Part 5 — ENCH 630

Potential Flow and Bernoulli Equation

Start with the Navier-Stokes equation:

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

Note that

$$\rho \frac{D\vec{v}}{Dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v}\cdot\vec{\nabla}\vec{v}$$

$$\vec{v}\cdot\vec{\nabla}\vec{v} = \vec{\nabla}\frac{\overbrace{(\vec{v}\cdot\vec{v})}^{|v|^2}}{2} - \vec{v}\times\overbrace{(\vec{\nabla}\times\vec{v})}^{\vec{\omega}\ \leftarrow\ \text{vorticity}}$$

$$\nabla^2\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}\times\underbrace{(\vec{\nabla}\times\vec{v})}_{\vec{\omega}}$$

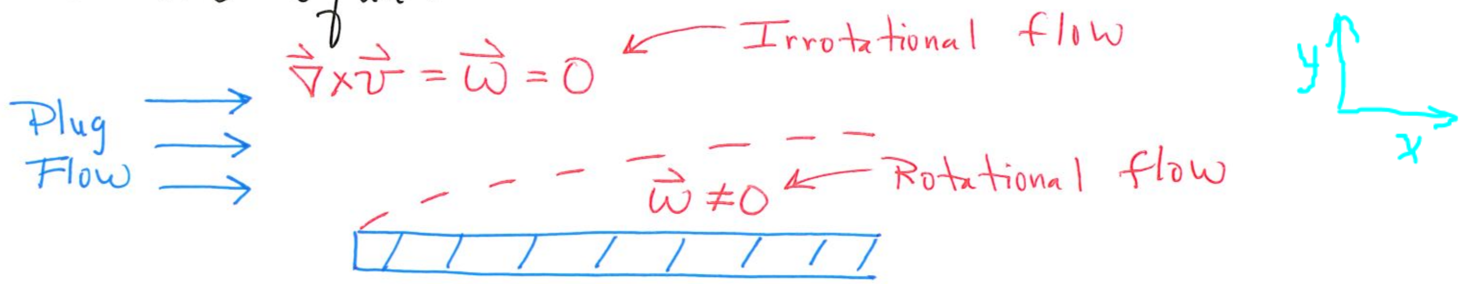
Assume ρ = constant, $\vec{\nabla}\cdot\vec{v} = 0$
and $\vec{g}$ is in the negative z direction so $\vec{\nabla}|\vec{g}|z = -\vec{g}$
The following then results

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{P}{\rho} + \frac{1}{2}|v|^2 + |\vec{g}|z\right) =$$

$$\vec{v}\times\vec{\omega} - \mu/\rho\,(\vec{\nabla}\times\vec{\omega})$$

This is sometimes called Crocco's Theorem
(L. Crocco, Z. angew Math. Mech., 17, 1, 1937).

Consider the full solution to the Navier-Stokes equation close to a solid surface



For 2D flow in the x-y plane, $\vec{\omega}$ is given as

$$\vec{\omega} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$v_z = 0$

Derivatives with respect to z are zero since no gradients in z direction

It can be shown that irrotation flow ($\vec{\nabla}\times\vec{v} = 0$) implies that a velocity potential ϕ exists so that $\vec{\nabla}\phi = \vec{v}$. The converse statement that potential flow implies irrotation flow is also true and is easier to prove:

Proof that $\vec{\nabla}\phi = \vec{v}$ implies $\vec{\nabla}\times\vec{v} = 0$ where ϕ is a scalar function:

$$\vec{\nabla}\times(\vec{\nabla}\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \begin{matrix} \vec{i}\left(\frac{\partial}{\partial z}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial z}\right) \\ -\vec{j}\left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial x}\right) \\ \vec{k}\left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}\right) \end{matrix}$$

Also note that $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi = \vec{\nabla}\cdot\vec{v} = 0$

And $\vec{v} = \vec{\nabla}\phi = \vec{i}\underbrace{\frac{\partial\phi}{\partial x}}_{v_x} + \vec{j}\underbrace{\frac{\partial\phi}{\partial y}}_{v_y}$

Also, if $\vec{\omega} = 0$, we have from page 1:

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{v}|^2 + P/\rho + |\vec{g}|z\right) = 0$$

or

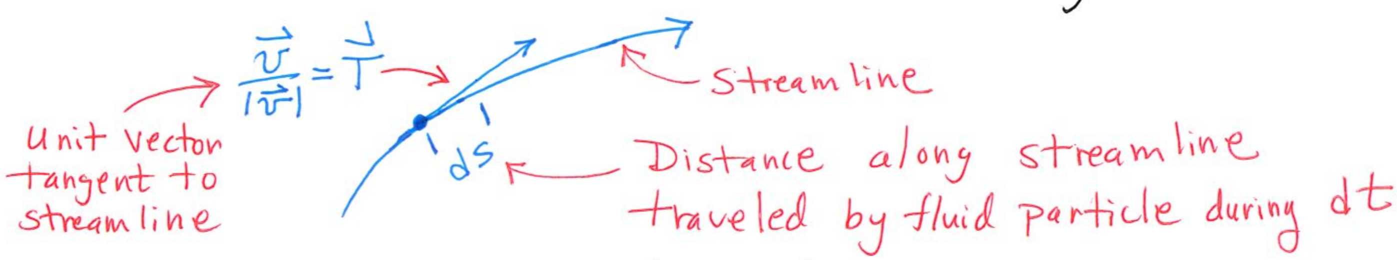
$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{v}|^2 + P/\rho + |\vec{g}|z = \text{constant}$$

Therefore, far from a solid surface and where no vorticity enters from the boundaries, it follows that $\vec{\omega} = 0$. This then means the flow field is independent of viscosity and the Bernoulli equation applies

Rotational Flow

Inviscid Flow

Consider the case where $\vec{\omega} \neq 0$ but $\mu = 0$



From page 1, at steady state:

$$\vec{\nabla}\left(P/\rho + \tfrac{1}{2}|\vec{v}|^2 + |g|z\right) = \vec{v} \times \vec{\omega}$$

Therefore: $\vec{T} \cdot \vec{\nabla}\left(P/\rho + \tfrac{1}{2}|\vec{v}|^2 + |g|z\right) = \vec{T} \cdot (\vec{v} \times \vec{\omega})$

$\vec{T} = \vec{v}/|\vec{v}|$ is always perpendicular to $\vec{v} \times \vec{\omega}$

$$\frac{d}{ds}\left(P/\rho + \tfrac{1}{2}|\vec{v}|^2 + |g|z\right) = 0$$

Conclusion: $P/\rho + \frac{1}{2}|\vec{v}|^2 + |g|z = \text{constant}$ along a streamline

Can also define a "stream function"

$$\frac{\partial \psi}{\partial y} = +v_x \; ; \; \frac{\partial \psi}{\partial x} = -v_y$$

BSL reverses the + and − signs

Stream function automatically satisfies continuity equation ($\vec{\nabla} \cdot \vec{v} = 0$)

Also $$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Along a path of constant ψ, $d\psi = 0$, so

$$0 = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$0 = -v_y \, dx + v_x \, dy$$

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{dy_{\text{fluid particle}}/dt}{dx_{\text{fluid particle}}/dt} = \frac{dy_{\text{fluid particle}}}{dx_{\text{fluid particle}}}$$

Stream function is therefore aligned with velocity vector.

Similarly, for constant ϕ:

$$d\phi = 0 = \underbrace{\frac{\partial \phi}{\partial x}}_{v_x} dx + \underbrace{\frac{\partial \phi}{\partial y}}_{v_y} dy$$

$\dfrac{dy}{dx} = \dfrac{-v_x}{v_y} \Rightarrow$ Curves of constant ϕ and ψ are at right angles.

It also follows that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \; ; \; \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

These are the "Cauchy - Rieman" equations which are satisfied by the real and imaginary parts of any function $W(z)$

$x + iy$ $\quad \sqrt{-1}$

$$w(z) = \phi(x,y) + i\,\psi(x,y)$$

Consider $\omega(z) = A\,(z + B/z)$

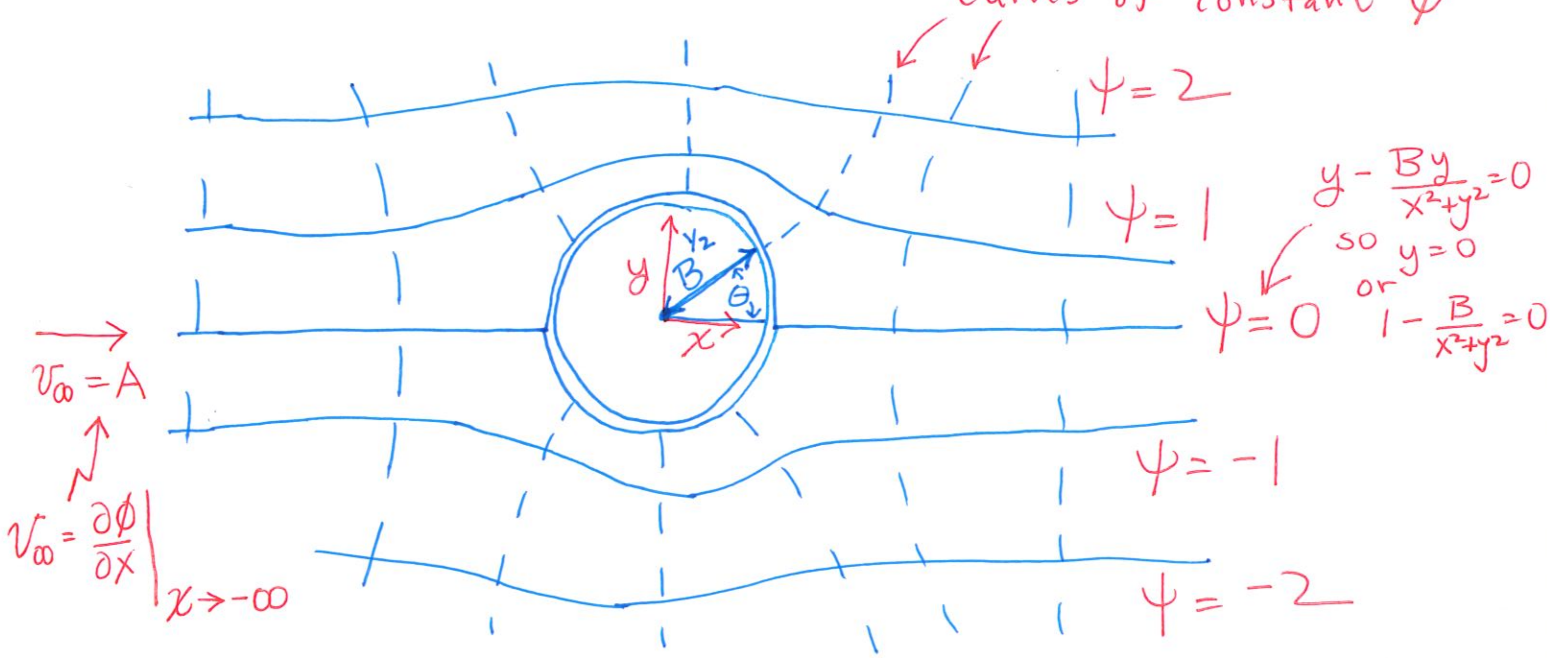
Substitute $z = x + iy$ ($i = \sqrt{-1}$)

$$\omega(z) = A\left(x+iy + \frac{B}{x+iy}\cdot\underbrace{\frac{x-iy}{x-iy}}_{\text{Unity}}\right)$$

$$= A\left(x+iy + \frac{B}{x^2+y^2}(x-iy)\right)$$

$$= A\left(\underbrace{x + \frac{Bx}{x^2+y^2}}_{\phi(x,y)} + i\underbrace{\left(y - \frac{By}{x^2+y^2}\right)}_{\psi(x,y)}\right)$$

Plot the stream function



Can also be interpreted as flow around a cylinder: