HANDBOOK OF MATHEMATICAL FUNCTIONS

WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES

Edited by
Milton Abramowitz and Irene A. Stegun

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9. Bessel Functions of Integer Order

Mathematical Properties

Notation

The tables in this chapter are for Bessel functions of integer order; the text treats general orders. The conventions used are:

$z=x+iy$; x, y real.

n is a positive integer or zero.

ν, μ are unrestricted except where otherwise indicated; ν is supposed real in the sections devoted to Kelvin functions **9.9**, **9.10**, and **9.11**.

The notation used for the Bessel functions is that of Watson [9.15] and the British Association and Royal Society Mathematical Tables. The function $Y_\nu(z)$ is often denoted $N_\nu(z)$ by physicists and European workers.

Other notations are those of:
Aldis, Airey:

$$G_n(z) \text{ for } -\tfrac{1}{2}\pi Y_n(z), K_n(z) \text{ for } (-)^n K_n(z).$$

Clifford:

$$C_n(x) \text{ for } x^{-\frac{1}{2}n} J_n(2\sqrt{x}).$$

Gray, Mathews and MacRobert [9.9]:

$$Y_n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z),$$

$$\overline{Y}_\nu(z) \text{ for } \pi e^{\nu\pi i}\sec(\nu\pi)Y_\nu(z),$$

$$G_\nu(z) \text{ for } \tfrac{1}{2}\pi i H_\nu^{(1)}(z).$$

Jahnke, Emde and Lösch [9.32]:

$$\Lambda_\nu(z) \text{ for } \Gamma(\nu+1)(\tfrac{1}{2}z)^{-\nu}J_\nu(z).$$

Jeffreys:

$$Hs_\nu(z) \text{ for } H_\nu^{(1)}(z),\ Hi_\nu(z) \text{ for } H_\nu^{(2)}(z),$$

$$Kh_\nu(z) \text{ for } (2/\pi)K_\nu(z).$$

Heine:

$$K_n(z) \text{ for} -\tfrac{1}{2}\pi Y_n(z).$$

Neumann:

$$Y^n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z).$$

Whittaker and Watson [9.18]:

$$K_\nu(z) \text{ for } \cos(\nu\pi)K_\nu(z).$$

Bessel Functions J and Y

9.1. Definitions and Elementary Properties

Differential Equation

9.1.1
$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0$$

Solutions are the Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $Y_\nu(z)$ (also called Weber's function) and of the third kind $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (also called the Hankel functions). Each is a regular (holomorphic) function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq 0)$ each is an entire (integral) function of ν. When $\nu=\pm n$, $J_\nu(z)$ has no branch point and is an entire (integral) function of z.

Important features of the various solutions are as follows: $J_\nu(z)(\mathscr{R}\nu\geq 0)$ is bounded as $z\to 0$ in any bounded range of arg z. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν.

$H_\nu^{(1)}(z)$ tends to zero as $|z|\to\infty$ in the sector $0<\arg z<\pi$; $H_\nu^{(2)}(z)$ tends to zero as $|z|\to\infty$ in the sector $-\pi<\arg z<0$. For all values of ν, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent.

Relations Between Solutions

9.1.2
$$Y_\nu(z)=\frac{J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.1.3

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)$$
$$=i\csc(\nu\pi)\{e^{-\nu\pi i}J_\nu(z)-J_{-\nu}(z)\}$$

9.1.4

$$H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)$$
$$=i\csc(\nu\pi)\{J_{-\nu}(z)-e^{\nu\pi i}J_\nu(z)\}$$

9.1.5 $J_{-n}(z)=(-)^n J_n(z)$ $\quad Y_{-n}(z)=(-)^n Y_n(z)$

9.1.6 $H_{-\nu}^{(1)}(z)=e^{\nu\pi i}H_\nu^{(1)}(z)$ $\quad H_{-\nu}^{(2)}(z)=e^{-\nu\pi i}H_\nu^{(2)}(z)$

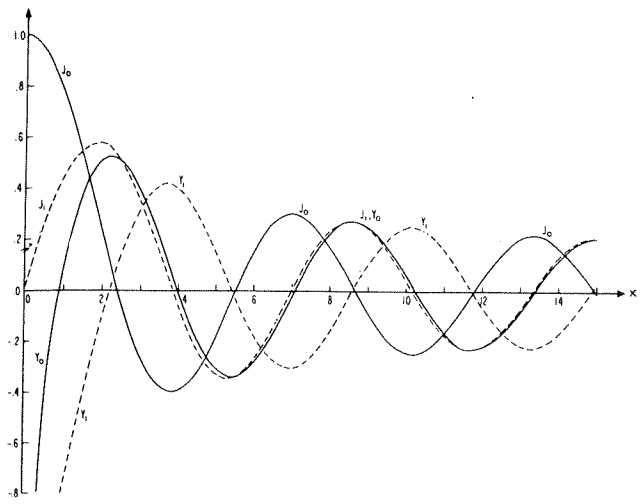

FIGURE 9.1. $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$.

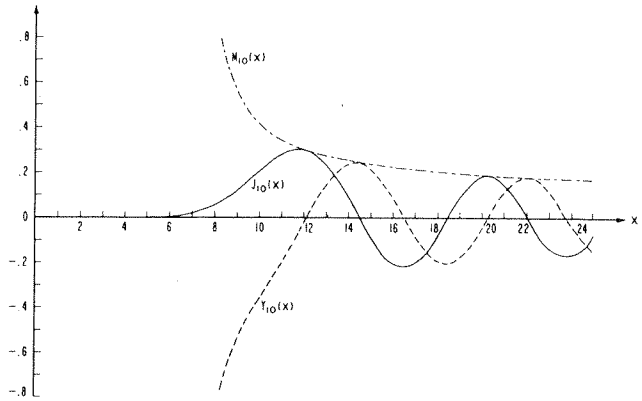

FIGURE 9.2. $J_{10}(x)$, $Y_{10}(x)$, *and* $M_{10}(x)=\sqrt{J_{10}^2(x)+Y_{10}^2(x)}$.

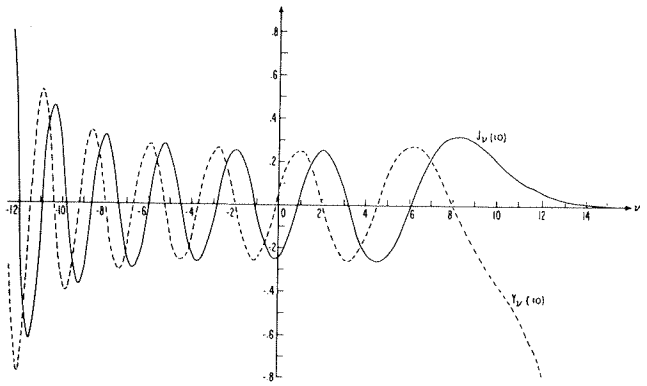

FIGURE 9.3. $J_\nu(10)$ *and* $Y_\nu(10)$.

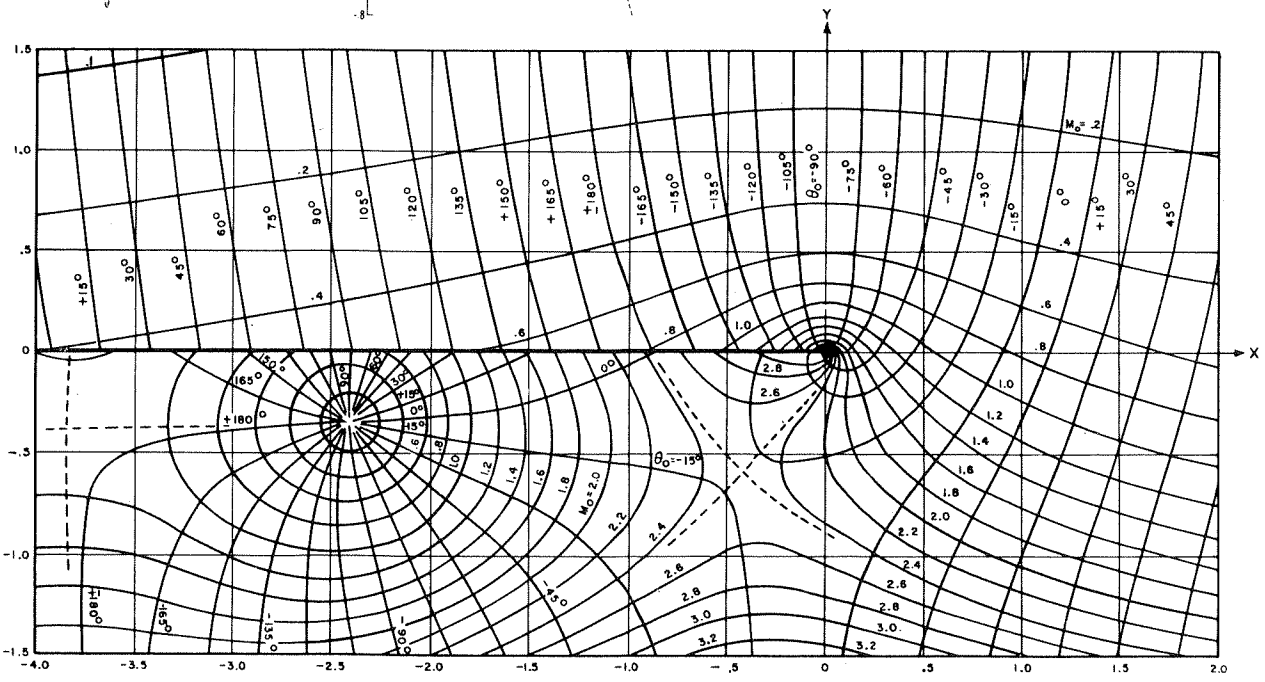

FIGURE 9.4. *Contour lines of the modulus and phase of the Hankel Function* $H_0^{(1)}(x+iy)=M_0e^{i\theta_0}$. From E. Jahnke, F. Emde, and F. Lösch, Tables of higher functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1960 (with permission).

Table 9.1 BESSEL FUNCTIONS—ORDERS 0, 1 AND 2

x	$J_0(x)$	$J_1(x)$	$J_2(x)$
0.0	1.00000 00000 00000	0.00000 00000	0.00000 00000
0.1	0.99750 15620 66040	0.04993 75260	0.00124 89587
0.2	0.99002 49722 39576	0.09950 08326	0.00498 33542
0.3	0.97762 62465 38296	0.14831 88163	0.01116 58619
0.4	0.96039 82266 59563	0.19602 65780	0.01973 46631
0.5	0.93846 98072 40813	0.24226 84577	0.03060 40235
0.6	0.91200 48634 97211	0.28670 09881	0.04366 50967
0.7	0.88120 08886 07405	0.32899 57415	0.05878 69444
0.8	0.84628 73527 50480	0.36884 20461	0.07581 77625
0.9	0.80752 37981 22545	0.40594 95461	0.09458 63043
1.0	0.76519 76865 57967	0.44005 05857	0.11490 34849
1.1	0.71962 20185 27511	0.47090 23949	0.13656 41540
1.2	0.67113 27442 64363	0.49828 90576	0.15934 90183
1.3	0.62008 59895 61509	0.52202 32474	0.18302 66988
1.4	0.56685 51203 74289	0.54194 77139	0.20735 58995
1.5	0.51182 76717 35918	0.55793 65079	0.23208 76721
1.6	0.45540 21676 39381	0.56989 59353	0.25696 77514
1.7	0.39798 48594 46109	0.57776 52315	0.28173 89424
1.8	0.33998 64110 42558	0.58151 69517	0.30614 35353
1.9	0.28181 85593 74385	0.58115 70727	0.32992 57277
2.0	0.22389 07791 41236	0.57672 48078	0.35283 40286
2.1	0.16660 69803 31990	0.56829 21358	0.37462 36252
2.2	0.11036 22669 22174	0.55596 30498	0.39505 86875
2.3	0.05553 97844 45602	0.53987 25326	0.41391 45917
2.4	+0.00250 76832 97244	0.52018 52682	0.43098 00402
2.5	-0.04838 37764 68198	0.49709 41025	0.44605 90584
2.6	-0.09680 49543 97038	0.47081 82665	0.45897 28517
2.7	-0.14244 93700 46012	0.44160 13791	0.46956 15027
2.8	-0.18503 60333 64387	0.40970 92469	0.47768 54954
2.9	-0.22431 15457 91968	0.37542 74818	0.48322 70505
3.0	-0.26005 19549 01933	0.33905 89585	0.48609 12606
3.1	-0.29206 43476 50698	0.30092 11331	0.48620 70142
3.2	-0.32018 81696 57123	0.26134 32488	0.48352 77001
3.3	-0.34429 62603 98885	0.22066 34530	0.47803 16865
3.4	-0.36429 55967 62000	0.17922 58517	0.46972 25683
3.5	-0.38012 77399 87263	0.13737 75274	0.45862 91842
3.6	-0.39176 89837 00798	0.09546 55472	0.44480 53988
3.7	-0.39923 02033 71191	0.05383 39877	0.42832 96562
3.8	-0.40255 64101 78564	+0.01282 10029	0.40930 43065
3.9	-0.40182 60148 87640	-0.02724 40396	0.38785 47125
4.0	-0.39714 98098 63847	-0.06604 33280	0.36412 81459
4.1	-0.38866 96798 35854	-0.10327 32577	0.33829 24809
4.2	-0.37655 70543 67568	-0.13864 69421	0.31053 47010
4.3	-0.36101 11172 36535	-0.17189 65602	0.28105 92288
4.4	-0.34225 67900 03886	-0.20277 55219	0.25008 60982
4.5	-0.32054 25089 85121	-0.23106 04319	0.21784 89837
4.6	-0.29613 78165 74141	-0.25655 28361	0.18459 31052
4.7	-0.26933 07894 19753	-0.27908 07358	0.15057 30295
4.8	-0.24042 53272 91183	-0.29849 98581	0.11605 03864
4.9	-0.20973 83275 85326	-0.31469 46710	0.08129 15231
5.0	-0.17759 67713 14338	-0.32757 91376	0.04656 51163
	$\left[\begin{matrix}(-4)6\\11\end{matrix}\right]$	$\left[\begin{matrix}(-4)5\\8\end{matrix}\right]$	$\left[\begin{matrix}(-4)3\\7\end{matrix}\right]$

$$J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$$

Compiled from British Association for the Advancement of Science, Bessel functions, Part II. Functions of positive integer order, Mathematical Tables, vol. X (Cambridge Univ. Press, Cambridge, England, 1952) and Harvard Computation Laboratory, Tables of the Bessel functions of the first kind of orders 0 through 135, vols. 3–14 (Harvard Univ. Press, Cambridge, Mass., 1947–1951) (with permission).

Table 9.5

ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

s	$j_{0,s}$	$J'_0(j_{0,s})$	$j_{1,s}$	$J'_1(j_{1,s})$	$j_{2,s}$	$J'_2(j_{2,s})$
1	2.40482 55577	−0.51914 74973	3.83171	−0.40276	5.13562	−0.33967
2	5.52007 81103	+0.34026 48065	7.01559	+0.30012	8.41724	+0.27138
3	8.65372 79129	−0.27145 22999	10.17347	−0.24970	11.61984	−0.23244
4	11.79153 44391	+0.23245 98314	13.32369	+0.21836	14.79595	+0.20654
5	14.93091 77086	−0.20654 64331	16.47063	−0.19647	17.95982	−0.18773
6	18.07106 39679	+0.18772 88030	19.61586	+0.18006	21.11700	+0.17326
7	21.21163 66299	−0.17326 58942	22.76008	−0.16718	24.27011	−0.16170
8	24.35247 15308	+0.16170 15507	25.90367	+0.15672	27.42057	+0.15218
9	27.49347 91320	−0.15218 12138	29.04683	−0.14801	30.56920	−0.14417
10	30.63460 64684	+0.14416 59777	32.18968	+0.14061	33.71652	+0.13730
11	33.77582 02136	−0.13729 69434	35.33231	−0.13421	36.86286	−0.13132
12	36.91709 83537	+0.13132 46267	38.47477	+0.12862	40.00845	+0.12607
13	40.05842 57646	−0.12606 94971	41.61709	−0.12367	43.15345	−0.12140
14	43.19979 17132	+0.12139 86248	44.75932	+0.11925	46.29800	+0.11721
15	46.34118 83717	−0.11721 11989	47.90146	−0.11527	49.44216	−0.11343
16	49.48260 98974	+0.11342 91926	51.04354	+0.11167	52.58602	+0.10999
17	52.62405 18411	−0.10999 11430	54.18555	−0.10839	55.72963	−0.10685
18	55.76551 07550	+0.10684 78883	57.32753	+0.10537	58.87302	+0.10396
19	58.90698 39261	−0.10395 95729	60.46946	−0.10260	62.01622	−0.10129
20	62.04846 91902	+0.10129 34989	63.61136	+0.10004	65.15927	+0.09882

s	$j_{3,s}$	$J'_3(j_{3,s})$	$j_{4,s}$	$J'_4(j_{4,s})$	$j_{5,s}$	$J'_5(j_{5,s})$
1	6.38016	−0.29827	7.58834	−0.26836	8.77148	−0.24543
2	9.76102	+0.24942	11.06471	+0.23188	12.33860	+0.21743
3	13.01520	−0.21828	14.37254	−0.20636	15.70017	−0.19615
4	16.22347	+0.19644	17.61597	+0.18766	18.98013	+0.17993
5	19.40942	−0.18005	20.82693	−0.17323	22.21780	−0.16712
6	22.58273	+0.16718	24.01902	+0.16168	25.43034	+0.15669
7	25.74817	−0.15672	27.19909	−0.15217	28.62662	−0.14799
8	28.90835	+0.14801	30.37101	+0.14416	31.81172	+0.14059
9	32.06485	−0.14060	33.53714	−0.13729	34.98878	−0.13420
10	35.21867	+0.13421	36.69900	+0.13132	38.15987	+0.12861
11	38.37047	−0.12862	39.85763	−0.12607	41.32638	−0.12366
12	41.52072	+0.12367	43.01374	+0.12140	44.48932	+0.11925
13	44.66974	−0.11925	46.16785	−0.11721	47.64940	−0.11527
14	47.81779	+0.11527	49.32036	+0.11343	50.80717	+0.11167
15	50.96503	−0.11167	52.47155	−0.10999	53.96303	−0.10838
16	54.11162	+0.10839	55.62165	+0.10685	57.11730	+0.10537
17	57.25765	−0.10537	58.77084	−0.10396	60.27025	−0.10260
18	60.40322	+0.10260	61.91925	+0.10129	63.42205	+0.10003
19	63.54840	−0.10004	65.06700	−0.09882	66.57289	−0.09765
20	66.69324	+0.09765	68.21417	+0.09652	69.72289	+0.09543

s	$j_{6,s}$	$J'_6(j_{6,s})$	$j_{7,s}$	$J'_7(j_{7,s})$	$j_{8,s}$	$J'_8(j_{8,s})$
1	9.93611	−0.22713	11.08637	−0.21209	12.22509	−0.19944
2	13.58929	+0.20525	14.82127	+0.19479	16.03777	+0.18569
3	17.00382	−0.18726	18.28758	−0.17942	19.55454	−0.17244
4	20.32079	+0.17305	21.64154	+0.16688	22.94517	+0.16130
5	23.58608	−0.16159	24.93493	−0.15657	26.26681	−0.15196
6	26.82015	+0.15212	28.19119	+0.14792	29.54566	+0.14404
7	30.03372	−0.14413	31.42279	−0.14055	32.79580	−0.13722
8	33.23304	+0.13727	34.63709	+0.13418	36.02562	+0.13127
9	36.42202	−0.13131	37.83872	−0.12859	39.24045	−0.12603
10	39.60324	+0.12606	41.03077	+0.12365	42.44389	+0.12137
11	42.77848	−0.12139	44.21541	−0.11924	45.63844	−0.11719
12	45.94902	+0.11721	47.39417	+0.11526	48.82593	+0.11342
13	49.11577	−0.11343	50.56818	−0.11166	52.00769	−0.10998
14	52.27945	+0.10999	53.73833	+0.10838	55.18475	+0.10684
15	55.44059	−0.10685	56.90525	−0.10537	58.35789	−0.10395
16	58.59961	+0.10396	60.06948	+0.10260	61.52774	+0.10129
17	61.75682	−0.10129	63.23142	−0.10003	64.69478	−0.09882
18	64.91251	+0.09882	66.39141	+0.09765	67.85943	+0.09652
19	68.06689	−0.09652	69.54971	−0.09543	71.02200	−0.09438
20	71.22013	+0.09438	72.70655	+0.09336	74.18277	+0.09237

Table 9.5

ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

s	$j'_{0,s}$	$J_0(j'_{0,s})$	$j'_{1,s}$	$J_1(j'_{1,s})$	$j'_{2,s}$	$J_2(j'_{2,s})$
1	0.00000 00000	+1.00000 00000	1.84118	+0.58187	3.05424	+0.48650
2	3.83170 59702	−0.40275 93957	5.33144	−0.34613	6.70613	−0.31353
3	7.01558 66698	+0.30011 57525	8.53632	+0.27330	9.96947	+0.25474
4	10.17346 81351	−0.24970 48771	11.70600	−0.23330	13.17037	−0.22088
5	13.32369 19363	+0.21835 94072	14.86359	+0.20701	16.34752	+0.19794
6	16.47063 00509	−0.19646 53715	18.01553	−0.18802	19.51291	−0.18101
7	19.61585 85105	+0.18006 33753	21.16437	+0.17346	22.67158	+0.16784
8	22.76008 43806	−0.16718 46005	24.31133	−0.16184	25.82604	−0.15720
9	25.90367 20876	+0.15672 49863	27.45705	+0.15228	28.97767	+0.14836
10	29.04682 85349	−0.14801 11100	30.60192	−0.14424	32.12733	−0.14088
11	32.18967 99110	+0.14060 57982	33.74618	+0.13736	35.27554	+0.13443
12	35.33230 75501	−0.13421 12403	36.88999	−0.13137	38.42265	−0.12879
13	38.47476 62348	+0.12861 66221	40.03344	+0.12611	41.56893	+0.12381
14	41.61709 42128	−0.12366 79608	43.17663	−0.12143	44.71455	−0.11937
15	44.75931 89977	+0.11924 98120	46.31960	+0.11724	47.85964	+0.11537
16	47.90146 08872	−0.11527 36941	49.46239	−0.11345	51.00430	−0.11176
17	51.04353 51836	+0.11167 04969	52.60504	+0.11001	54.14860	+0.10846
18	54.18555 36411	−0.10838 53489	55.74757	−0.10687	57.29260	−0.10544
19	57.32752 54379	+0.10537 40554	58.89000	+0.10397	60.43635	+0.10266
20	60.46945 78453	−0.10260 05671	62.03235	−0.10131	63.57989	−0.10008

s	$j'_{3,s}$	$J_3(j'_{3,s})$	$j'_{4,s}$	$J_4(j'_{4,s})$	$j'_{5,s}$	$J_5(j'_{5,s})$
1	4.20119	+0.43439	5.31755	+0.39965	6.41562	+0.37409
2	8.01524	−0.29116	9.28240	−0.27438	10.51986	−0.26109
3	11.34592	+0.24074	12.68191	+0.22959	13.98719	+0.22039
4	14.58585	−0.21097	15.96411	−0.20276	17.31284	−0.19580
5	17.78875	+0.19042	19.19603	+0.18403	20.57551	+0.17849
6	20.97248	−0.17505	22.40103	−0.16988	23.80358	−0.16533
7	24.14490	+0.16295	25.58976	+0.15866	27.01031	+0.15482
8	27.31006	−0.15310	28.76784	−0.14945	30.20285	−0.14616
9	30.47027	+0.14487	31.93854	+0.14171	33.38544	+0.13885
10	33.62695	−0.13784	35.10392	−0.13509	36.56078	−0.13256
11	36.78102	+0.13176	38.26532	+0.12932	39.73064	+0.12707
12	39.93311	−0.12643	41.42367	−0.12425	42.89627	−0.12223
13	43.08365	+0.12169	44.57962	+0.11973	46.05857	+0.11790
14	46.23297	−0.11746	47.73367	−0.11568	49.21817	−0.11402
15	49.38130	+0.11364	50.88616	+0.11202	52.37559	+0.11049
16	52.52882	−0.11017	54.03737	−0.10868	55.53120	−0.10728
17	55.67567	+0.10700	57.18752	+0.10563	58.68528	+0.10434
18	58.82195	−0.10409	60.33677	−0.10283	61.83809	−0.10163
19	61.96775	+0.10141	63.48526	+0.10023	64.98980	+0.09912
20	65.11315	−0.09893	66.63309	−0.09783	68.14057	−0.09678

s	$j'_{6,s}$	$J_6(j'_{6,s})$	$j'_{7,s}$	$J_7(j'_{7,s})$	$j'_{8,s}$	$J_8(j'_{8,s})$
1	7.50127	+0.35414	8.57784	+0.33793	9.64742	+0.32438
2	11.73494	−0.25017	12.93239	−0.24096	14.11552	−0.23303
3	15.26818	+0.21261	16.52937	+0.20588	17.77401	+0.19998
4	18.63744	−0.18978	19.94185	−0.18449	21.22906	−0.17979
5	21.93172	+0.17363	23.26805	+0.16929	24.58720	+0.16539
6	25.18393	−0.16127	26.54503	−0.15762	27.88927	−0.15431
7	28.40978	+0.15137	29.79075	+0.14823	31.15533	+0.14537
8	31.61788	−0.14317	33.01518	−0.14044	34.39663	−0.13792
9	34.81339	+0.13623	36.22438	+0.13381	37.62008	+0.13158
10	37.99964	−0.13024	39.42227	−0.12808	40.83018	−0.12608
11	41.17885	+0.12499	42.61152	+0.12305	44.03001	+0.12124
12	44.35258	−0.12035	45.79400	−0.11859	47.22176	−0.11695
13	47.52196	+0.11620	48.97107	+0.11460	50.40702	+0.11309
14	50.68782	−0.11246	52.14375	−0.11099	53.58700	−0.10960
15	53.85079	+0.10906	55.31282	+0.10771	56.76260	+0.10643
16	57.01138	−0.10596	58.47887	−0.10471	59.93454	−0.10352
17	60.16995	+0.10311	61.64239	+0.10195	63.10340	+0.10084
18	63.32681	−0.10049	64.80374	−0.09940	66.26961	−0.09837
19	66.48221	+0.09805	67.96324	+0.09704	69.43356	+0.09607
20	69.63635	−0.09579	71.12113	−0.09484	72.59554	−0.09393