

Combination of Variable Solutions

Combination of variables solutions to partial differential equations are suggested whenever the physical situation indicates that two independent variables can be combined to produce only one independent variable. There are general methods for finding the appropriate combined variable. However, in ChEN 430, you will be told what combined variable to use. The solution procedure is to substitute this combined variable into the differential equation for the original independent variables. If the method works, the resulting differential equation will involve only the combined variable. In this way, you will have converted a partial differential equation into an ordinary differential equation, which is then solved by solution methods for ordinary differential equations.

Example

Consider the dependent variable θ that depends on independent variables t and x as specified in the following partial differential equation and restricting conditions (initial and boundary conditions):

$$\frac{\partial \theta}{\partial t} = \gamma \frac{\partial^2 \theta}{\partial x^2} \quad (1)$$

$$\theta = \theta_0 \quad \text{at } t = 0 \quad \text{for } 0 < x < \infty \quad (2)$$

$$\theta = \theta_l \quad \text{at } x = 0 \quad \text{for } t > 0 \quad (3)$$

$$\theta = \theta_0 \quad \text{at } x \rightarrow \infty \quad \text{for } t > 0 \quad (4)$$

In this problem, the variable θ is forced to depart from its initial value of θ_0 . The distance x at which departures from θ_0 are felt grows as time increases. This suggests that it might be possible to combine (ratio) x and t in some way to obtain a combined variable (η) that will transform the partial differential equation in t and x into an ordinary differential equation in η . That is,

$$\theta = f(x, t) = g(\eta) \quad \text{where} \quad \eta = h(x, t)$$

where f and g are functions that will be determined when the problem is solved. The functional form of the combined variable, $h(x, t)$ will be suggested in the problem statement.

For the example specified in Eqs. (1) through (4), the combined variable is expected to be:

$$\eta = \frac{x}{\sqrt{4\gamma t}} \quad (5)$$

The first step in the solution procedure is to obtain the terms specified by the partial differential equation. The combination of variable solution will have worked if the resulting differential equation involves only η .

First, find $\partial\theta/\partial t$ in terms of η . Using chain rule,

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial t} \quad (6)$$

where

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t} \left[\frac{x}{\sqrt{4\gamma t}} \right] = \frac{x}{\sqrt{4\gamma}} \left[-\frac{1}{2} \frac{1}{t^{3/2}} \right] \quad (7)$$

Consequently,

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= -\frac{d\theta}{d\eta} \frac{x}{2t\sqrt{4\gamma t}} \\ &= -\frac{\eta}{2t} \frac{d\theta}{d\eta} \end{aligned} \quad (8)$$

Next find $\partial^2 \theta / \partial x^2$ in terms of η . Begin by finding $\partial \theta / \partial x$. From chain rule,

$$\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial x} \quad (9)$$

where

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x} \left[\frac{x}{\sqrt{4\gamma t}} \right] = \frac{1}{\sqrt{4\gamma t}} \quad (10)$$

Consequently,

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \frac{d\theta}{d\eta} \frac{1}{\sqrt{4\gamma t}} \\ &= \frac{\eta}{x} \frac{d\theta}{d\eta} \end{aligned} \quad (11)$$

Differentiating Eq. (11) a second time yields:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{d\theta}{d\eta} \frac{1}{\sqrt{4\gamma t}} \right] = \frac{d}{d\eta} \left[\frac{d\theta}{d\eta} \frac{1}{\sqrt{4\gamma t}} \right] \frac{\partial \eta}{\partial x} + \frac{d\theta}{d\eta} \frac{\partial}{\partial x} \left[\frac{1}{\sqrt{4\gamma t}} \right] \quad (12)$$

Since

$$\frac{\partial}{\partial x} \left[\frac{1}{\sqrt{4\gamma t}} \right] = 0$$

Eq. (12) simplifies to give,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{d^2 \theta}{d\eta^2} \frac{1}{4\gamma t} \quad (13)$$

Substituting Eqs. (8) and (13) into the differential equation (Eq. 1):

$$-\frac{\eta}{2t} \frac{d\theta}{d\eta} = \gamma \frac{1}{4\gamma t} \frac{d^2\theta}{d\eta^2} \quad (14)$$

which simplifies to yield the following ordinary differential equation:

$$\frac{d^2\theta}{d\eta^2} + 2\eta \frac{d\theta}{d\eta} = 0 \quad (15)$$

The recommended combination variable did transform the second-order partial differential equation in x and t (Eq. 1) into a second-order ordinary differential equation in η .

The combination variable also must be introduced into the restricting conditions (Eqs. 2-4):

$$\theta = \theta_o \quad \text{at } t = 0 \quad \text{or} \quad \text{at } \eta \rightarrow \infty \quad (17)$$

$$\theta = \theta_l \quad \text{at } x = 0 \quad \text{or} \quad \text{at } \eta = 0 \quad (18)$$

$$\theta = \theta_o \quad \text{at } x \rightarrow \infty \quad \text{or} \quad \text{at } \eta \rightarrow \infty \quad (19)$$

Notice that the conditions defined by Eq. (17) and (19) become the same. This seems reasonable since the second-order differential equation requires only two conditions on η .

Eq. (15) is a linear, but non-constant coefficient equation. It can be solved by reduction of order. Substitute for $p = \partial\theta/\partial\eta$ into Eq. (15) to obtain:

$$\frac{dp}{d\eta} + 2\eta p = 0 \quad (20)$$

which can be separated and integrated:

$$\int \frac{dp}{p} = - \int 2\eta d\eta \quad (21)$$

to yield:

$$\ln p = -\eta^2 + \hat{I}_1 \quad (22)$$

After exponentiation Eq. (21) becomes:

$$p = \frac{d\theta}{d\eta} = I_1 \exp(-\eta^2) \quad (23)$$

where $I_1 = \exp(\hat{I}_1)$. Integrating Eq. (23) yields

$$\theta = \int d\theta = I_1 \int_0^{\eta} \exp(-\hat{\eta}^2) d\hat{\eta} + I_2 \quad (24)$$

Solving for I_2 using Eq. (18) yields $I_2 = \theta_l$. Solving for I_1 using Eq. (17),

$$\theta_o = I_1 \int_0^{\infty} \exp(-\hat{\eta}^2) d\hat{\eta} + \theta_1 \quad (25)$$

where

$$\int_0^{\infty} \exp(-\hat{\eta}^2) d\hat{\eta} = \frac{\sqrt{\pi}}{2} \quad (26)$$

After simplification, Eq. (25) yields:

$$I_1 = (\theta_o - \theta_1) \frac{2}{\sqrt{\pi}} \quad (27)$$

Substituting for I_1 and I_2 into Eq. (24):

$$\theta = (\theta_o - \theta_1) \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-\hat{\eta}^2) d\hat{\eta} + \theta_1 \quad (28)$$

Recall that the error function [(i.e., erf(x)] is defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (29)$$

As a consequence, Eq. (28) can be written simply as:

$$\theta = \theta_1 - (\theta_1 - \theta_o) \text{erf}(\eta) \quad (30)$$

Substituting for the definition of η , we obtain the final result:

$$\theta = \theta_1 - (\theta_1 - \theta_o) \text{erf}(x/\sqrt{4\gamma t}) \quad (31)$$