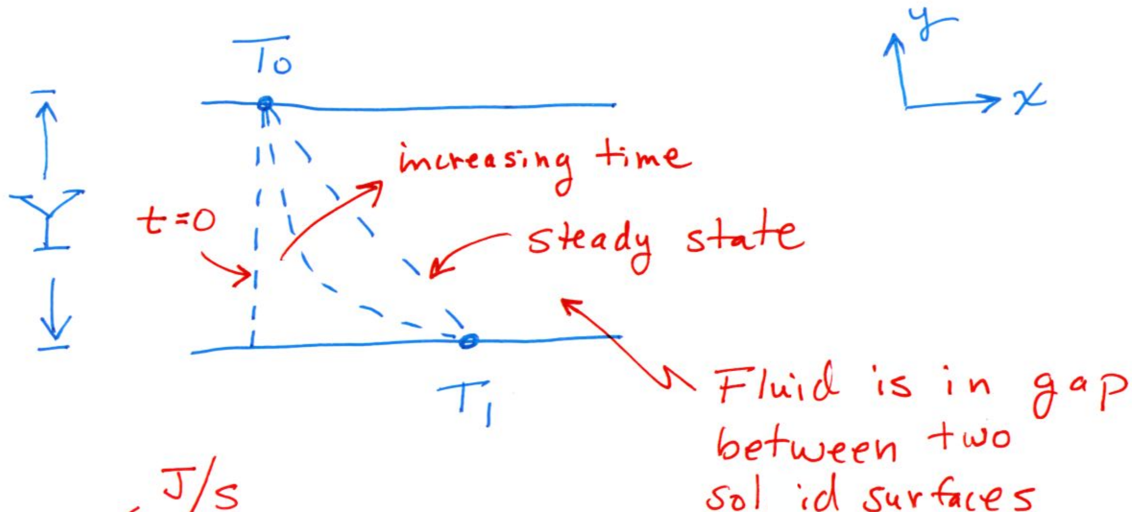
Heat transfer

Consider a solid slab with a sudden change in temperature at one side:



$$\frac{\dot{Q}}{A} = k \frac{\Delta T}{Y}$$

J/s

Fourier's law of heat conduction

k: thermal conductivity

Microscopic form:

$$q_y = -k \frac{dT}{dy}$$

q_y: Heat flux in J/(s·m²)

In three dimensions:

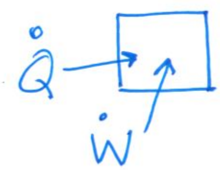
$$\vec{q} = -k \vec{\nabla} T$$

$$q_x \vec{i} + q_y \vec{j} + q_z \vec{k} = -k \left(\vec{i} \frac{\partial T}{\partial x} + \vec{j} \frac{\partial T}{\partial y} + \vec{k} \frac{\partial T}{\partial z} \right)$$

Rectangular coordinates

Equation of energy (see BSL for details)

1^{st} law for a control mass:



Include gravity in $\dot{W}$

$$\dot{Q} + \dot{W} = \frac{d}{dt}\left(U + \frac{1}{2}v^2\right)$$

For a control volume:

$$\underbrace{\text{Production of } U + \tfrac{1}{2}v^2}_{\text{outflow} - \text{inflow} + \text{accumulation}} = \dot{Q} + \dot{W}$$

Subtract mechanical energy balance to yield the "thermal energy" balance:

$$\rho \frac{D\hat{U}}{Dt} = \underbrace{-\vec{\nabla}\cdot\vec{q}}_{\substack{\text{heat efflux}\\ \text{per unit volume}}} \underbrace{- P(\vec{\nabla}\cdot\vec{v})}_{\substack{\text{Internal energy}\\ \text{increase by}\\ \text{compression}}} \underbrace{- (\bar{\bar{\tau}} : \vec{\nabla}\vec{v})}_{\substack{\text{Internal energy}\\ \text{increase by}\\ \text{shear stress}}}$$

Use
$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$
$$= C_V\, dT + \left(-P + T\left(\frac{\partial P}{\partial T}\right)_V\right) dV$$

Rearrange, and also use $\vec{q} = -k\vec{\nabla}T$

$$\rho \hat{C}_V \frac{DT}{Dt} = k\nabla^2 T - T\left(\frac{\partial P}{\partial T}\right)_V (\vec{\nabla}\cdot\vec{v}) - \bar{\bar{\tau}} : \vec{\nabla}v$$

for ideal gas $\left(\frac{\partial P}{\partial T}\right)_V = P/T$

viscous dissipation

For a fluid at constant P, we have

$$d\hat{u} = -P\,d\hat{V} + \hat{C}_p\,dT$$

and above equations become (neglecting viscous dissipation):

Source term

$$\rho \hat{C}_p \frac{DT}{Dt} = k\nabla^2 T + S_v$$

The above equation also holds for fluids where ρ is independent of T, i.e., $(\partial\rho/\partial T)_p = 0$. Thus above equation applies to gases for P = constant and to liquids even when P varies.

Definition of heat transfer coefficient:

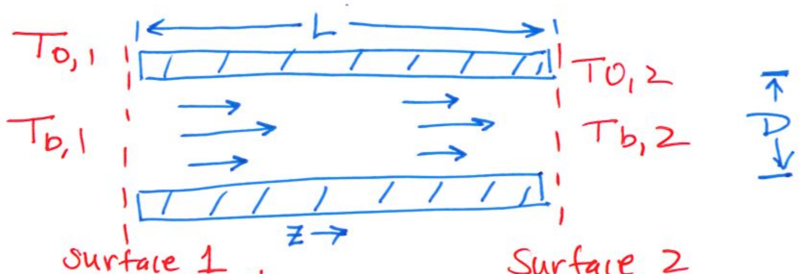
$$\dot{Q} = h\,A\,\Delta T$$

Newtons law of cooling

$(J/(s\,m^2 K))$

Heat transfer rate (J/s)

Flow in a conduit:



$$\dot{Q} = h_1 \overbrace{(\pi D L)}^{\text{Area}} (T_{0,1} - T_{b,1})$$

Based on initial temperature difference

$$\dot{Q} = h_a (\pi D L)\left[\frac{(T_{0,1} - T_{b,1}) + (T_{0,2} - T_{b,2})}{2}\right]$$

Based on average temperature driving force

$$\dot{Q} = h_{ln}\,(\pi D L)\left[\frac{(T_{0,1}-T_{b,1})-(T_{0,2}-T_{b,2})}{\ln (T_{01}-T_{b1})/(T_{0,2}-T_{b,2})}\right]$$

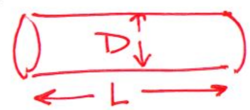
Based on log mean driving force

$$d\dot{Q} = h_{loc}\,(\pi D\, dz)\,(T_0 - T_b)\Big|_z$$

local heat transfer coefficient

h_{ln} tends to be more constant (e.g., less dependent on L/D) and more predictable

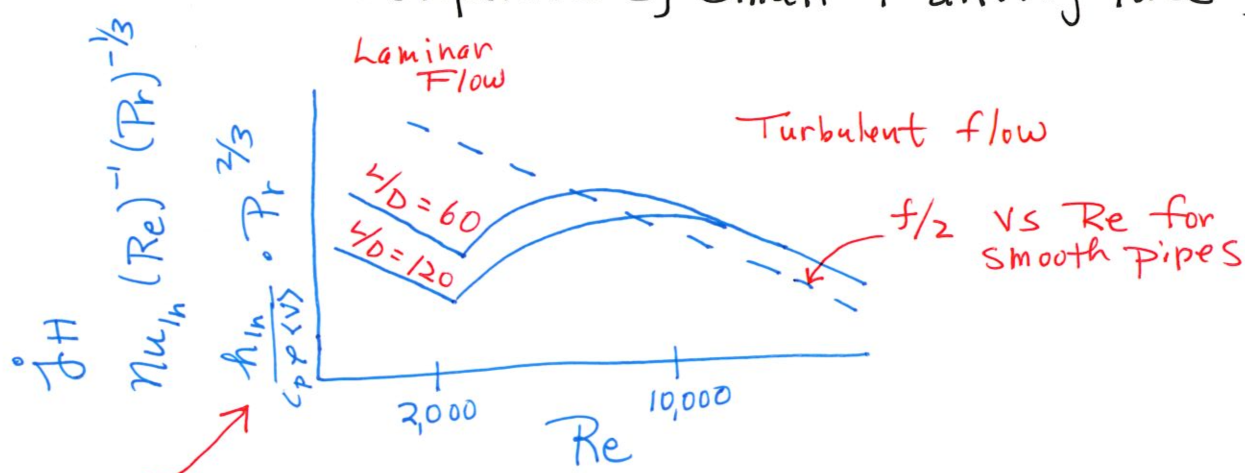
For flow in a pipe, assuming small temperature driving force, dimensional analysis yields:



$$\underbrace{\frac{h_{ln} D}{k}}_{Nu_{ln}} = f_{ln}\left(Re, Pr, L/D\right)$$

also f_1 and f_a $\frac{\rho\langle v\rangle D}{\mu}$ $\frac{\hat{C}_p \mu}{k}$

Sieder and Tate correlation (smooth walls, constant wall temperature, small T driving force)



$$\frac{h_{ln}}{C_p \rho \langle v\rangle} = St = Nu/Re\,Pr$$

Heat source terms for thermal energy equations:

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \underbrace{S_v}_{\text{Source term}}$$

Examples:

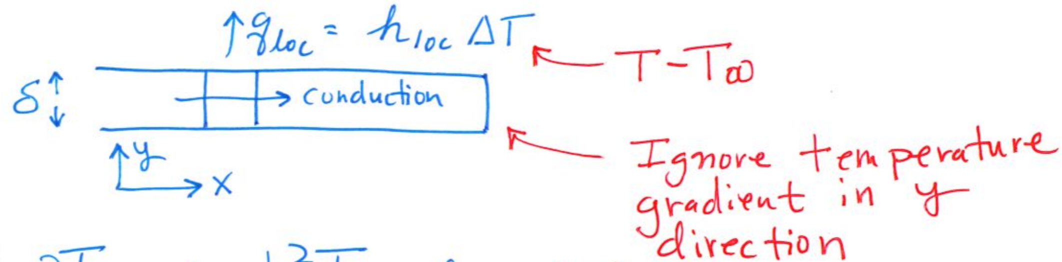
1. Chemical reaction: $S_v = r \, \Delta \tilde{H}$

rate of reaction [moles/s·cm³]

enthalpy of reaction

Pseudo heat source:

2. Interphase heat transfer (e.g., heat transfer in a "fin"):



$$\rho \hat{C}_p \frac{\partial T}{\partial t} = k \frac{d^2 T}{dx^2} + \underbrace{\underbrace{h_{loc} \Delta T}_{\text{heat transfer per unit area}} \cdot 1/\delta}_{S_v}$$

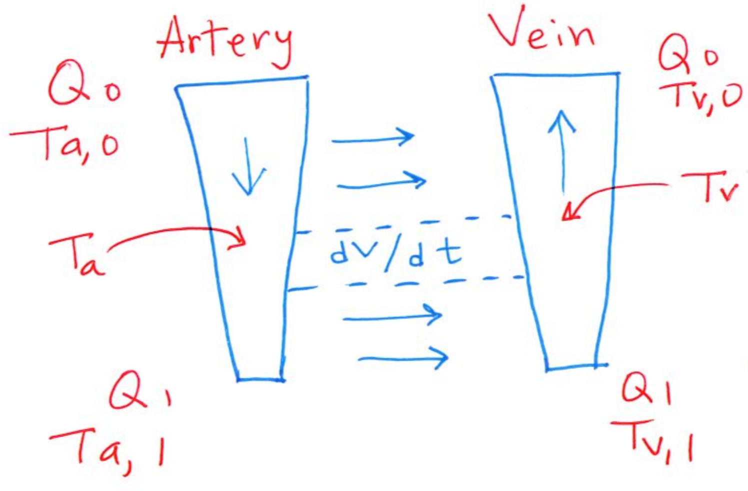
$1/\delta$: Area per unit volume

3. "Pennes" bioheat transfer equation for conduction of heat in tissue

$$\rho \hat{C}_p \frac{\partial T_t}{\partial t} = k_t \nabla^2 T_t + q_{blood} + q_m$$

q_m = rate of metabolic heat generation

q_{blood} = heat source term due to blood flow in capillaries



blood flow rate per unit volume:

$$\frac{dV/dt}{\text{volume of tissure}}$$

Heat deposited in tissue per unit volume of blood:

$$\rho_b \, C_{p,b} \, (T_a - T_v)$$

Heat deposited in tissue per unit volume of tissue per unit time:

$$q_{blood} = \rho_b \, C_{p,b} \, (T_a - T_v) \underbrace{\frac{dV/dt}{\text{vol. tissue}}}_{\omega}$$

Assume constant ($T_a = 37°C$)

Temperature of surrounding tissue ($T_v = T_t$)