ENCH 630 -- Problem Set 1: Review of Mathematical Methods

Note: Assume that a Cartesian coordinate system applies when working these problems.

1. In the derivation of the transport equations for potential flow, the following identity is used:

$$\frac{\partial |\boldsymbol{v}|^2}{\partial t} = 2\boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial t}$$

In the above equation $\boldsymbol{v}$ is the time dependent velocity vector which can be written as $v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$ where v_x, v_y, and v_z are functions of x, y, and z and t. Note also that **i**, **j**, and **k** are the unit vectors in the x, y, and z directions and that $|\boldsymbol{v}|$ is the magnitude of the velocity.. Prove this relation.

2. Prove the following relations:

$$\nabla\varphi_1 \times \nabla\varphi_2 = \nabla \times (\varphi_1 \nabla\varphi_2)$$

$$\boldsymbol{\nabla} \bullet (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0$$

$$\boldsymbol{\nabla} \bullet \left(\boldsymbol{\nabla}\,\varphi_1 \times \boldsymbol{\nabla}\,\varphi_2\right) = 0$$

In the above relations $\varphi_1(x,y,z)$ and $\varphi_2(x,y,z)$ are scalar variables, $\boldsymbol{v}$(x,y,z) is a vector variable which can be written as $v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$, ∇ is the del operator, and $\bullet$ and $\times$ are the dot and cross products. If necessary, in your proof you may assume that none of the variables in this problem are dependent on time.

3. Prove by using the chain rule (or any other appropriate method) that the dot product of a unit vector in a particular direction and the gradient of a scalar function yields the derivative of the scalar function in the direction of the unit vector.

4. Consider the following partial differential equation:

$$\frac{\partial \theta}{\partial t} = \gamma \frac{\partial^2 \theta}{\partial x^2} + \frac{x^2}{4\gamma\, t^2}$$

where θ is a function of x and t , and γ is a constant Use the chain rule to transform the above equation so that the original set of independent variables (x and t) are replaced by a new independent variable η which is defined as follows:

$$\eta = \frac{x}{\sqrt{4\,\gamma\,t}}$$

5. Consider the following form of the Leibniz rule for differentiating a one-dimensional integral:

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x,t)\ dx + f(\beta,t)\frac{d\beta}{dt} - f(\alpha,t)\frac{d\alpha}{dt}$$

Show that this result can be derived directly from the chain rule for partial derivatives by recognizing that the integral $\int_{\alpha(t)}^{\beta(t)} f(x,t)dx$ can be considered to be either a function of t alone (i.e., $g(t)$) or it can be considered to be a function of α, β, and t (i.e., $g(\alpha,\beta,t)$) where α and β are functions of t. Then, write d $g(t)$ /dt in terms of $\partial g(\alpha,\beta,t)/\partial t$, $\partial g(\alpha,\beta,t)/\partial\alpha$, and $\partial g(\alpha,\beta,t)/\partial\beta$.

6. Consider the following relation, which is one form of the Reynold's transport theorem:

$$\frac{d}{dt}\int_V \rho\ s\ dV = \int_V \rho\ \frac{Ds}{Dt} dV$$

Note that s is a scalar variable that depends on both time and position and that s corresponds to amy scalar property of a fluid per unit mass of fluid, and ρ is the fluid density. Also note that the surface of the volume V in the above expression is assumed to move at the fluid velocity (so that $v_{\text{surface}} = v$ at all locations on the surface of V). Prove the above relation by starting with the Leibniz formula for differentiating a volume integral and applying this formula to the case where the surface enclosing the volume moves at the local fluid velocity (i.e., $v_s = v$ in eqn. A.5-5 of BSL). Then incorporate the Gauss divergence theorem, the continuity equation, and other appropriate vector identities (see also eqns. A.5.5. A.5.1, 3.1-4, and A.4-18 to A.4-28 of BSL).