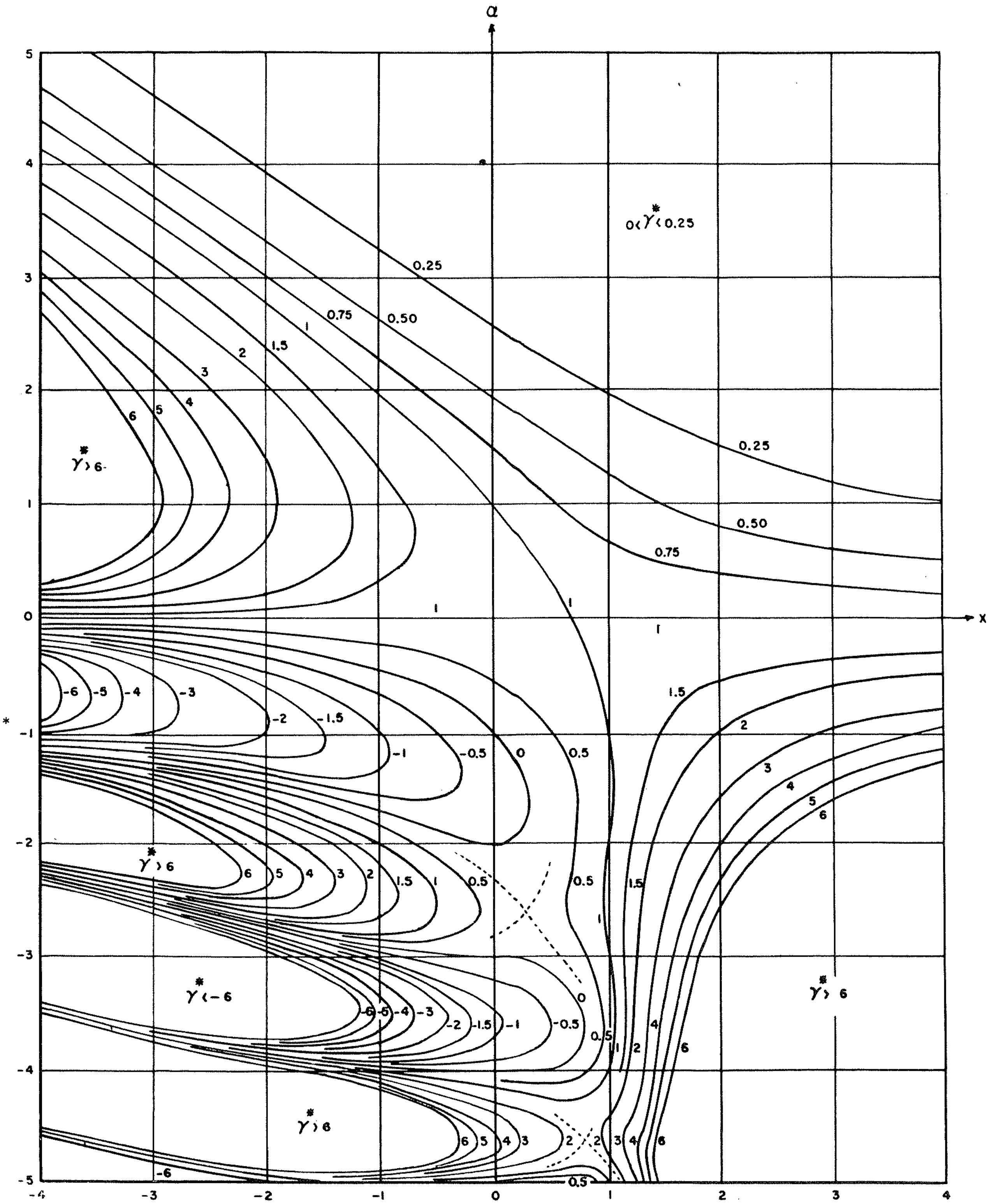


FIGURE 6.3. *Incomplete gamma function.*

$$\gamma^*(a,x)=\frac{x^{-a}}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}dt$$

From F. G. Tricomi, Sulla funzione gamma incompleta, Annali di Matematica, IV, 33, 1950 (with permission).

*See page II.

**6.5.5**

**Probability Integral of the $\chi^2$-Distribution**

$$P(\chi^2|\nu)=\frac{1}{2^{\frac{1}{2}\nu}\Gamma\left(\frac{\nu}{2}\right)}\int_0^{\chi^2} t^{\frac{1}{2}\nu-1}e^{-\frac{t}{2}}\,dt$$

**6.5.6**

**(Pearson's Form of the Incomplete Gamma Function)**

$$I(u,p)=\frac{1}{\Gamma(p+1)}\int_0^{u\sqrt{p+1}} e^{-t}t^p\,dt$$
$$=P(p+1,\ u\sqrt{p+1})$$

**6.5.7** $$C(x,a)=\int_x^\infty t^{a-1}\cos t\,dt \qquad (\mathscr{R}a<1)$$

**6.5.8** $$S(x,a)=\int_x^\infty t^{a-1}\sin t\,dt \qquad (\mathscr{R}a<1)$$

**6.5.9**

$$E_n(x)=\int_1^\infty e^{-xt}t^{-n}\,dt=x^{n-1}\Gamma(1-n,x)$$

**6.5.10**

$$\alpha_n(x)=\int_1^\infty e^{-xt}t^n\,dt=x^{-n-1}\Gamma(1+n,x)$$

**6.5.11** $$e_n(x)=\sum_{j=0}^{n}\frac{x^j}{j!}$$

**Incomplete Gamma Function as a Confluent Hypergeometric Function** (see chapter **13**)

**6.5.12** $$\gamma(a,x)=a^{-1}x^ae^{-x}M(1,\ 1+a,x)$$
$$=a^{-1}x^a\,M(a,\ 1+a,-x)$$

**Special Values**

**6.5.13**

$$P(n,x)=1-\left(1+x+\frac{x^2}{2!}+\ \dots\ +\frac{x^{n-1}}{(n-1)!}\right)e^{-x}$$
$$=1-e_{n-1}(x)\,e^{-x}$$

For relation to the Poisson distribution, see **26.4.**

**6.5.14** $$\gamma^*(-n,\ x)=x^n$$

**6.5.15** $$\Gamma(0,\ x)=\int_x^\infty e^{-t}t^{-1}\,dt=E_1(x)$$

**6.5.16** $$\gamma(\tfrac{1}{2},\ x^2)=2\int_0^x e^{-t^2}\,dt=\sqrt{\pi}\ \mathrm{erf}\ x$$

**6.5.17** $$\Gamma(\tfrac{1}{2},x^2)=2\int_x^\infty e^{-t^2}dt=\sqrt{\pi}\ \mathrm{erfc}\ x$$

**6.5.18** $$\tfrac{1}{2}\sqrt{\pi}\ x\,\gamma^*(\tfrac{1}{2},-x^2)=\int_0^x e^{t^2}dt$$

**6.5.19** $$\Gamma(-n,x)=\frac{(-1)^n}{n!}\left[E_1(x)-e^{-x}\sum_{j=0}^{n-1}\frac{(-1)^j j!}{x^{j+1}}\right]$$

**6.5.20** $$\Gamma(a,ix)=e^{\frac{1}{2}\pi i a}\,[C(x,a)-iS(x,a)]$$

**Recurrence Formulas**

**6.5.21** $$P(a+1,\ x)=P(a,\ x)-\frac{x^ae^{-x}}{\Gamma(a+1)}$$

**6.5.22** $$\gamma(a+1,x)=a\gamma(a,x)-x^ae^{-x}$$

**6.5.23** $$\gamma^*(a-1,x)=x\gamma^*(a,x)+\frac{e^{-x}}{\Gamma(a)}$$

**Derivatives and Differential Equations**

**6.5.24**

$$\left(\frac{\partial\gamma^*}{\partial\alpha}\right)_{\alpha=0}=-\int_x^\infty\frac{e^{-t}dt}{t}-\ln x=-E_1(x)-\ln x$$

**6.5.25** $$\frac{\partial\gamma(a,x)}{\partial x}=-\frac{\partial\Gamma(a,x)}{\partial x}=x^{a-1}e^{-x}$$

**6.5.26**

$$\frac{\partial^n}{\partial x^n}[x^{-a}\Gamma(a,x)]=(-1)^nx^{-a-n}\Gamma(a+n,x) \qquad (n=0,1,2,\ldots)$$

**6.5.27**

$$\frac{\partial^n}{\partial x^n}[e^xx^a\gamma^*(a,x)]=e^xx^{a-n}\gamma^*(a-n,\ x) \qquad (n=0,1,2,\ldots)$$

**6.5.28** $$x\frac{\partial^2\gamma^*}{\partial x^2}+(a+1+x)\frac{\partial\gamma^*}{\partial x}+a\gamma^*=0$$

**Series Developments**

**6.5.29**

$$\gamma^*(a,z)=e^{-z}\sum_{n=0}^\infty\frac{z^n}{\Gamma(a+n+1)}=\frac{1}{\Gamma(a)}\sum_{n=0}^\infty\frac{(-z)^n}{(a+n)n!} \qquad (|z|<\infty)$$

**6.5.30**

$$\gamma(a, x+y)-\gamma(a, x)$$
$$=e^{-x}x^{a-1}\sum_{n=0}^{\infty}\frac{(a-1)(a-2)\ldots(a-n)}{x^n}[1-e^{-y}e_n(y)]$$
$$(|y|<|x|)$$

### Continued Fraction

**6.5.31**

$$\Gamma(a,x)=e^{-x}x^a\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right)$$
$$(x>0, |a|<\infty)$$

### Asymptotic Expansions

**6.5.32**

$$\Gamma(a, z)\sim z^{a-1}e^{-z}\left[1+\frac{a-1}{z}+\frac{(a-1)(a-2)}{z^2}+\cdots\right]$$
$$\left(z\to\infty \text{ in } |\arg z|<\frac{3\pi}{2}\right)$$

Suppose $R_n(a,z)=u_{n+1}(a,z)+\ldots$ is the remainder after $n$ terms in this series. Then if $a,z$ are real, we have for $n>a-2$

$$|R_n(a,z)|\leq|u_{n+1}(a,z)|$$

and sign $R_n(a,z)=$ sign $u_{n+1}(a,z)$.

**6.5.33** $$\gamma(a,z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n z^{a+n}}{(a+n)n!} \qquad (a\to+\infty)$$

**6.5.34** $$\lim_{n\to\infty}\frac{e_n(\alpha n)}{e^{\alpha n}}=\begin{cases}0 \text{ for } \alpha>1\\ \frac{1}{2} \text{ for } \alpha=1\\ 1 \text{ for } 0\leq\alpha<1\end{cases}$$

**6.5.35**

$$\Gamma(z+1,z)\sim e^{-z}z^z\left(\sqrt{\frac{\pi}{2}}z^{\frac{1}{2}}+\frac{2}{3}+\frac{\sqrt{2\pi}}{24}\frac{1}{z^{\frac{1}{2}}}+\cdots\right)$$
$$(z\to\infty \text{ in } |\arg z|<\tfrac{1}{2}\pi)$$

### Definite Integrals

**6.5.36**

$$\int_0^\infty e^{-at}\,\Gamma(b,ct)\,dt=\frac{\Gamma(b)}{a}\left[1-\frac{c^b}{(a+c)^b}\right]$$ *
$$(\mathscr{R}(a+c)>0, \mathscr{R}b>-1)$$

**6.5.37**

$$\int_0^\infty t^{a-1}\Gamma(b,t)\,dt=\frac{\Gamma(a+b)}{a}$$
$$(\mathscr{R}(a+b)>0, \quad \mathscr{R}a>0)$$

## 6.6. Incomplete Beta Function

**6.6.1** $$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt$$

**6.6.2** $$I_x(a,b)=B_x(a,b)/B(a,b)$$

For statistical applications, see **26.5**.

### Symmetry

**6.6.3** $$I_x(a,b)=1-I_{1-x}(b,a)$$

### Relation to Binomial Expansion

**6.6.4** $$I_p(a,n-a+1)=\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

For binomial distribution, see **26.1**.

### Recurrence Formulas

**6.6.5** $$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

**6.6.6** $$(a+b-ax)I_x(a,b)$$ *
$$=a(1-x)I_x(a+1,b-1)+bI_x(a,b+1)$$

**6.6.7** $$(a+b)I_x(a,b)=aI_x(a+1,b)+bI_x(a,b+1)$$

### Relation to Hypergeometric Function

**6.6.8** $$B_x(a,b)=a^{-1}x^aF(a,1-b;\,a+1;\,x)$$

# Numerical Methods

## 6.7. Use and Extension of the Tables

**Example 1.** Compute $\Gamma(6.38)$ to 8S. Using the recurrence relation **6.1.16** and **Table 6.1** we have,

$$\Gamma(6.38)=[(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$
$$=232.43671.$$

**Example 2.** Compute $\ln\Gamma(56.38)$, using **Table 6.4** and linear interpolation in $f_2$. We have

$$\ln\Gamma(56.38)=(56.38-\tfrac{1}{2})\ln(56.38)-(56.38)+f_2(56.38)$$

The error of linear interpolation in the table of the function $f_2$ is smaller than $10^{-7}$ in this region. Hence, $f_2(56.38)=.92041\ 67$ and $\ln\Gamma(56.38)=169.85497\ 42$.

Direct interpolation in **Table 6.4** of $\log_{10}\Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10}\Gamma(n)$ is obtained with a relative error of $10^{-5}$.

*See page II.