ENCH 630 - Lecture 1

Basic Mathematical Relations

Vector-tensor notation (see Appendix A of BSL)

Three basic types of dependent variables:

scalar = 0 order tensor; $3^0 = 1$ elements
vector = 1st order tensor; $3^1 = 3$ elements
tensor = 2nd order tensor; $3^2 = 9$ elements

# of elements in 3D space

Examples ?

Types of multiplication

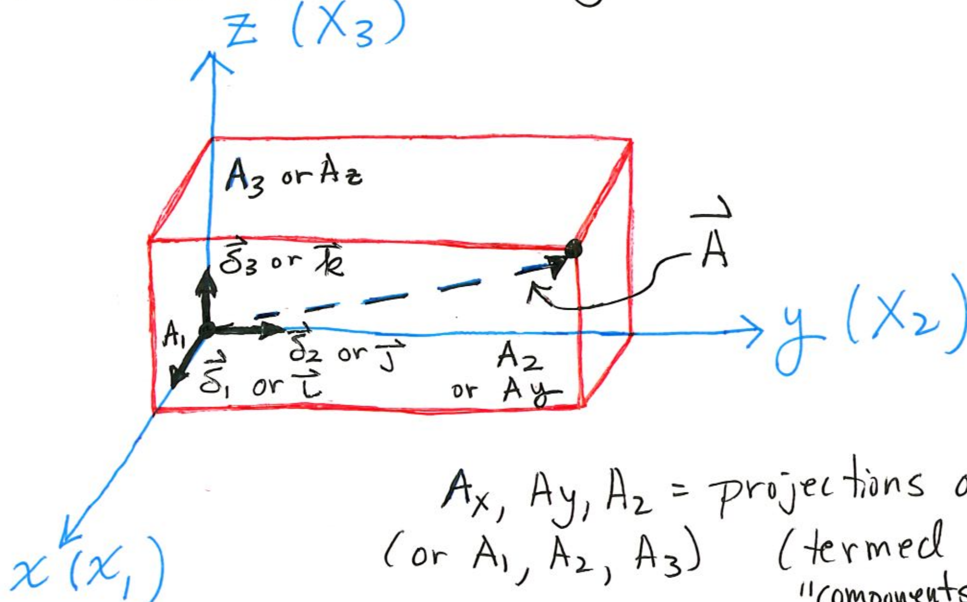
Multiplication sign	Product Order
None	Σ
X (cross)	$\Sigma - 1$
$\bullet$ (dot)	$\Sigma - 2$
: (double dot)	$\Sigma - 4$

(Σ = Sum of orders being multiplied)

Examples ?

Graphical representation of vector operations

Consider the Cartesian system:

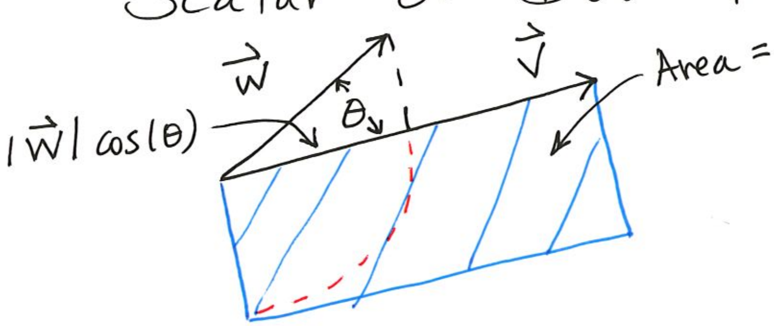


A_x, A_y, A_z = projections on x, y, z axis (or A_1, A_2, A_3) (termed x, y, z "components" or "elements")

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$
$$\vec{A} = A_1 \vec{\delta}_1 + A_2 \vec{\delta}_2 + A_3 \vec{\delta}_3$$

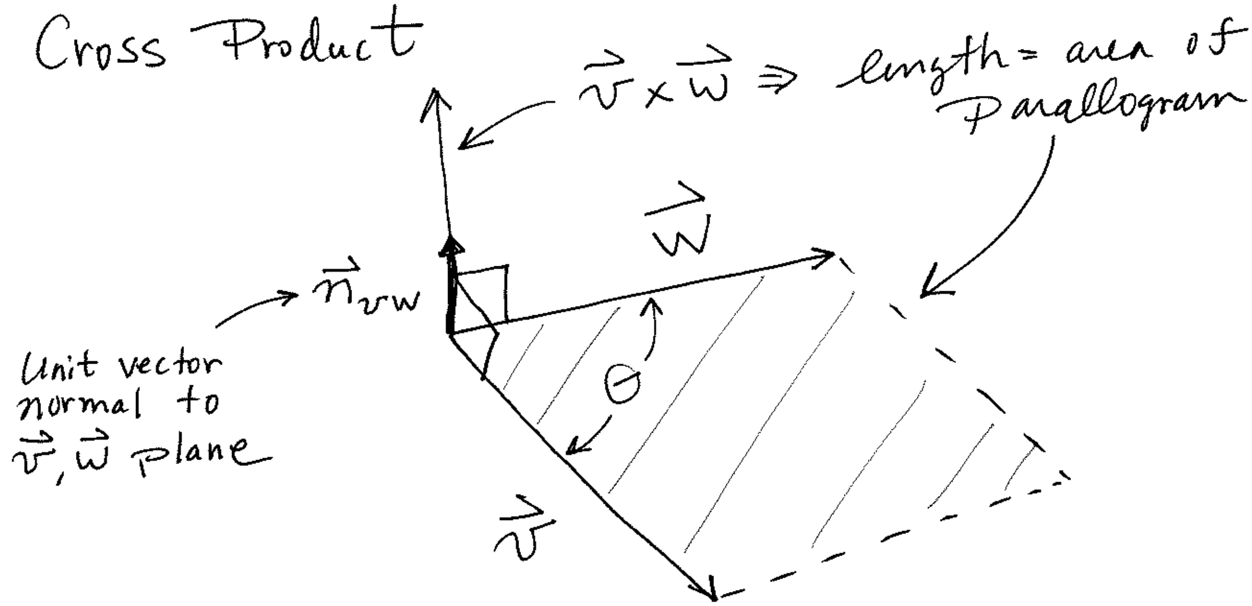
Magnitude $\Rightarrow$ $|\vec{A}| = \sqrt{\sum_{i=1}^{3} A_i^2}$

Scalar or Dot Product



$$\vec{W} \cdot \vec{V} = |\vec{W}|\,|\vec{V}| \cos\theta$$
$$= W_1 V_1 + W_2 V_2 + W_3 V_3$$
$$= W_x V_x + W_y V_y + W_z V_z$$

Cross Product



$$\vec{v}\times\vec{w} = |\vec{v}|\,|\vec{w}|\sin\theta\,\vec{n}_{vw}$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

evaluate by expanding in terms of cofactors of a row or column

determinant of "minor"

$$\vec{v}\times\vec{w} =$$

Kronecker delta function (δ_{ij})

$$\vec{\delta}_i\cdot\vec{\delta}_j = \delta_{ij}$$

$$\left.\begin{aligned} \delta_{ij} &= 1 \quad \text{if} \quad i=j \\ &= 0 \quad \text{if} \quad i\neq j \end{aligned}\right\} \quad \begin{aligned} \delta_{11} &= 1 \\ \delta_{12} &= 0 \\ &\text{etc.} \end{aligned}$$

Vector differential operators

$\vec{\nabla}$ = "del" operator

In Cartesian coordinates

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Gradient of a scalar variable (or "field")

$$\vec{\nabla} s = \vec{i}\frac{\partial s}{\partial x} + \vec{j}\frac{\partial s}{\partial y} + \vec{k}\frac{\partial s}{\partial z}$$

Divergence of a vector variable

$$\vec{\nabla}\cdot\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \underbrace{\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}}_{\substack{1,2,3 \text{ notation} \\ (x_1 = x,\ x_2 = y,\ \text{etc.})}}$$

Curl of a vector variable

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = ?$$

Laplacian of a scalar variable

$$\vec{\nabla}\cdot(\vec{\nabla} s) = (\vec{\nabla}\cdot\vec{\nabla}) s = \nabla^2 s \quad \leftarrow \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$$

Laplacian of a vector variable

$$\nabla^2 \vec{v} = \vec{i}\,\nabla^2 v_x + \vec{j}\,\nabla^2 v_y + \vec{k}\,\nabla^2 v_z$$

Consider also the total derivative:

$$ds = \left(\frac{\partial s}{\partial t}\right)_{x,y,z} dt + \left(\frac{\partial s}{\partial x}\right)_{t,y,z} dx + \left(\frac{\partial s}{\partial y}\right)_{t,x,z} dy + \left(\frac{\partial s}{\partial z}\right)_{t,x,y} dz$$

Divide by dt:

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial s}{\partial z}\frac{dz}{dt}$$

velocity of movement

Total time derivative (how s changes with time for a moving observed)

If the movement velocity is the fluid velocity, the "substantial" derivative results.

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} v_x + \frac{\partial s}{\partial y} v_y + \frac{\partial s}{\partial z} v_z$$

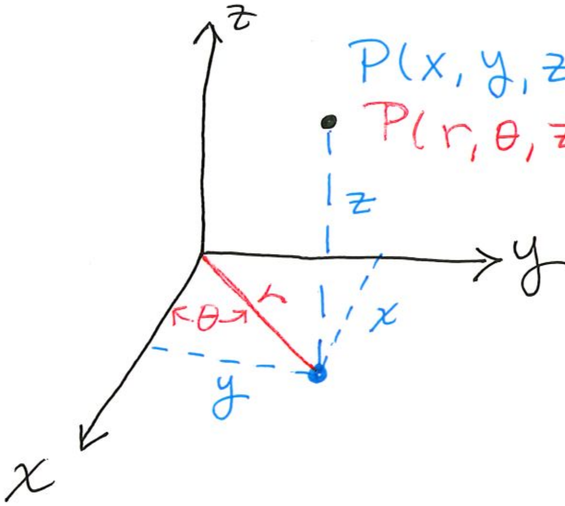
$$= \frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla} s$$

velocity of fluid

Define $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla}) \leftarrow$ Substantial derivative operator

Substantial derivative of a vector variable (see BSL)

Cylindrical coordinates



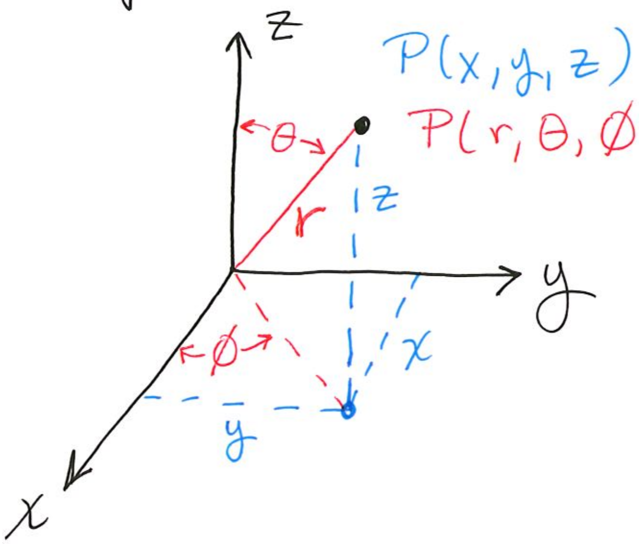
See BSL Table A-7 for definitions of $\vec{\nabla}, \vec{v}, \vec{\nabla}\cdot\vec{v}$ etc. in various coordinate systems

Cylindrical coordinates:

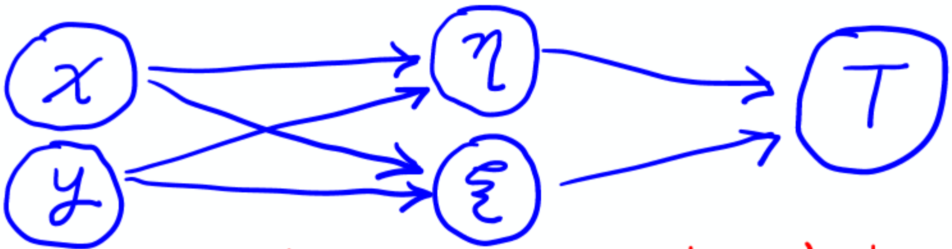
Divergence $\rightarrow \vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$

Laplacian $\rightarrow \nabla^2 S = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{\partial^2 S}{\partial z^2}$

Spherical coordinates



Chain rule of partial derivatizes. Consider the scalar dependent variable T (e.g., "temperature")



$\eta = \eta(x,y)$
$\xi = \xi(x,y)$

"original" independent variables

"new" independent variables

How to transform an equation containing

$$T(x,y), \quad \frac{\partial T(x,y)}{\partial x}, \quad \frac{\partial T(x,y)}{\partial y}, \quad \text{etc.}$$

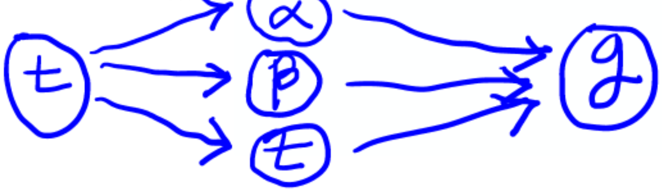
Into an equation containing

$$T(\eta,\xi), \quad \frac{\partial T(\eta,\xi)}{\partial \xi}, \quad \frac{\partial T(\eta,\xi)}{\partial \eta}, \quad \text{etc.}$$

$$\left(\frac{\partial T}{\partial x}\right)_y = \left(\frac{\partial T}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial x}\right)_y + \left(\frac{\partial T}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial x}\right)_y$$

$$\left(\frac{\partial T}{\partial y}\right)_x = \left(\frac{\partial T}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial y}\right)_x \left(\frac{\partial T}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial y}\right)_x$$

Example: $\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$ can be considered as a function of t (i.e., $g(t)$) or as a function of α, β and t (i.e., $g(\alpha,\beta,t)$). Determine $\frac{d}{dt} g(t)$ using the chain rule:



Result leads to Liebniz rule.