# Part 7

## Special Topics in Fluid Mechanics

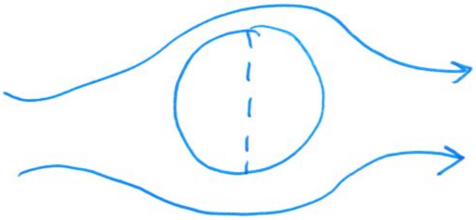
### Topic 1: Interphase momentum transfer

$$f = \text{friction factor} = \frac{\text{Force}}{\text{area} \cdot \frac{1}{2}\rho v^2}$$

$$= \text{Drag coefficient} = C_D$$

$$f = 16/Re \quad \text{(laminar flow)}$$

"Fanning" friction factor
(not the "Moody" friction factor)



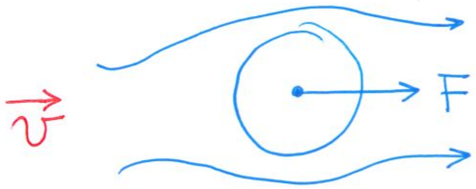
$$f = C_D = 24/Re \quad \text{(Creeping flow)}$$

$$f = C_D = \frac{4}{3}\frac{1}{\sqrt{Re_L}} \quad \text{(laminar flow)}$$

### Topic 2: Dimensional Analysis

Theoretical result: $C_D = \frac{F}{\pi R^2 \cdot \frac{1}{2}\rho v^2} = \frac{24}{\frac{\rho v D}{\mu}}$

Experimental approach:



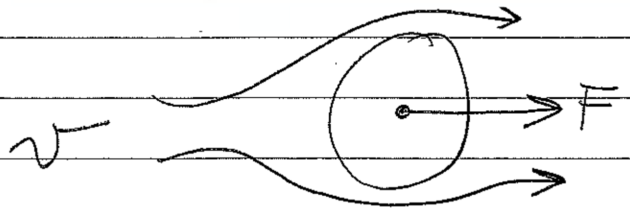
F depends on $v, \rho, \mu, D$

fluid velocity

Diameter

Dimensional analysis (a brief review and "proof" of the Buckingham $\pi$ theorem)

Consider flow past a sphere:



F depends on
$v, \rho, \mu, D$

$D$ ← Diameter

Propose the following:

$$F = \overset{\text{dimensionless}}{A}\, v^{\alpha} \rho^{\beta} \mu^{\gamma} D^{\delta}$$

or $\log(F) = \log(A) + \alpha \log(v) + \beta \log(\rho) + \gamma \log(\mu) + \delta \log(D)$

But the above equation needs to be dimensionally homogeneous

$F \; [=] \; ML/t^2$
$v \; [=] \; L/t$
$D \; [=] \; L$
$\rho \; [=] \; M/L^3$
$\mu \; [=] \; M/Lt$

Therefore, considering M:

$$1 = \beta + \gamma$$

Considering L:

$$1 = \alpha + \delta - 3\beta - \gamma$$

Considering t:

$$-2 = +\alpha + \gamma$$

Solve for $\beta, \gamma, \delta$ in terms of $\alpha$:

$$\beta = \alpha + 1$$
$$\gamma = 2 - \alpha$$
$$\delta = \alpha$$

This yields:

$$F = A v^{\alpha} \rho^{\alpha-1} \mu^{2-\alpha} D^{\alpha}$$

$$F = A\left(\rho v D/\mu\right)^{\alpha}\left(\mu^2/\rho\right)$$

multiply each side by $(\rho v D/\mu)^{-2}$ and rearrange

$$\underbrace{\frac{F}{\rho v^2 D^2}}_{\text{proportional to } C_D} = A\,(\underbrace{\rho v D/\mu}_{Re})^{\alpha-2}$$

Experimentally, the above applies at low values of $\rho v D/\mu$ with $\alpha = 1$ and $A = 3\pi$, which yields

$$\overbrace{\frac{F}{\pi (D/2)^2 \cdot \frac{1}{2}\rho v^2}}^{C_D} = \frac{24}{Re}$$

at high values of $\rho v D/\mu$, the above eqn applies with $\alpha = 2$ and $C_D = 0.44$

More generally if

$$\bigcirc = f(u_1, u_2, u_3, \ldots u_n)$$

$u_i$ denotes both dependent and independent variables

and there are $m$ "primary dimensions"
such as M, L, t, etc.

Then we have

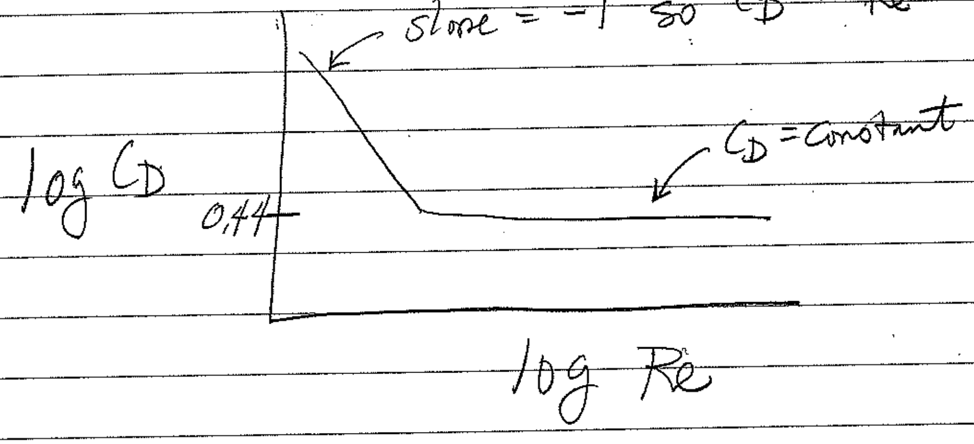
$$0 = g(\Pi_1, \ldots, \Pi_{n-m})$$

Two especially interesting cases

$n-m = 1 \Rightarrow$ only one dimensionless group which must equal a constant

$n-m = 2 \Rightarrow$ two dimensionless groups

Plot on a log-log scale and look for limiting cases



Consider again fluid flow around a sphere. A general procedure for determining the dimensionless groups can be based on identifying "repeating parameters" as shown next.

① Identify parameters : $\overset{\swarrow \text{ dependent variable}}{F}, \underbrace{v, \rho, \mu, D}_{\text{independent variables}}$

② Select primary dimensions ( M, L, t )

③ Write units for each parameter

| $F$ | $v$ | $D$ | $\rho$ | $\mu$ |
|---|---|---|---|---|
| $\frac{ML}{t^2}$ | $\frac{L}{t}$ | $L$ | $\frac{M}{L^3}$ | $\frac{M}{Lt}$ |

④ Select a number of "repeating" parameters equal to number of primary dimensions ($= r$) that include all primary dimensions. (Don't include dependent variable.)

E.g.: $\rho, v, D$

⑤ Form combinations with remaining variables to yield dimensionless variables, thus, # of $\pi$ groups = # of variables $- r$

$$\pi_1 = \rho^a v^b D^c F$$

$$= \left(\frac{M}{L^3}\right)^a \left(\frac{L}{t}\right)^b L^c \frac{ML}{t^2} = M^0 L^0 t^0$$

$M: a+1=0 \Rightarrow a=-1$

$L: -3a+b+c+1=0 \Rightarrow b+c=-4$

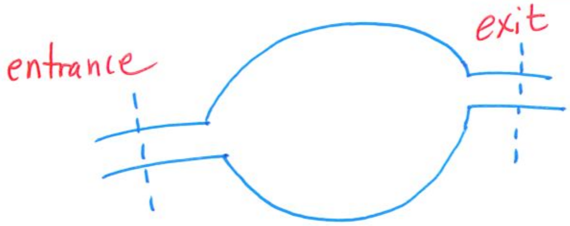
$t: -b-2=0 \Rightarrow b=-2 \Rightarrow c=-2$

Thus $\pi_1 = F/\rho v^2 D^2 \leftarrow C_D$; Similarly, $\pi_2 = \frac{\rho v D}{\mu}$

# Topic 3. Macroscopic mechanical energy balance

$\vec{v}\cdot$( Navier-Stokes Eqn. )

Integrate "microscopic mechanical energy balance" over volume, assume $\rho$ = constant, steady state, and solid wall boundaries



$\Delta$: exit value − entrance value

$$\underbrace{\Delta \frac{1}{2} \underbrace{\frac{\langle v^3\rangle}{\langle v\rangle}}_{\sim\langle v\rangle^2}}_{\text{Change in Kinetic energy per unit mass (velocity head)}} + \underbrace{g\Delta h}_{\text{Change in Potential energy per unit mass (gravity head)}} + \underbrace{\frac{\Delta P}{\rho}}_{\text{Change in pressure head}} + \hat{W} + \underbrace{\hat{E}_V}_{\text{Frictional loss per unit mass flowing through system}} = 0$$

$$\langle v^3\rangle = \frac{\int_0^R 2\pi r v^3 dr}{\pi R^2}$$

$\hat{W}$: rate at which system does work on surroundings per unit mass flowing through system

What is $\hat{E}_v$? Consider a straight pipe with fully developed flow 

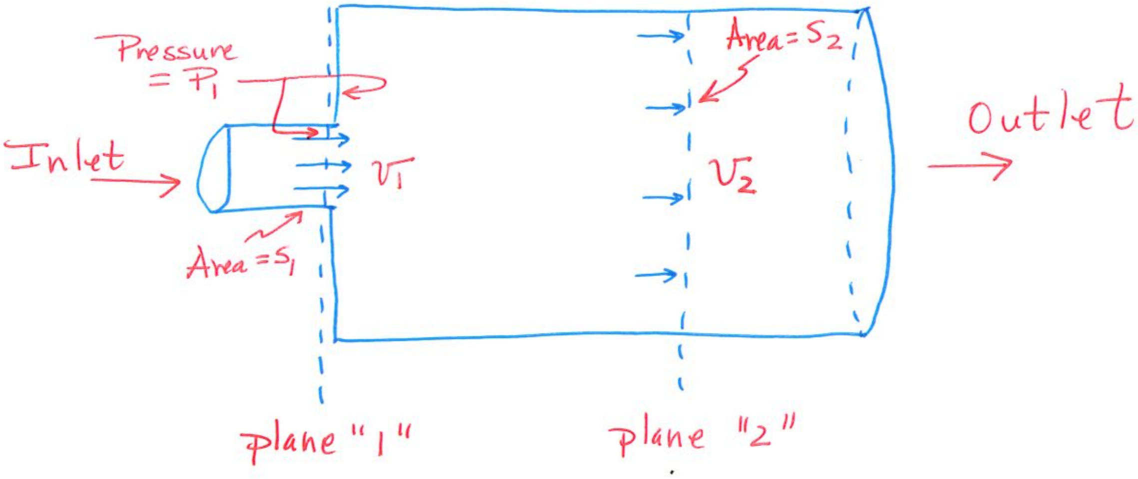
$$F_{shear} = (P_1 - P_2)\,S \quad \leftarrow \text{Cross-sectional area}$$

But $\hat{E}_v = P_1 - P_2/\rho \Rightarrow F_{shear} = \rho\,\hat{E}_v S$

So $\hat{E}_v = F_{shear}/\rho S = \dfrac{f \cdot 2\pi R L \cdot \frac{1}{2}\rho\langle v\rangle^2}{\rho\,\pi R^2}$

Final Result: $\hat{E}_v = \underbrace{f\frac{L}{D}4}_{\text{friction loss factor}} \cdot \frac{1}{2}\langle v\rangle^2$

# Energy loss factor for a sudden enlargement



Mass balance:

$$\cancel{\rho}\, v_1 S_1 = \cancel{\rho}\, v_2 S_2$$

Momentum balance:

$$\rho v_2 v_2 S_2 - \rho v_1 v_1 S_1 = (P_1 - P_2) S_2$$
$$= -(\Delta P) S_2$$

Combine to yield:

$$\Delta P / \rho = \tfrac{1}{2} v_1^2 \left( 2 v_2 / v_1 - 2 v_2^2 / v_1^2 \right)$$

Mechanical energy balance:

$$-E_v = \Delta P / \rho + \tfrac{1}{2} \left( v_2^2 - v_1^2 \right)$$

Substitute to eliminate $\Delta P / \rho$ :

$$E_v = \tfrac{1}{2} v_1^2 \underbrace{\left( 1 - S_1 / S_2 \right)^2}_{\text{friction loss factor}}$$

Final form of equation:

$$\Delta \tfrac{1}{2}\langle v\rangle^2 + g\,\Delta h + \Delta P/\rho + \hat{W}$$

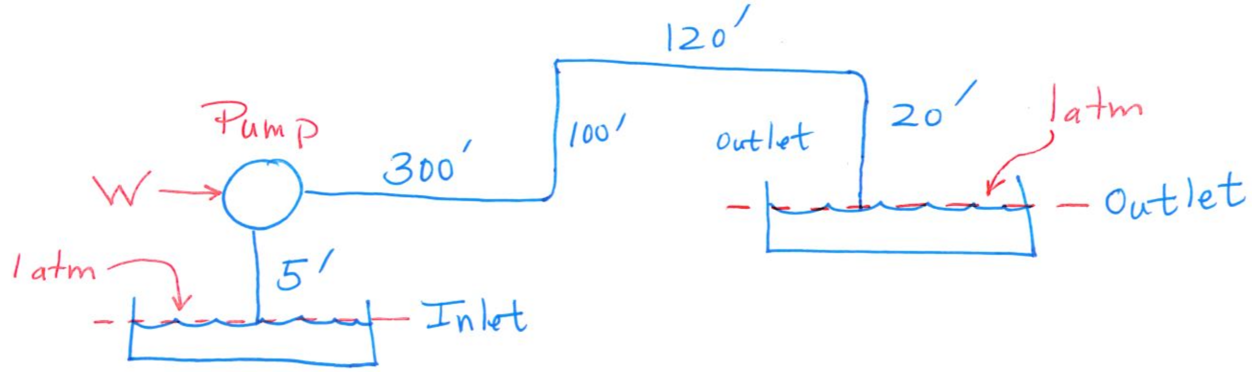
$$+ \sum\left(f\,\frac{L}{D}\cdot 2\cdot\langle v\rangle^2\right)$$

straight sections of pipe

$$+ \sum \frac{1}{2}\langle v\rangle^2 e_i$$

loss factors for fittings etc.

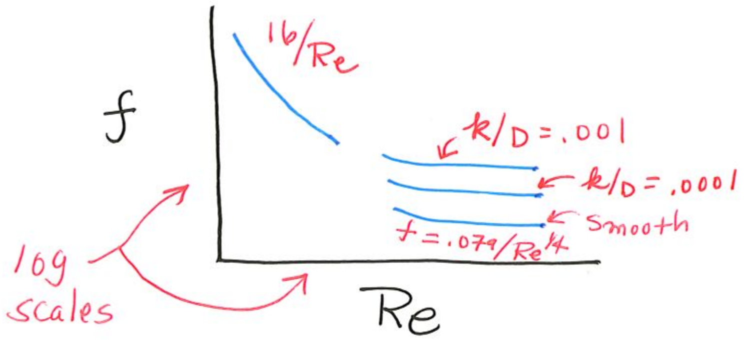
Typical problem:



What is the power needed to achieve 12 ft³/min in 4" pipe?

$$\langle v\rangle = Q/\pi R^2 = 2.3\ \text{ft/s} \Rightarrow Re = \frac{\rho v D}{\mu} = 7\times10^4$$

Note that $\Delta \frac{1}{2}\langle v\rangle^2 = \Delta P/\rho = 0$



at $Re = 7\times10^4$

$f = .0049$

$e_i$ for 90° elbow $\cong$ 1/2

$e_i$ for sudden contraction = 0.45

$e_i$ for sudden expansion = 1