ENCH 630 – Transport Phenomena

Notes on a Linear Algebra approach to dimensional analysis, and its application to solving PDEs.

Consider the case where the dependent variable "u" is a function of the independent variables $W_1, \ldots W_5$. We can define the follow linear algebra problem (where the primary variables are L, M, t, T):

$$\underbrace{\begin{array}{c|cccccc} & W_1 & W_2 & W_3 & W_4 & W_5 & u \\ \hline L & x & x & x & x & x & x \\ M & x & x & x & x & x & x \\ t & x & x & x & x & x & x \\ T & x & x & x & x & x & x \end{array}}_{\text{dimension matrix } [g]} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ a \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

dimension vector for W_1

vector of exponents for a variable grouping (denote as $\vec{b}$)

dimension vector for a variable grouping (denote as $\vec{e}$)

For example, if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$, then

the variable grouping is $W_3^2 W_5 / u^2$

and the units are $[g] \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix} = \vec{e}$

For dimensionless variables, $\vec{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

so that the search for the complete set of dimensionless variables becomes equivalent to finding the "null space" of [g].

First, we find the "row rank" of [g] by reducing it to row eschelon form using Gaussian elimination.‡

$$\begin{bmatrix} x & x & x & x & x & x \\ 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x \\ 0 & 0 & 0 & x & x & x \end{bmatrix}$$

For the above case, the row rank is 4, and since the number of columns is 6, the null space is $6-4=2$, i.e., two dimensional.

To find a basis for the null space (i.e., the dimensionless variables) we take the starting matrix and select "repeating" variables that include all dimensions, with the number of repeating variables equal to the row rank.

‡ Due to the sparseness of [g], row eschelon form can often be achieved by the simple interchange of rows and columns, which corresponds to reordering units and variables.

$$\begin{array}{c|cccccc} & W_1 & W_2 & W_3 & W_4 & W_5 & U \\ \hline L & \times & \times & \times & \times & \times & \times \\ M & \times & \times & \times & \times & \times & \times \\ t & \times & \times & \times & \times & \times & \times \\ T & \times & \times & \times & \times & \times & \times \end{array}$$

$$\underbrace{\text{r} \quad \text{r} \quad \text{r} \quad \text{r}}_{\text{repeating parameters}}$$

Then form the combination of all repeating parameters with remaining non-repeating parameters (one at a time) to find dimensionless groups, i.e.:

$$\pi_1 = W_1 W_2^{e_2} W_3^{e_3} W_4^{e_4} W_5^{e_5},$$

$$\pi_2 = U W_2^{e_2'} W_3^{e_3'} W_4^{e_4'} W_5^{e_5'}$$

The two sets $e_2, \ldots, e_5$ and $e_2' \ldots e_5'$ are both found by solving a set of 4 linear equations for 4 unknowns.

Equivalently π_1 and π_2 can be obtained by solving the following systems

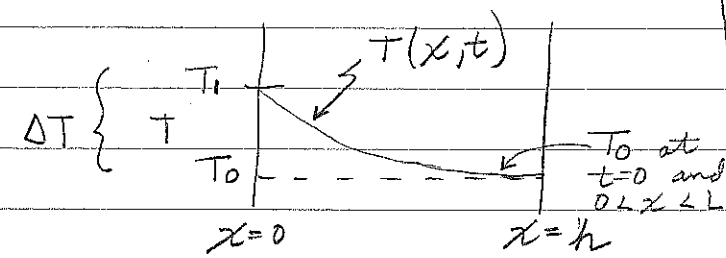
$$[g] \cdot \begin{bmatrix} 1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[g] \cdot \begin{bmatrix} 0 \\ e_2' \\ e_3' \\ e_4' \\ e_5' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Some illustrative examples related to solving PDEs.

Example 1:

Consider unsteady state heat transfer in a slab of width h with a sudden change in T of ΔT at $x=0$ and $t=0$:



The PDE and B.C.'s in terms of the normalized variable $T-T_0$ are:

$$\frac{\partial}{\partial t}(T-T_0) = \frac{k}{\rho C_p}\frac{\partial^2 (T-T_0)}{\partial X^2}$$

$T-T_0 = 0$ at $t=0$ and $0<X<h$
$T-T_0 = \Delta T$ at $t>0$ and $X=0$
$T-T_0 = 0$ at $t>0$ and $x=h$

We suppose that (based on the known pde and B.C.'s) (assume also the $k/\rho C_p$ is a constant):

$$T-T_0 = f_1\left(t, X, \frac{k}{\rho C_p}, h, \Delta T\right)$$

The matrix [g] becomes:

$$\begin{array}{c} \\ L \\ t \\ T \\ \\ \end{array}
\begin{array}{cccccc} x & t & k/\rho C_p & h & \Delta T & T-T_0 \\ \hline 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ r & & r & & r & \end{array}$$

By exchanging columns 3 and 5 we see the row rank is 3 and there are 6-3=3 π groups. Using the 3 repeating parameters shown, a system of dimensionless variables is

$$\frac{T-T_0}{\Delta T} = f_2\left(\frac{x}{h}, \frac{k}{\rho C_p}\frac{t}{x^2}\right)$$

This means we still need to solve a PDE even in dimensionless form.

Consider again the previous example. We presume that as $h \to \infty$ (the semi-infinite slab problem) there is an asymptotic solution where $T-T_0/\Delta T = f_2\left(0, \frac{k}{\rho C_p}\frac{t}{x^2}\right) = f_3\left(\frac{kt}{\rho C_p x^2}\right)$

When $h = \infty$, the B.C.'s become $T - T_0 = 0$ at $t = 0$ and $0 < x < \infty$, $T - T_0 = \Delta T$ at $t > 0$ and $x = 0$, and $T - T_0 = 0$ at $t > 0$ and $0 < x < \infty$. h is therefore no longer a parameter, and the matrix [g] becomes:

$$\begin{array}{c|ccccc} & x & t & k/\rho C_p & \Delta T & T-T_0 \\ \hline L & 1 & 0 & 2 & 0 & 0 \\ t & 0 & 1 & -1 & 0 & 0 \\ T & 0 & 0 & 0 & 1 & 1 \\ & r & & r & r & \end{array}$$

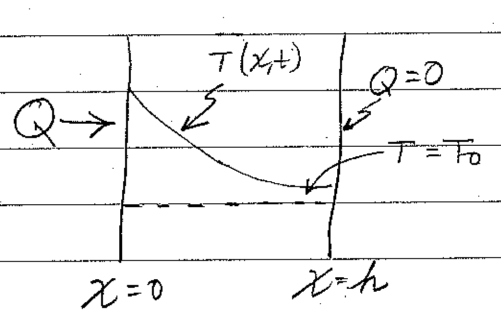
By exchanging columns 3 and 4, we see the row rank is 3 and the number of dimensionless variables is $5 - 3 = 2$. Using the repeating parameters shown above, we get as expected:

$$\frac{T-T_0}{\Delta T} = f_3\left(\frac{k}{\rho C_p}\frac{t}{X^2}\right)$$

By transforming the starting P.D.E. and B.C.'s into the above dimensionless variables, we see the PDE becomes an ODE that can be more easily solved. This is the basis for the "combination of variables" method for solving the governing PDE. The solution obtained is also termed a "self-similar" solution or "automodel" solution. The term "self similar" comes from the property that a single dimensional solution can be plotted as $\frac{T-T_0}{\Delta T} = f\left(\frac{t}{x^2}\frac{k}{\rho C_p}\right)$ and the plot used to describe all solutions.

Example 2:

Let us now change the boundary conditions at $x=0$ and $x=h$: constant heat flux, Q, instead of a constant temperature change ΔT at $x=0$, and also $Q=0$ at $x=h$ (insulated surface):



The PDE and BC's are:

$$\frac{\partial (T-T_0)}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 (T-T_0)}{\partial x^2}$$

$T-T_0 = 0$ at $t=0$ and $0<x<h$

$\dfrac{\partial (T-T_0)}{\partial x} = -\dfrac{Q}{k}$ at $t>0$ and $x=0$

$\dfrac{\partial (T-T_0)}{\partial x} = 0$ at $t>0$ and $x=h$

We assume $T-T_0 = f_1(x, t, k/\rho C_p, h, Q/k)$

The matrix $[g]$ then becomes:

$$\begin{array}{c c} & \begin{array}{cccccc} x & t & k/\rho C_p & h & Q/k & T-T_0 \end{array} \\ \begin{array}{c} L \\ t \\ T \end{array} & \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Exchanging columns 3 and 6 indicated rank is 3, and the number of dimensionless variables is $6-3=3$. The governing equation is therefore still a PDE after transforming to dimensionless variables.

As in the previous example, let $h \to \infty$ so that h is no longer a parameter. The matrix $[g]$ then becomes:

$$\begin{array}{c c} & \begin{array}{ccccc} x & t & k/\rho C_p & Q/k & T-T_0 \end{array} \\ \begin{array}{c} L \\ t \\ T \end{array} & \left[\begin{array}{ccccc} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \\ & \begin{array}{ccccc} & r & r & r & \end{array} \end{array}$$

Here again the row rank is 3, and the number of dimensionless variables becomes 5 - 3 = 2. Using the repeating parameters shown we get as dimensionless groups:

$$\frac{(T-T_0)^2}{t\,\frac{k}{\rho C_p}\left(\frac{Q}{k}\right)^2} \quad \text{and} \quad \left(\frac{k}{\rho C_p}\frac{t}{X^2}\right)$$

or, taking the square root of the first dimensionless variable to form a new dimensionless variable, we get:

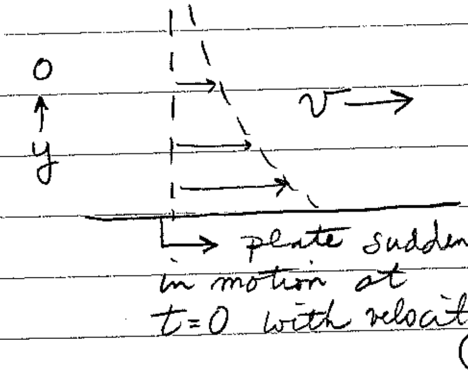
$$T - T_0 = \frac{Q\sqrt{t}}{\sqrt{\rho C_p k}}\, f_2\left(\frac{k}{\rho C_p}\frac{t}{X^2}\right)$$

The above can again be used to develop a "self-similar" solution by solving an ODE instead of a PDE.

See Appendices A, B and C for details related to the conversion of the PDEs for examples 1 and 2 into ODEs.

Let us also consider the fluid mechanics version of the semi-infinite slab version of example 1:

Example 3: Sudden motion of a plate in a semi-infinite fluid



PDE

$$\frac{\partial v}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 v}{\partial y^2}$$

BC's

$v=0$ at $t=0$ and $0<y<\infty$

$v=0$ at $t>0$ and $y=\infty$

$v=U$ at $t>0$ and $y=0$

We assume:

$$v = f_1\left(\mu/\rho,\ t,\ x;\ U\right)$$

The matrix [g] becomes:

$$\begin{array}{c c}
 & \begin{array}{ccccc} \mu/\rho & t & x & U & v \end{array} \\
\begin{array}{c} L \\ t \end{array} & \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 & -1 \end{array}\right]
\end{array}$$

The row rank is 2, so the number of dimensionless groups is $5-2=3$, so that dimensional analysis does not directly yield an ODE.

On comparing Examples 1 and 3, we see that the main difference is that in the former case the dependent variable (T) has is own unique primary unit (temperature) whereas in Example 3 the dependent variable (v) has units of

L / t. This suggests the idea that for example 3 we define an "artificial primary unit" (APU) of "velocity" so that examples 1 and 3 are exactly analogous.

With the APU of "velocity" we get in the present case

$$\begin{array}{r|c} & \begin{matrix} \mu/\rho & t & x & U & v \end{matrix} \\ \begin{matrix} L \\ t \\ \text{APU} \rightarrow \text{velocity} \end{matrix} & \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\ & \begin{matrix} & r & r & r & \end{matrix} \end{array}$$

The row rank here is 3, so there are $5-3=2$ dimensionless groups.

Note that in the above matrix we ignore the fact that the APU "velocity" has units of L/t. We also need each primary unit to occur in at least two variables, so both v and U are given units of "velocity".

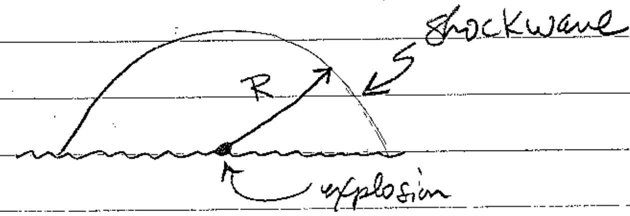
Using the repeating parameters shown above, we have

$$\frac{v}{U} = f\left(\frac{\mu}{\rho}\frac{t}{x^2}\right)$$

Although the use here of the APU "velocity" is relatively obvious, this concept can also be applied in less obvious situations as well.

Development of an empirical model based on dimensionless variables

Example: Model for determining the energy released in a nuclear explosion (adapted from Bluman and Kumei, and based on a study originally by G.I. Taylor)



Video camera gives $R(t)$, and it is presumed the parameters are E, ρ_0, and P_0 (the energy, the initial air density, and the initial air pressure). This yields the assumption

$$R = f_1(E, t, \rho_0, P_0)$$

The matrix $[G]$ becomes:

$$\begin{array}{c|ccccc} & E & t & \rho_0 & P_0 & R \\ \hline L & 2 & 0 & -3 & -1 & 1 \\ M & 1 & 0 & 1 & 1 & 0 \\ t & -2 & 1 & 0 & 2 & 0 \\ & r & r & r & & \end{array}$$

The row rank is 3, so there are $5-3=2$ dimensionless groups. Using the above repeating parameters we get:

$$\Pi_1 = P_0 \left[\frac{t^6}{E^2 \rho_0^3} \right]^{1/5}$$

$$\Pi_2 = R \left[\frac{E t^2}{\rho_0} \right]^{-1/5}$$

and $\Pi_2 = f_2(\Pi_1)$

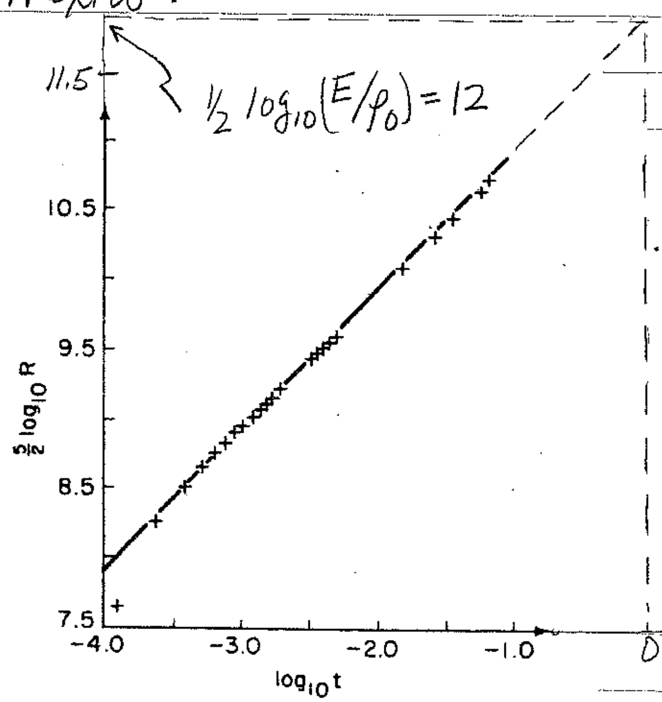
Experimental studies with small explosions indicates that $f_2(\Pi_1) \approx$ constant ≈ 1 for the range of Π_1 of interest. The desired model therefore becomes

$$R \left[\frac{E t^2}{\rho_0} \right]^{-1/5} = 1$$

or $\quad \frac{5}{2} \log_{10} R = \frac{1}{2} \log_{10} \frac{E}{\rho_0} + \log_{10} t$

Thus a plot of $\frac{5}{2} \log_{10} R$ versus $\log_{10} t$ should yield a line with slope $= 1$ and an intercept of $\frac{1}{2} \log_{10} E/\rho_0$

Actual data from 1945 atomic test in New Mexico:



Appendix A: Transformation of the PDE in Example 1 to an ODE (for the case where $h \to \infty$)

To transform the PDE in example 1 to an ODE, use the chain rule and let $T^* = (T - T_0)/\Delta T$ and $\lambda = \frac{k}{\rho C_p}\frac{t}{x^2}$. The algebra is simplest if a new dimensionless variable is defined as $\eta = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\lambda}} = \frac{x}{\sqrt{2\alpha t}}$ where $\alpha = k/\rho C_p$.

Then:

$$\left(\frac{\partial T^*}{\partial t}\right)_x = \frac{dT^*}{d\eta}\cdot\left(\frac{\partial \eta}{\partial t}\right)_x = \frac{dT^*}{d\eta}\cdot\frac{x}{\sqrt{2\alpha}}\left(\frac{-1/2}{t^{3/2}}\right)$$

$$\left(\frac{\partial T^*}{\partial x}\right)_t = \frac{dT^*}{d\eta}\cdot\left(\frac{\partial \eta}{\partial x}\right)_t = \frac{dT^*}{d\eta}\frac{1}{\sqrt{2\alpha t}}$$

$$\left(\frac{\partial^2 T^*}{\partial x^2}\right)_t = \frac{1}{\sqrt{2\alpha t}}\cdot\frac{d^2 T^*}{d\eta^2}\cdot\left(\frac{\partial \eta}{\partial x}\right)_t = \frac{d^2 T^*}{d\eta^2}\frac{1}{2\alpha t}$$

Substituting into the governing PDE on page 4 (with $h \to \infty$) yields:

$$\frac{dT^*}{d\eta}\cdot\frac{x}{\sqrt{2\alpha t}}\frac{(-1/2)}{\cancel{t}} = \cancel{\alpha}\cdot\frac{d^2T^*}{d\eta^2}\frac{1}{2\cancel{\alpha}\cancel{t}}$$

Which leads to:

$$0 = \frac{d^2T^*}{d\eta^2} + \eta\frac{dT^*}{d\eta}$$

with BC's $T^* = 0$ when $\eta = \infty$ and $T^* = 1$ when $\eta = 0$

Solution is

$$T^* = \operatorname{erfc}(\eta)$$

or

$$\frac{T - T_0}{\Delta T} = \operatorname{erfc}\left(\frac{x}{\sqrt{2\alpha t}}\right)$$

Appendix B: Transforming the PDE in example 2 to an ODE (for the case where $h \to \infty$)

The governing PDE is

$$\frac{\partial T^*}{\partial t} = \alpha\, \partial^2 T^* / \partial x^{*2}$$

Using the chain rule (with $\eta = \alpha t / x^2$ and $T^* = \beta \sqrt{\alpha}\sqrt{t}\, f_2(\eta)$)

$$\left(\frac{\partial T^*}{\partial t}\right)_x = \left(\frac{\partial T^*}{\partial t}\right)_\eta \cancelto{1}{\left(\frac{\partial t}{\partial t}\right)_x} + \left(\frac{\partial T^*}{\partial \eta}\right)_t \left(\frac{\partial \eta}{\partial t}\right)_x$$

$$= \tfrac{1}{2}\beta\sqrt{\alpha}\, t^{-1/2} f_2 + \beta\sqrt{\alpha}\, t^{1/2} f_2' \,\alpha x^{-2}$$

$$\left(\frac{\partial T^*}{\partial x}\right)_t = \left(\frac{\partial T^*}{\partial t}\right)_\eta \cancelto{0}{\left(\frac{\partial t}{\partial x}\right)_t} + \left(\frac{\partial T^*}{\partial \eta}\right)_t \underbrace{\left(\frac{\partial \eta}{\partial x}\right)_t}_{-2\alpha t x^{-3}}$$

$$= \beta\sqrt{\alpha}\sqrt{t}\, f_2' \cdot (-2\alpha t x^{-3})$$

$$= -2\beta f_2' \eta^{3/2}$$

$$\left(\frac{\partial^2 T^*}{\partial x^2}\right)_t = -2\beta \left(\frac{\partial}{\partial x}\left(f_2' \eta^{3/2}\right)\right)_t = -2\beta \frac{\partial}{\partial \eta}\left(f_2' \eta^{3/2}\right)\left(\frac{\partial \eta}{\partial x}\right)_t$$

$$= -2\beta \left(f_2' \tfrac{3}{2}\eta^{1/2} + \eta^{3/2} f''\right) \cdot (-2\alpha t x^{-3})$$

Substituting into the first equation on this page yields:

$$\tfrac{1}{2}\beta\sqrt{\alpha}\, t^{-1/2} f_2 + \beta\sqrt{\alpha}\, t^{1/2} f_2' \alpha x^{-2} = \alpha \cdot (-2\beta) \cdot f_2' \tfrac{3}{2}\eta^{1/2} + \eta^{3/2} f'' \cdot (-2\alpha t x^{-3})$$

Using $\eta = \alpha t / x^2$ and simplifying yields:

$$\tfrac{1}{2} f_2 + f_2'(\eta - 4\eta^2) - \eta f_2'' = 0$$

This is the ODE for f_2 that can be solved.

Appendix C: A different approach

There is another option to determine the similarity parameter, which is the "try it and see" approach. Returning to the semi-infinite domain version of Example 1, we can <u>hypothesize</u> that

$$T^* = h(t) \cdot f\left(\frac{x}{g(t)}\right)$$

By examining the BC that $T^* = 1$ when $x = 0$, we have $T^*(0) = h(t) \cdot f(0) = 1$, so we have $h(t) = 1$ and we can also set $f(0) = 1$.

Now substitute the above equation into the governing PDE (i.e., $\partial T^*/\partial t = \alpha\, \partial^2 T^*/\partial x^2$) and use the chain rule as follows (let $\eta = \frac{x}{g(t)}$)

$$T^*(x, t) \longrightarrow T^*(\underset{x,t}{\underset{\uparrow}{\eta}})$$

We get:

$$\frac{d^2 f}{d\eta^2} + \frac{g g'}{\alpha} \eta \frac{df}{d\eta} = 0$$

To eliminate the variable "t" in the above equation, we need to have $g g' = \text{constant}$. The most useful constant is 2α, in which case we have $g g' = 2\alpha$ so $g(t) = (2\alpha t)^{1/2}$. This yield the same result as earlier where

$$\eta = \frac{x}{\sqrt{2\alpha t}}$$