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1

Dimensional Analysis, Modelling, and Invariance

1.1 Introduction

In this chapter we introduce the ideas of invariance concretely through a thorough treatment of dimensional analysis. We show how dimensional analysis is connected to modelling and the construction of solutions obtained through invariance for boundary value problems for partial differential equations.

Often for a quantity of interest one knows at most the independent quantities it depends upon, say n in total, and the dimensions of all $n+1$ quantities. The application of dimensional analysis usually reduces the number of essential independent quantities. This is the starting point of modelling where the objective is to reduce significantly the number of experimental measurements. In the following sections we will show that dimensional analysis can lead to a reduction in the number of independent variables appearing in a boundary value problem for a partial differential equation. Most importantly we show that for partial differential equations the reduction of variables through dimensional analysis is a special case of reduction from invariance under groups of scaling (stretching) transformations.

1.2 Dimensional Analysis—Buckingham Pi-Theorem

The basic theorem of dimensional analysis is the so-called *Buckingham Pi-theorem,* attributed to the American engineering scientist Buckingham (1914, 1915a, b). General references on the subject include those of Birkhoff (1950), Bridgman (1931), Barenblatt (1979), Sedov (1959), and Bluman (1983a). A historical perspective is given by Görtler (1975). For a detailed mathematical perspective see Curtis, Logan, and Parker (1982).

The following assumptions and conclusions of dimensional analysis constitute the Buckingham Pi-theorem.

1.2.1 ASSUMPTIONS BEHIND DIMENSIONAL ANALYSIS

Essentially no real problem violates the following assumptions:

(i) A quantity u is to be determined in terms of n *measurable quantities* (variables and parameters), $(W_1, W_2, \ldots, W_n)$:

$$u = f(W_1, W_2, \ldots, W_n), \tag{1.1}$$

where f is an unknown function of $(W_1, W_2, \ldots, W_n)$.

(ii) The quantities $(u, W_1, W_2, \ldots, W_n)$ involve m *fundamental dimensions* labelled by $L_1, L_2, \ldots, L_m$. For example in a mechanical problem these are usually the mechanical fundamental dimensions $L_1 =$ length, $L_2 =$ mass, and $L_3 =$ time.

(iii) Let Z represent any of $(u, W_1, W_2, \ldots, W_n)$. Then the *dimension of* Z, denoted by $[Z]$, is a product of powers of the fundamental dimensions, in particular

$$[Z] = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_m^{\alpha_m} \tag{1.2}$$

for some real numbers, usually rational, $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ which are the dimension exponents of Z. The *dimension vector of* Z is the column vector

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}. \tag{1.3}$$

A quantity Z is said to be *dimensionless* if and only if $[Z] = 1$, i.e. all dimension exponents are zero. For example, in terms of the mechanical fundamental dimensions, the dimension vector of energy E is

$$\alpha(E) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Let

$$\mathbf{b}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix} \tag{1.4}$$

be the dimension vector of W_i, $i = 1, 2, \ldots, n$, and let

$$\mathrm{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \tag{1.5}$$

be the $m \times n$ *dimension matrix* of the given problem.

(iv) For any set of fundamental dimensions one can choose a *system of units* for measuring the value of any quantity Z. A change from one system of units to another involves a positive *scaling* of each fundamental dimension which in turn induces a scaling of each quantity Z. For example for the mechanical fundamental dimensions the common systems of units are MKS, c.g.s. or British foot-pounds. In changing from c.g.s. to MKS units, L_1 is scaled by 10^{-2}, L_2 is scaled by 10^{-3}, L_3 is unchanged, and hence the value of energy E is scaled by 10^{-7}. Under a change of system of units the value of a dimensionless quantity is unchanged, i.e. its value is *invariant* under an arbitrary scaling of any fundamental dimension. Hence it is meaningful to deem dimensionless quantities as large or small. The last assumption of dimensional analysis is that formula (1.1) acts as a dimensionless equation in the sense that (1.1) is invariant under an arbitrary scaling of any fundamental dimension, i.e. (1.1) is independent of the choice of system of units.

1.2.2 Conclusions from Dimensional Analysis

The assumptions of the Buckingham Pi-theorem stated in Section 1.2.1 lead to:

(i) Formula (1.1) can be expressed in terms of dimensionless quantities.

(ii) The number of dimensionless quantities is $k + 1 = n + 1 - r(\mathrm{B})$ where $r(\mathrm{B})$ is the rank of matrix B. Precisely k of these dimensionless quantities depend on the measurable quantities $(W_1, W_2, \ldots, W_n)$.

(iii) Let

$$\mathbf{x}^{(i)} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix}, \quad i = 1, 2, \ldots, k, \tag{1.6}$$

represent the $k = n - r(\mathrm{B})$ linearly independent solutions $\mathbf{x}$ of the system

$$\mathrm{B}\mathbf{x} = 0. \tag{1.7}$$

Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \tag{1.8}$$

be the dimension vector of u and let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{1.9}$$

represent a solution of the system

$$\mathbf{B}\mathbf{y} = -\mathbf{a}. \tag{1.10}$$

Then formula (1.1) simplifies to

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k) \tag{1.11}$$

where π, π_i, are dimensionless quantities,

$$\pi = u W_1^{y_1} W_2^{y_2} \cdots W_n^{y_n}, \tag{1.12a}$$

$$\pi_i = W_1^{x_{1i}} W_2^{x_{2i}} \cdots W_n^{x_{ni}}, \quad i = 1, 2, \ldots, k, \tag{1.12b}$$

and g is an unknown function of its arguments. In particular (1.1) becomes

$$u = W_1^{-y_1} W_2^{-y_2} \cdots W_n^{-y_n} g(\pi_1, \pi_2, \ldots, \pi_k). \tag{1.13}$$

[In terms of experimental modelling formula (1.13) is "cheaper" than (1.1) by $r(\mathrm{B})$ orders of magnitude.]

1.2.3 PROOF OF THE BUCKINGHAM PI-THEOREM

First of all,

$$[u] = L_1^{a_1} L_2^{a_2} \cdots L_m^{a_m}, \tag{1.14a}$$

$$[W_i] = L_1^{b_{1i}} L_2^{b_{2i}} \cdots L_m^{b_{mi}}, \quad i = 1, 2, \ldots, n. \tag{1.14b}$$

Next we use assumption (iv) and consider the invariance of (1.1) under arbitrary scalings of the fundamental dimensions by taking each fundamental dimension in turn. We scale L_1 by letting

$$L_1^* = e^{\epsilon} L_1, \quad \epsilon \in \mathbb{R}. \tag{1.15}$$

Then accordingly

$$u^* = e^{\epsilon a_1} u, \tag{1.16a}$$

$$W_i^* = e^{\epsilon b_{1i}} W_i, \quad i = 1, 2, \ldots, n. \tag{1.16b}$$

Equations (1.16a,b) define a one-parameter (ϵ) Lie group of scaling transformations of the $n + 1$ quantities $(u, W_1, W_2, \ldots, W_n)$ with $\epsilon = 0$ corresponding to the identity transformation. This group is induced by the one-parameter group of scalings (1.15) of the fundamental dimension L_1.

From assumption (iv), formula (1.1) holds if and only if

$$u^* = f(W_1^*, W_2^*, \ldots, W_n^*),$$

i.e.,

$$e^{\epsilon a_1} u = f(e^{\epsilon b_{11}} W_1, e^{\epsilon b_{12}} W_2, \ldots, e^{\epsilon b_{1n}} W_n), \quad \text{for all } \epsilon \in \mathbb{R}. \tag{1.17}$$

Case I. $b_{11} = b_{12} = \cdots = b_{1n} = a_1 = 0$. Here L_1 is not a fundamental dimension of the problem or, in other words, formula (1.1) is *dimensionless with respect to* L_1.

Case II. $b_{11} = b_{12} = \cdots = b_{1n} = 0$, $a_1 \neq 0$. It follows that $u \equiv 0$, a trivial situation.

Hence it follows that $b_{1i} \neq 0$ for some $i = 1, 2, \ldots, n$. Without loss of generality we assume $b_{11} \neq 0$. We define new measurable quantities

$$X_{i-1} = W_i W_1^{-b_{1i}/b_{11}}, \quad i = 2, 3, \ldots, n, \tag{1.18}$$

and let

$$X_n = W_1. \tag{1.19}$$

We choose as the new unknown

$$v = u W_1^{-a_1/b_{11}}. \tag{1.20}$$

In terms of the quantities (1.18)–(1.20), formula (1.1) is equivalent to

$$v = F(X_1, X_2, \ldots, X_n) \tag{1.21}$$

where F is an unknown function of $(X_1, X_2, \ldots, X_n)$, and the group of transformations (1.16a,b) becomes

$$v^* = v, \tag{1.22a}$$

$$X_i^* = X_i, \quad i = 1, 2, \ldots, n-1, \tag{1.22b}$$

$$X_n^* = e^{\epsilon b_{11}} X_n, \tag{1.22c}$$

so that $(v, X_1, X_2, \ldots, X_{n-1})$ are *invariants* of (1.16a,b). Moreover the quantities $(v, X_1, X_2, \ldots, X_n)$ satisfy assumption (iii), and formula (1.21) satisfies assumption (iv). Hence

$$v = F(X_1, X_2, \ldots, X_{n-1}, e^{\epsilon b_{11}} X_n), \tag{1.23}$$

for all $\epsilon \in \mathbb{R}$. Consequently F is independent of X_n. Moreover the measurable quantities $(X_1, X_2, \ldots, X_{n-1})$ are products of powers of $(W_1, W_2, \ldots, W_n)$ and v is a product of u and powers of $(W_1, W_2, \ldots, W_n)$. Formula (1.1) reduces to

$$v = G(X_1, X_2, \ldots, X_{n-1}), \tag{1.24}$$

where $(v, X_1, X_2, \ldots, X_{n-1})$ are dimensionless with respect to L_1 and G is an unknown function of its $n-1$ arguments.

Continuing in turn with the other $m-1$ fundamental dimensions, we reduce formula (1.1) to a dimensionless formula

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k), \tag{1.25}$$

where $[\pi] = [\pi_i] = 1$, g is an unknown function of $(\pi_1, \pi_2, \ldots, \pi_k)$,

$$\pi = u W_1^{y_1} W_2^{y_2} \cdots W_n^{y_n}, \tag{1.26a}$$

and

$$\pi_i = W_1^{x_{1i}} W_2^{x_{2i}} \cdots W_n^{x_{ni}}, \tag{1.26b}$$

for some real numbers $\{y_j, x_{ji}\}$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$.

Next we show that the number of measurable dimensionless quantities is $k = n - r(\mathrm{B})$. This follows immediately since

$$[W_1^{x_1} W_2^{x_2} \cdots W_n^{x_n}] = 1$$

if and only if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

satisfies (1.7). Equation (1.7) has $k = n - r(\mathrm{B})$ linearly independent solutions $\mathbf{x}^{(i)}$ given by (1.6). The real numbers

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

follow from setting

$$[u W_1^{y_1} W_2^{y_2} \cdots W_n^{y_n}] = 1,$$

leading to $\mathbf{y}$ satisfying (1.10). □

Note that the proof of the Buckingham Pi-theorem makes no assumption about the continuity of the unknown function f, and hence of g, with respect to any of their arguments.

1.2.4 Examples

(1) *The Atomic Explosion of 1945*

Sir Geoffrey Taylor (1950) deduced the approximate energy released by the first atomic explosion in New Mexico from motion picture records of J.E.

Mack declassified in 1947. But the amount of energy released by the blast was still classified in 1947! [Taylor carried out the analysis for his deduction in 1941.] A dimensional analysis argument of Taylor's deduction follows:

An atomic explosion is approximated by the release of a large amount of energy E from a "point." A consequence is an expanding spherical fireball whose edge corresponds to a powerful shock wave. Let $u = R$ be the radius of the shock wave. We treat R as the unknown and assume that

$$R = f(W_1, W_2, W_3, W_4) \tag{1.27}$$

where

$W_1 = E$, the energy released by the explosion,
$W_2 = t$, the elapsed time after the explosion takes place,
$W_3 = \rho_0$, the initial or ambient air density,

and

$W_4 = P_0$, the initial or ambient air pressure.

For this problem we use the mechanical fundamental dimensions. The corresponding dimension matrix is

$$\mathrm{B} = \begin{bmatrix} 2 & 0 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ -2 & 1 & 0 & -2 \end{bmatrix}. \tag{1.28}$$

$r(\mathrm{B}) = 3$ and hence $k = n - r(\mathrm{B}) = 4 - 3 = 1$. The general solution of $\mathrm{B}\mathbf{x} = 0$ is $x_1 = -\frac{2}{5}x_4$, $x_2 = \frac{6}{5}x_4$, $x_3 = -\frac{3}{5}x_4$ where x_4 is arbitrary. Setting $x_4 = 1$, we get the measurable dimensionless quantity

$$\pi_1 = P_0 \left[\frac{t^6}{E^2\rho_0^3}\right]^{1/5}. \tag{1.29}$$

The dimension vector of R is

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.30}$$

The general solution of $\mathrm{B}\mathbf{y} = -\mathbf{a}$ is

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x} \tag{1.31}$$

where $\mathbf{x}$ is the general solution of $\mathrm{B}\mathbf{x} = 0$. Setting $\mathbf{x} = 0$ in (1.31), we obtain the dimensionless unknown

$$\pi = R\left[\frac{Et^2}{\rho_0}\right]^{-1/5}. \tag{1.32}$$

Thus from dimensional analysis

$$R = \left[\frac{Et^2}{\rho_0}\right]^{1/5} g(\pi_1) \tag{1.33}$$

where g is an unknown function of π_1.

Now we assume that $g(\pi_1)$ is continuous at $\pi_1 = 0$ so that $g(\pi_1) \simeq g(0)$ if $\pi_1 << 1$. Moreover, we assume that $g(0) \neq 0$. This leads to Taylor's approximation formula

$$R = At^{2/5} \tag{1.34}$$

where

$$A = \left(\frac{E}{\rho_0}\right)^{1/5} g(0). \tag{1.35}$$

Plotting $\log R$ versus $\log t$ for a light explosives experiment, one can determine that $g(0) \simeq 1$. Using Mack's motion picture for the first atomic explosion, Taylor plotted $\frac{5}{2}\log_{10} R$ versus $\log_{10} t$ with R and t measured in c.g.s. units. [See Figure 1.2.4-1 where the motion picture data is indicated by +.] This verified the use of the approximation $g(\pi_1) \simeq g(0)$ and led to an accurate estimation of the classified energy E of the explosion!

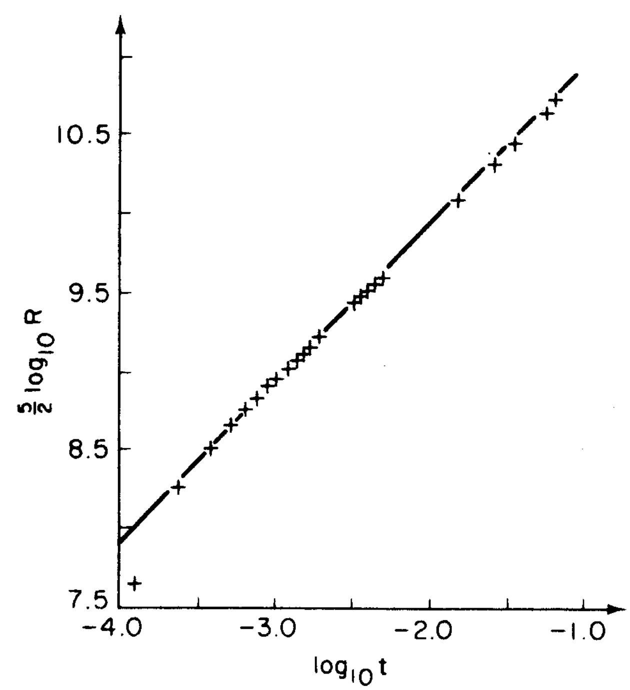

Figure 1.2.4-1

(2) *An Example in Heat Conduction Illustrating the Choice of Fundamental Dimensions*

Consider the standard problem of one-dimensional heat conduction in an "infinite" bar with constant thermal properties, initially heated by a point source of heat. Let u be the temperature at any point of the bar. We assume that

$$u = f(W_1, W_2, W_3, W_4, W_5, W_6) \tag{1.36}$$

where

$W_1 = x$, the distance along the bar from the point source of heat,
$W_2 = t$, the elapsed time after the initial heating,
$W_3 = \rho$, the mass density of the bar,
$W_4 = c$, the specific heat of the bar,
$W_5 = K$, the thermal conductivity of the bar,
$W_6 = Q$, the strength of the heat source measured in energy units per (length units)^2.

It is interesting to consider the effects of dimensional analysis in simplifying (1.36) with two different choices of fundamental dimensions.

Choice I (*Dynamical Units*). Here we let L_1 = length, L_2 = mass, L_3 = time, and L_4 = temperature. Correspondingly, the dimension matrix is

$$\mathrm{B}_I = \begin{bmatrix} 1 & 0 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 & -3 & -2 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}. \tag{1.37}$$

$r(\mathrm{B}_I) = 4$ and hence $k = 6 - 4 = 2$ is the number of measurable dimensionless quantities. One can choose two linearly independent solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ of $\mathrm{B}_I\mathbf{x} = 0$ such that π_1 is linear in x and independent of t; π_2 is linear in t and independent of x. Then

$$\pi_1 = \xi = \frac{\rho c^2 Q}{K^2} x, \tag{1.38a}$$

$$\pi_2 = \tau = \frac{\rho c^3 Q^2}{K^3} t. \tag{1.38b}$$

It is convenient to choose as the dimensionless quantity π a solution of

$$\mathrm{B}_I\mathbf{y} = -\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

where $y_1 = y_2 = 0$, so that π is independent of x and t. Consequently

$$\pi = \frac{K^2}{Q^2 c} u. \tag{1.39}$$

Hence dimensional analysis with dynamical units reduces (1.36) to

$$u = \frac{Q^2 c}{K^2} F(\xi, \tau) \tag{1.40}$$

where F is an unknown function of ξ and τ.

Choice II (*Thermal Units*). Motivated by the implicit assumption that in the posed problem there is no conversion of heat energy to mechanical energy, we refine the dynamical units by introducing a thermal unit $L_5 =$ "calories." The corresponding dimension matrix is

$$\mathrm{B}_{II} = \begin{bmatrix} 1 & 0 & -3 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}. \tag{1.41}$$

$r(\mathrm{B}_{II}) = 5$ and hence there is only one measurable dimensionless quantity. It is convenient to choose as dimensionless quantities

$$\pi_1 = \eta = \frac{x}{\sqrt{\kappa t}}, \quad \text{where } \kappa = \frac{K}{\rho c}, \tag{1.42a}$$

and

$$\pi = \frac{\sqrt{\rho c K t}}{Q} u. \tag{1.42b}$$

Thus dimensional analysis with thermal units reduces (1.36) to

$$u = \frac{Q}{\sqrt{\rho c K t}} G(\eta) \tag{1.43}$$

where G is an unknown function of η.

Note that equation (1.43) is a special case of equation (1.40) where

$$\eta = \frac{\xi}{\sqrt{\tau}} \quad \text{and} \quad F(\xi, \tau) = \frac{1}{\sqrt{\tau}} G\left(\frac{\xi}{\sqrt{\tau}}\right).$$

[In terms of thermal units each of the quantities, ξ, τ, $\dfrac{K^2 u}{Q^2 c}$ is not dimensionless.]

Obviously, if it is correct, equation (1.43) is a great simplification of equation (1.40). By conducting experiments or associating a properly-posed boundary value problem to determine u, one can show that thermal units are justified. In turn thermal units can then be used for other heat (diffusion) problems where the governing equations are not completely known.

Exercises 1.2

1. Use dimensional analysis to prove the Pythagoras theorem. [Hint: Drop a perpendicular to the hypotenuse of a right-angle triangle and consider the resulting similar triangles.]

2. How would you use dimensional analysis and experimental modelling to find the time of flight of a body dropped vertically from a height h?

3. Given that in c.g.s. units $\rho_0 = 1.3 \times 10^{-3}$, and $P_0 = 1.0 \times 10^6$, use Figure 1.2.4-1 to estimate the domain of π_1 and E.

1.3 Application of Dimensional Analysis to Partial Differential Equations

Consider the use of dimensional analysis where the quantities $(u, W_1, W_2, \ldots, W_n)$ arise in a boundary value problem for a partial differential equation which has a unique solution. Then the unknown u (the *dependent variable* of the partial differential equation) is the solution of the BVP and $(W_1, W_2, \ldots, W_n)$ denote all *independent variables* and *constants* appearing in the BVP. From the Buckingham Pi-theorem it follows that such a BVP can always be re-expressed in dimensionless form where π is a dimensionless dependent variable and $(\pi_1, \pi_2, \ldots, \pi_k)$ are dimensionless independent variables and dimensionless constants.

Say $(W_1, W_2, \ldots, W_\ell)$ are the ℓ independent variables and $(W_{\ell+1}, W_{\ell+2}, \ldots, W_n)$ are the $n - \ell$ constants appearing in the BVP. Let

$$\mathrm{B}_1 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & b_{2\ell} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{m\ell} \end{bmatrix} \tag{1.44a}$$

be the dimension matrix of the independent variables and let

$$\mathrm{B}_2 = \begin{bmatrix} b_{1,\ell+1} & b_{1,\ell+2} & \cdots & b_{1n} \\ b_{2,\ell+1} & b_{2,\ell+2} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m,\ell+1} & b_{m,\ell+2} & \cdots & b_{mn} \end{bmatrix} \tag{1.44b}$$

be the dimension matrix of the constants. The dimension matrix of the BVP is

$$\mathrm{B} = \left[\mathrm{B}_1 \ \vdots \ \mathrm{B}_2\right]. \tag{1.45}$$

A dimensionless π_i quantity is called a *dimensionless constant* if it does not depend on $(W_1, W_2, \ldots, W_\ell)$, i.e., in equation (1.26b), $x_{ji} = 0$, $j = 1, 2, \ldots, \ell$. A dimensionless π_i quantity is a *dimensionless variable* if $x_{ji} \neq 0$ for some $j = 1, 2, \ldots, \ell$. An important objective in applying dimensional analysis to a BVP is to reduce the number of independent variables. The rank of B_2, i.e. $r(\mathrm{B}_2)$, represents the reduction in the number of constants through dimensional analysis. Consequently the reduction in the number of independent variables is $\rho = r(\mathrm{B}) - r(\mathrm{B}_2)$. In particular the number of dimensionless measurable quantities is $k = n - r(\mathrm{B}) = [\ell - \rho] + [(n - \ell) - r(\mathrm{B}_2)]$ where $\ell - \rho$ of the π_i quantities are dimensionless independent variables and $(n - \ell) - r(\mathrm{B}_2)$ are dimensionless constants.

If $r(\mathrm{B}) = r(\mathrm{B}_2)$, then dimensional analysis reduces the given BVP to a dimensionless BVP with $(n - \ell) - r(\mathrm{B}_2)$ dimensionless constants. In this case the number of independent variables is not reduced. Nonetheless this is useful as a starting point for perturbation analysis.

If $\ell \geq 2$, $\ell - \rho = 1$, then the resulting solution of the BVP is called a *self-similar solution* or *automodel solution.*

1.3.1 Examples

(1) *Source Problem for Heat Conduction*

Consider the unknown temperature u of the heat conduction problem of Section 1.2.4 as the solution $u(x,t)$ of the BVP:

$$\rho c \frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \; t > 0, \tag{1.46a}$$

$$u(x,0) = \frac{Q}{\rho c}\delta(x), \tag{1.46b}$$

$$\lim_{x \to \pm\infty} u(x,t) = 0. \tag{1.46c}$$

In equation (1.46b) $\delta(x)$ is the Dirac delta function.

The use of dimensional analysis with dynamical units reduces (1.46a–c) to

$$\frac{\partial F}{\partial \tau} - \frac{\partial^2 F}{\partial \xi^2} = 0, \quad -\infty < \xi < \infty, \; \tau > 0, \tag{1.47a}$$

$$F(\xi, 0) = \delta(\xi), \tag{1.47b}$$

$$\lim_{\xi \to \pm\infty} F(\xi, \tau) = 0, \tag{1.47c}$$

with u defined in terms of $F(\xi,\tau)$ by (1.40) and ξ, τ given by (1.38a,b). Consequently there is no essential progress in solving BVP (1.46a–c).

We now justify the use of dimensional analysis with thermal units to solve (1.46a–c) as follows:

First note that from equations (1.47a,c) we have

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} F(\xi,\tau)d\xi = \int_{-\infty}^{\infty} \frac{\partial^2 F}{\partial \xi^2}(\xi,\tau)d\xi = 0.$$

Then from this equation and (1.47b) we get the conservation law

$$\int_{-\infty}^{\infty} F(\xi,\tau)d\xi = 1, \quad \text{valid for all } \tau > 0.$$

Consequently the substitution $F(\xi,\tau) = \frac{1}{\sqrt{\tau}}G(\frac{\xi}{\sqrt{\tau}})$, which results from using dimensional analysis with thermal units, reduces (1.47a–c) and hence (1.46a–c) to a BVP for an ordinary differential equation with independent variable $\eta = \frac{\xi}{\sqrt{\tau}}$ and dependent variable $G(\eta)$:

$$2\frac{d^2G}{d\eta^2} + \eta\frac{dG}{d\eta} + G = 0, \quad -\infty < \eta < \infty, \tag{1.48a}$$

$$\int_{-\infty}^{\infty} G(\eta)d\eta = 1, \tag{1.48b}$$

$$G(\pm\infty) = 0. \tag{1.48c}$$

This reduction of (1.46a–c) to a BVP for an ordinary differential equation is obtained much more naturally and easily in Section 1.4 from invariance of (1.46a–c) under a one-parameter group of scalings of its variables.

(2) *Prandtl–Blasius Problem for a Flat Plate*

Consider the Prandtl boundary layer equations for flow past a semi-infinite flat plate:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1.49a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.49b}$$

$0 < x < \infty$, $0 < y < \infty$, with boundary conditions

$$u(x,0) = 0, \tag{1.49c}$$

$$v(x,0) = 0, \tag{1.49d}$$

$$u(x,\infty) = U, \tag{1.49e}$$

$$u(0,y) = U. \tag{1.49f}$$

In BVP (1.49a–f), x is the distance along the plate surface from its edge (tangential coordinate), y is the distance from the plat surface (normal coordinate), u is the x-component of velocity, v is the y-component of velocity, ν is the kinematic viscosity and U is the velocity of the incident flow [Figure 1.3.1-1].

Our aim is to calculate the shear at the plate (skin friction) $\frac{\partial u}{\partial y}(x,0)$ which leads to the determination of the viscous drag on the plate.

We look at the problem of determining $\frac{\partial u}{\partial y}(x,0)$ as defined through BVP (1.49a–f) from three analytical perspectives:

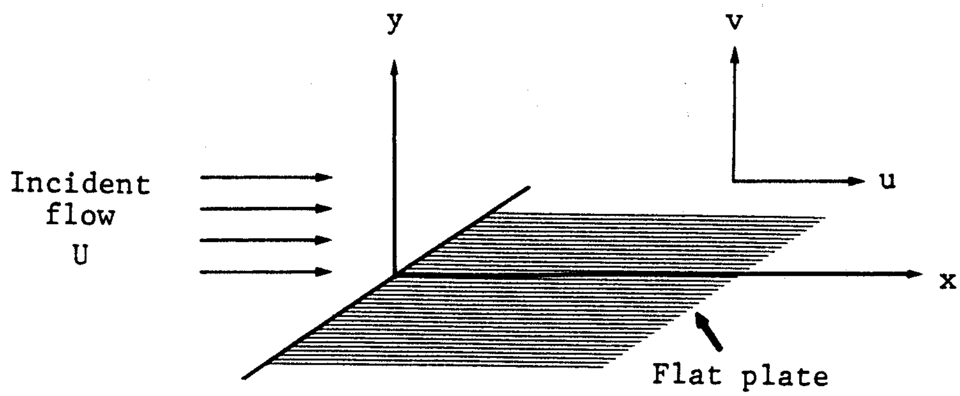

Figure 1.3.1-1

(i) *Dimensional Analysis.* From (1.49a–f) it follows that

$$\frac{\partial u}{\partial y}(x,0) = f(x,U,\nu) \tag{1.50}$$

with the unknown f to be determined as a function of measurable quantities x, U, ν. The fundamental dimensions are $L =$ length and $T =$ time. Then in terms of these fundamental dimensions:

$$\left[\frac{\partial u}{\partial y}(x,0)\right] = T^{-1}, \tag{1.51a}$$

$$[x] = L, \tag{1.51b}$$

$$[U] = LT^{-1}, \tag{1.51c}$$

$$[\nu] = L^2T^{-1}. \tag{1.51d}$$

Consequently $r(\mathrm{B}) = 2$. Dimensionless quantities are

$$\pi_1 = \frac{Ux}{\nu}, \tag{1.52a}$$

and

$$\pi = \frac{\nu}{U^2}\frac{\partial u}{\partial y}(x,0). \tag{1.52b}$$

Hence dimensional analysis leads to

$$\frac{\partial u}{\partial y}(x,0) = \frac{U^2}{\nu}g\left(\frac{Ux}{\nu}\right) \tag{1.53}$$

where g is an unknown function of $\frac{Ux}{\nu}$.

(ii) *Scalings of Quantities Followed by Dimensional Analysis.* Consider a linear transformation of the variables of (1.49a–f) given by $x = aX$, $y = bY$, $u = UQ$, $v = cR$ where (a,b,c) are undetermined positive constants, U is the velocity of the incident flow and (X,Y,Q,R) represent new (dimensional) independent and dependent variables: $Q = Q(X,Y)$, $R = R(X,Y)$; $u(x,y) = UQ(X,Y) = UQ(\frac{x}{a}, \frac{y}{b})$, $v(x,y) = cR(X,Y) = cR(\frac{x}{a}, \frac{y}{b})$.

Consequently

$$\frac{\partial u}{\partial y}(x,0) = \frac{U}{b}\frac{\partial Q}{\partial Y}(X,0) \tag{1.54}$$

and the BVP (1.49a–f) transforms to

$$\frac{U}{a}Q\frac{\partial Q}{\partial X} + \frac{c}{b}R\frac{\partial Q}{\partial Y} = \frac{\nu}{b^2}\frac{\partial^2 Q}{\partial Y^2}, \tag{1.55a}$$

$$\frac{U}{a}\frac{\partial Q}{\partial X} + \frac{c}{b}\frac{\partial R}{\partial Y} = 0, \tag{1.55b}$$

$0 < X < \infty$, $0 < Y < \infty$, with

$$Q(X,0) = 0, \tag{1.55c}$$

$$R(X,0) = 0, \tag{1.55d}$$

$$Q(X,\infty) = 1, \tag{1.55e}$$

$$Q(0,Y) = 1. \tag{1.55f}$$

From the form of (1.55a,b) it is convenient to choose (a,b,c) so that

$$\frac{U}{a} = \frac{c}{b} = \frac{\nu}{b^2}.$$

Hence we set $c = 1$, $b = \nu$, $a = U\nu$. As a result equations (1.55a,b) become cleared of constants:

$$Q\frac{\partial Q}{\partial X} + R\frac{\partial Q}{\partial Y} = \frac{\partial^2 Q}{\partial Y^2}, \tag{1.56a}$$

$$\frac{\partial Q}{\partial X} + \frac{\partial R}{\partial Y} = 0, \tag{1.56b}$$

$0 < X < \infty$, $0 < Y < \infty$. Moreover

$$\frac{\partial u}{\partial y}(x,0) = \frac{U}{\nu}\frac{\partial Q}{\partial Y}(X,0) = \frac{U}{\nu}\frac{\partial Q}{\partial Y}\left(\frac{x}{U\nu}, 0\right). \tag{1.57}$$

But now it follows that since $Q(X,Y)$ results from the solution of (1.56a,b), (1.55c–f), we have

$$\frac{\partial Q}{\partial Y}(X,0) = h(X), \tag{1.58}$$

for some unknown function h. We apply dimensional analysis to (1.58):

$$\left[\frac{\partial Q}{\partial Y}\right] = LT^{-1}, \tag{1.59a}$$

$$[X] = L^{-2}T^2. \tag{1.59b}$$

Hence (1.58) reduces to

$$h(X) = \sigma X^{-1/2} \tag{1.60}$$

for some fixed dimensionless constant σ to be determined. Thus (1.53) simplifies further to $g(\frac{Ux}{\nu}) = \sigma(\frac{Ux}{\nu})^{-1/2}$ so that

$$\frac{\partial u}{\partial y}(x,0) = \sigma\left(\frac{U^3}{x\nu}\right)^{1/2}. \tag{1.61}$$

(iii) *Further Use of Dimensional Analysis on the Full BVP.* We now apply dimensional analysis to the BVP (1.56a,b), (1.55c–f), to reduce it to a BVP for an ordinary differential equation. It is convenient (but not necessary) to introduce a *potential (stream function)* $\psi(X,Y)$ from the form of (1.56b). Let $Q = \frac{\partial\psi}{\partial Y}$, $R = -\frac{\partial\psi}{\partial X}$. Then in terms of the single dependent variable ψ, BVP (1.56a,b), (1.55c–f), becomes:

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial X\partial Y} - \frac{\partial\psi}{\partial X}\frac{\partial^2\psi}{\partial Y^2} = \frac{\partial^3\psi}{\partial Y^3}, \tag{1.62a}$$

$0 < X < \infty$, $0 < Y < \infty$, with

$$\frac{\partial\psi}{\partial Y}(X,0) = 0, \tag{1.62b}$$

$$\frac{\partial\psi}{\partial X}(X,0) = 0, \tag{1.62c}$$

$$\frac{\partial\psi}{\partial Y}(X,\infty) = 1, \tag{1.62d}$$

$$\frac{\partial\psi}{\partial Y}(0,Y) = 1. \tag{1.62e}$$

Moreover we get

$$\frac{\partial Q}{\partial Y}(X,0) = \frac{\partial^2\psi}{\partial Y^2}(X,0) = \sigma X^{-1/2}. \tag{1.63}$$

We apply dimensional analysis to simplify $\psi(X,Y)$. Since BVP (1.62a–e) has no constants, we have

$$\psi = F(X,Y), \tag{1.64}$$

for some unknown function F. We see that

$$[\psi] = [Y] = L^{-1}T, \tag{1.65a}$$

$$[X] = L^{-2}T^2. \tag{1.65b}$$

Consequently there is only one measurable dimensionless quantity. It is convenient to choose as dimensionless quantities

$$\pi_1 = \eta = \frac{Y}{\sqrt{X}}, \tag{1.66a}$$

and

$$\pi = \frac{\psi}{\sqrt{X}}. \tag{1.66b}$$

Hence

$$\psi(X, Y) = \sqrt{X}\, G(\eta) \tag{1.67}$$

where $G(\eta)$ solves a BVP for an ordinary differential equation which is obtained by substituting (1.67) into (1.62a–e). Moreover from (1.67) and (1.63) it follows that

$$\sigma = G''(0). \tag{1.68}$$

[A prime denotes differentiation with respect to η.] Note that

$$\frac{\partial \psi}{\partial Y} = G'(\eta), \; \frac{\partial \psi}{\partial X} = \frac{1}{2\sqrt{X}}[G - \eta G'],$$

$$\frac{\partial^2 \psi}{\partial Y^2} = \frac{1}{\sqrt{X}} G'', \; \frac{\partial^3 \psi}{\partial Y^3} = \frac{1}{X} G''',$$

$$\frac{\partial^2 \psi}{\partial X \partial Y} = \frac{1}{2X}[-\eta G''];$$

$0 < X < \infty$, $0 < Y < \infty$ leads to $0 < \eta < \infty$; $Y = 0$ leads to $\eta = 0$; $Y \to \infty$ leads to $\eta \to \infty$; $X = 0$ leads to $\eta \to \infty$. Correspondingly BVP (1.62a–e) reduces to solving the third order ordinary differential equation known as the *Blasius equation* for $G(\eta)$,

$$2\frac{d^3 G}{d\eta^3} + G\frac{d^2 G}{d\eta^2} = 0, \quad 0 < \eta < \infty, \tag{1.69a}$$

with boundary conditions

$$G(0) = G'(0) = 0, \quad G'(\infty) = 1. \tag{1.69b}$$

The aim is to find $\sigma = G''(0)$.

A numerical procedure for solving BVP (1.69a,b) is the shooting method where one considers the auxiliary initial value problem

$$2\frac{d^3 H}{dz^3} + H\frac{d^2 H}{dz^2} = 0, \quad 0 < z < \infty, \tag{1.70a}$$

$$H(0) = H'(0) = 0, \quad H''(0) = A, \tag{1.70b}$$

for some initial guess A. One integrates out the IVP (1.70a,b) and determines that $H'(\infty) = B$ for some number, $B = B(A)$. One continues shooting with different values of A until B is close enough to 1.

It turns out that the invariance of (1.70a) and the initial conditions $H(0) = H'(0) = 0$ under a one-parameter family of scalings (one-parameter Lie group of scaling transformations) leads to determining σ with only one shooting:

The transformation

$$z = \frac{\eta}{\alpha}, \tag{1.71a}$$

$$H(z) = \alpha G(\eta), \tag{1.71b}$$

with $\alpha > 0$ an arbitrary constant, maps (1.70a,b) to (1.69a) with initial conditions

$$G(0) = G'(0) = 0, \quad G''(0) = \frac{A}{\alpha^3}. \tag{1.72}$$

Moreover $H'(\infty) = B$ implies that

$$G'(\infty) = \frac{B}{\alpha^2}. \tag{1.73}$$

Hence we pick α so that $\alpha^2 = B$, i.e. $\alpha = \sqrt{B}$. Then

$$\sigma = G''(0) = \frac{A}{B^{3/2}}. \tag{1.74}$$

One can show that

$$\sigma = 0.332\ldots\ . \tag{1.75}$$

Exercises 1.3

1. For the heat conduction problem (1.46a–c), show that $r(\mathrm{B}_2) = 4$ for both dynamical and thermal units.

2. Derive (1.47a–c).

3. Derive (1.48a–c).

4. The BVP (1.46a–c) in effect has only two constants: $\kappa = \frac{K}{\rho c}$ (diffusivity) and $\lambda = \frac{Q}{\rho c}$. Use dimensional analysis with dynamical units to reduce (1.46a–c) where now $W_1 = x$, $W_2 = t$, $W_3 = \kappa$, $W_4 = \lambda$.

5. Consider the *Rayleigh flow problem* [see Schlichting (1955)] where an infinite flat plate is immersed in an incompressible fluid at rest. The plate is instantaneously accelerated so that it moves parallel to itself with constant velocity U.

   Let u be the fluid velocity in the direction of U (x-direction). Let the y-direction be the direction normal to the plate. The situation is illustrated in Figure 1.3.1.

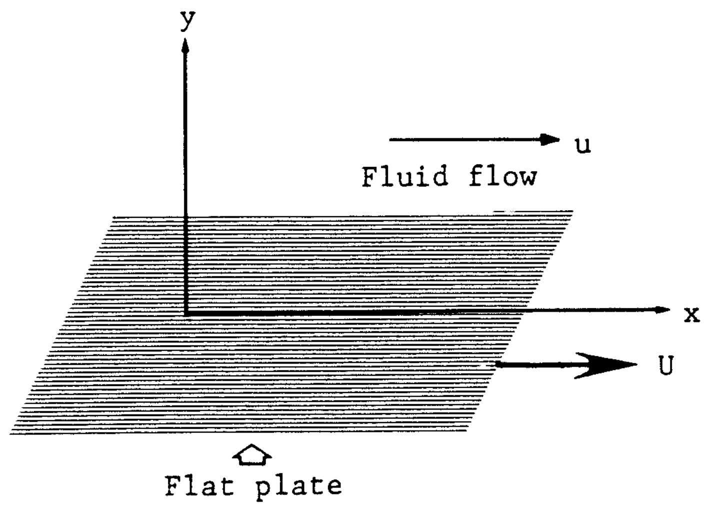

Figure 1.3.1

From symmetry considerations the Navier–Stokes equations governing this problem reduce to the viscous diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \quad 0 < t < \infty, \; 0 < y < \infty, \tag{1.76a}$$

with boundary conditions

$$u(y, 0) = 0, \tag{1.76b}$$

$$u(0, t) = U, \tag{1.76c}$$

$$u(\infty, t) = 0. \tag{1.76d}$$

(a) Use dimensional analysis to simplify BVP (1.76a–d).

(b) Use scalings of quantities followed by dimensional analysis to further simplify (1.76a–d). Find the explicit self-similar solution $u(y, t)$ of (1.76a–d).

1.4 Generalization of Dimensional Analysis—Invariance of Partial Differential Equations Under Scalings of Variables

In both examples of Section 1.3.1 the use of dimensional analysis to reduce a BVP for a partial differential equation to a BVP for an ordinary differential equation is rather cumbersome and should make the reader feel uneasy. For the heat conduction problem the use of dimensional analysis depends on either making the right choice of fundamental dimensions (thermal units) or combining effectively the constants before using dynamical units [cf.

Exercise 1.3-4]. For the Prandtl–Blasius problem we used scaled variables before applying dimensional analysis.

A much easier way to accomplish such a reduction for a BVP is to consider the invariance property of the BVP under a one-parameter family of scalings (one-parameter Lie group of scaling transformations) when its variables are scaled but the constants of the BVP are not scaled. If the BVP is invariant under such a family of scaling transformations, then the number of independent variables is reduced constructively by one. We show that if, for some choice of fundamental dimensions, dimensional analysis leads to a reduction of the number of independent variables of a BVP, then such a reduction is always possible through invariance of the BVP under scalings applied strictly to its variables. [Recall that dimensional analysis involves scalings of *both* variables and constants.] Moreover there exist BVP's for which the number of independent variables is reduced from invariance under a one-parameter family of scalings of its variables but the number of independent variables is not reduced from the use of dimensional analysis for any *known* choice of fundamental dimensions. [One could argue that this is a way of determining new sets of fundamental dimensions!] Hence, for the purpose of reducing the number of independent variables of a BVP, invariance of a BVP under a one-parameter family of scalings of its variables is a generalization of dimensional analysis.

Zel'dovich (1956) [see also Barenblatt and Zel'dovich (1972) and Barenblatt (1979)] calls a ***self-similar solution of the first kind*** a solution of a BVP obtained by reduction through dimensional analysis and a ***self-similar solution of the second kind*** a solution to a BVP obtained by reduction through invariance under scalings of the variables when this reduction is not possible through dimensional analysis. The two examples of Section 1.3.1 show that these distinctions are somewhat blurred.

Before proving a general theorem relating dimensional analysis and invariance under scalings of variables, we consider the invariance property of the heat conduction problem (1.46a–c) under scalings of its variables.

Consider the family of scaling transformations

$$x^* = \alpha x, \tag{1.77a}$$

$$t^* = \beta t, \tag{1.77b}$$

$$u^* = \gamma u, \tag{1.77c}$$

where α, β, γ are arbitrary positive constants.

Definition 1.4-1. A transformation of the form (1.77a–c) *leaves BVP* (1.46a–c) *invariant* (*is admitted by BVP* (1.46a–c)) if and only if for any solution $u = \Theta(x,t)$ of (1.46a–c) it follows that

$$v(x^*,t^*) = u^* = \gamma u = \gamma\Theta(x,t) \tag{1.78}$$

solves the BVP

$$\rho c \frac{\partial v}{\partial t^*} - K \frac{\partial^2 v}{\partial x^{*2}} = 0, \quad -\infty < x^* < \infty,\ t^* > 0, \tag{1.79a}$$

$$v(x^*, 0) = \frac{Q}{\rho c}\delta(x^*), \tag{1.79b}$$

$$\lim_{x^* \to \pm\infty} v(x^*, t^*) = 0. \tag{1.79c}$$

[Implicitly it is assumed that the domain $-\infty < x^* < \infty$, $t^* > 0$ corresponds to the domain $-\infty < x < \infty$, $t > 0$; $t^* = 0$ corresponds to $t = 0$; $x^* \to \pm\infty$ corresponds to $x \to \pm\infty$, i.e. (1.77a–c) leaves the boundary of BVP (1.46a–c) invariant.]

Lemma 1.4-1. *If a scaling* (1.77a–c) *leaves BVP* (1.46a–c) *invariant, and* $u = \Theta(x,t)$ *solves* (1.46a–c), *then* $u = \gamma\Theta(\frac{x}{\alpha}, \frac{t}{\beta})$ *also solves* (1.46a–c).

Proof. See Exercise 1.4-1. □

In order that (1.77a–c) leaves BVP (1.46a–c) invariant, it is sufficient to leave each of these three equations separately invariant. Invariance of (1.46a), i.e. $u = \Theta(x,t)$ solves (1.46a) if and only if $v = \gamma\Theta(x,t)$ solves (1.79a), leads to $\beta = \alpha^2$ and invariance of (1.46b,c) leads to $\gamma = \frac{1}{\alpha}$. Hence the one-parameter ($\alpha > 0$) family of scaling transformations

$$x^* = \alpha x, \tag{1.80a}$$

$$t^* = \alpha^2 t, \tag{1.80b}$$

$$u^* = \frac{1}{\alpha} u, \tag{1.80c}$$

is admitted by (1.46a–c).

Clearly if $u = \Theta(x,t)$ solves (1.46a–c) then

$$v(x^*, t^*) = \Theta(x^*, t^*) = \Theta(\alpha x, \alpha^2 t) \tag{1.81}$$

solves (1.79a–c). Hence a transformation (1.80a–c) maps any solution $v = \Theta(x^*, t^*)$ of (1.79a–c) to a solution

$$v = \frac{1}{\alpha}\Theta(x,t) = \frac{1}{\alpha}\Theta\left(\frac{x^*}{\alpha}, \frac{t^*}{\alpha^2}\right)$$

of (1.79a–c) or, equivalently, maps any solution $u = \Theta(x,t)$ of (1.46a–c) to a solution $u = \frac{1}{\alpha}\Theta(\frac{x}{\alpha}, \frac{t}{\alpha^2})$ of (1.46a–c).

The solution of (1.46a–c) and hence (1.79a–c) is unique. As a result the solution $u = \Theta(x,t)$ of (1.46a–c) satisfies the functional equation (arising from the uniqueness of the solution to this BVP)

$$\Theta(x^*, t^*) = \frac{1}{\alpha}\Theta(x,t). \tag{1.82}$$

Such a solution of a partial differential equation, arising from invariance under a one-parameter Lie group of transformations, is called a *similarity solution* or *invariant solution.* The functional equation (1.82), satisfied by the invariant solution, is called the *invariant surface condition.* An invariant solution arising from invariance under a one-parameter Lie group of scalings such as (1.80a–c) is also called a *self-similar solution* or *automodel solution.*

From (1.80a,b), the invariant surface condition (1.82) satisfied by $\Theta(x,t)$ is

$$\Theta(\alpha x, \alpha^2 t) = \frac{1}{\alpha}\Theta(x,t). \tag{1.83}$$

In order to solve (1.83), let $z = \frac{x}{\sqrt{t}}$ and $\Theta(x,t) = \frac{1}{\sqrt{t}}\phi(z,t)$. Then in terms of z, t, $\phi(z,t)$, equation (1.83) becomes

$$\frac{1}{\sqrt{t}}\phi(z,t) = \frac{\alpha}{\sqrt{\alpha^2 t}}\phi(z,\alpha^2 t) = \frac{\phi(z,\alpha^2 t)}{\sqrt{t}}.$$

Hence $\phi(z,t)$ satisfies the functional equation

$$\phi(z,t) = \phi(z,\alpha^2 t) \quad \text{for any } \alpha > 0. \tag{1.84}$$

Thus $\phi(z,t)$ does not depend on t. This leads to the *invariant form* (*similarity form*)

$$u = \Theta(x,t) = \frac{1}{\sqrt{t}}F(z) \tag{1.85}$$

for the solution of BVP (1.46a–c); z is called the *similarity variable.* The substitution of (1.85) into (1.46a–c) leads to a BVP for an ordinary differential equation with unknown $F(z)$. The details are left to Exercise 1.4-2.

Now consider the following theorem connecting dimensional analysis and invariance under scalings of variables.

Theorem 1.4-1. *If the number of independent variables appearing in a BVP for a partial differential equation can be reduced by ρ through dimensional analysis, then the number of variables can be reduced by ρ through invariance of the BVP under a ρ-parameter family of scaling transformations of its variables.*

Proof. Consider the dimension matrices B, B_1 and B_2 defined by (1.44a,b), (1.45). Through dimensional analysis the number of independent variables of the given BVP is reduced by $\rho = r(B) - r(B_2)$.

An arbitrary scaling of any fundamental dimension is represented by the m-parameter family of scaling transformations

$$L_j^* = e^{\epsilon_j} L_j, \quad j = 1, 2, \ldots, m \tag{1.86}$$

where $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ are arbitrary real numbers. Let the row vector

$$\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_m]. \tag{1.87}$$

The scaling (1.86) induces a scaling of the value of each measurable quantity W_i:

$$W_i^* = e^{\sum_{j=1}^{m} \epsilon_j b_{ji}} W_i = e^{(\epsilon \mathrm{B})_i} W_i, \quad i = 1, 2, \ldots, n, \tag{1.88}$$

where $(\epsilon \mathrm{B})_i$ is the ith component of the n-component row vector $\epsilon \mathrm{B}$; the value of u scales to

$$u^* = e^{\sum_{j=1}^{m} \epsilon_j a_j} u. \tag{1.89}$$

From assumption (iv) of the Buckingham Pi-theorem, the family of scaling transformations (1.88), (1.89), induced by the m-parameter family of scalings of the fundamental dimensions (1.86), leaves the given BVP invariant. Our aim is to find the number of essential parameters in the subfamily of transformations of the form (1.88), (1.89) for which the constants are all invariant, i.e. we aim to find the dimension of the vector space of all vectors $\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_m]$ such that

$$W_i^* = W_i, \quad i = \ell + 1, \ell + 2, \ldots, n, \tag{1.90a}$$

and

$$W_j^* \neq W_j \quad \text{for some } j = 1, 2, \ldots, \ell. \tag{1.90b}$$

Equation (1.90a) holds if and only if

$$\epsilon \mathrm{B}_2 = 0. \tag{1.91}$$

The number of essential parameters is the number of linearly independent solutions ϵ of (1.91) such that $\epsilon \mathrm{B}_1 \neq 0$.

It is helpful to introduce a few definitions and some notation:

Let A be a matrix linear transformation acting on vector space V such that if $\mathbf{v} \in V$ then $\mathbf{v}\mathrm{A}$ is the action of A on $\mathbf{v}$. The ***null space*** of A is the vector space $V_{(\mathrm{A})_N} = \{\epsilon \in V : \epsilon \mathrm{A} = 0\}$; the ***range space*** of A is the vector space $V_{(\mathrm{A})_R} = \{z : z = \epsilon \mathrm{A} \text{ for some } \epsilon \in V\}$; dim V is the dimension of the vector space V. It follows that

$$\dim V = \dim V_{(\mathrm{A})_R} + \dim V_{(\mathrm{A})_N}.$$

Consider the matrices B, B_1, and B_2 defined by (1.44a,b), (1.45). Let V be R^m, where m is the number of rows of each of these three matrices, so that $\dim V = m$. Then dim $V_{(\mathrm{B})_N}$ is the number of linearly independent solutions ϵ of the set of equations $\epsilon \mathrm{B} = 0$, and dim $V_{(\mathrm{B}_2)_N}$ is the number of linearly independent solutions ϵ of $\epsilon \mathrm{B}_2 = 0$. It follows that

$$\dim V_{(\mathrm{B}_2)_N} = m - r(\mathrm{B}_2); \quad \dim V_{(\mathrm{B})_N} = m - r(\mathrm{B}) = m - r(\mathrm{B}_2) - \rho.$$

Since $V_{(\mathrm{B}_2)_N(\mathrm{B}_1)_N} = V_{(\mathrm{B})_N}$, it follows that

$$\begin{aligned} \dim V_{(\mathrm{B}_2)_N} &= \dim V_{(\mathrm{B}_2)_N(\mathrm{B}_1)_N} + \dim V_{(\mathrm{B}_2)_N(\mathrm{B}_1)_R} \\ &= \dim V_{(\mathrm{B})_N} + \dim V_{(\mathrm{B}_2)_N(\mathrm{B}_1)_R}. \end{aligned}$$

Hence $\dim V_{(B_2)_N(B_1)_R} = \rho$. But $\dim V_{(B_2)_N(B_1)_R}$ is the number of linearly independent solutions ϵ of the system $\epsilon B_2 = 0$ such that $\epsilon B_1 \neq 0$. Hence the number of essential parameters is ρ, completing the proof of the theorem. □

Exercises 1.4

1. Prove Lemma 1.4-1.

2. Set up the BVP for $F(z)$ as defined by equation (1.85). Put this BVP in dimensionless form using
   (a) dynamical units;
   (b) thermal units. Explain.

3. Consider diffusion in a half-space with a concentration dependent diffusion coefficient which is directly proportional to the concentration of a substance $C(x,t)$. Initially and far from the front face $x = 0$ the concentration is assumed to be zero. The concentration is fixed on the front face. The aim is to find the concentration flux on the front face, $\frac{\partial C}{\partial x}(0,t)$. In special units $C(x,t)$ satisfies the BVP

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(C\frac{\partial C}{\partial x}\right), \quad 0 < x < \infty,\ 0 < t < \infty, \tag{1.92a}$$

   where

$$C(x,0) = C(\infty,t) = 0; \quad C(0,t) = A. \tag{1.92b}$$

   (a) Exploit similarity to determine $\frac{\partial C}{\partial x}(0,t)$ as effectively as possible.
   (b) Use scaling invariance to reduce the BVP (1.92a,b) to a BVP for an ordinary differential equation.
   (c) Discuss a numerical procedure to determine $\frac{\partial C}{\partial x}(0,t)$ based on the scaling property of the reduced BVP derived in (b).

4. For boundary layer flow over a semi-infinite wedge at zero angle of attack, the governing partial differential equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U(x)\frac{dU}{dx} = \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad 0 < x < \infty,\ 0 < y < \infty,$$

   with boundary conditions $u(x,0) = v(x,0) = 0$, $\lim_{y\to\infty} u(x,y) = U(x)$; $U(x) = Ax^\ell$ where A, ℓ are constants with $\ell = \frac{\beta}{2-\beta}$ corresponding to the opening angle $\pi\beta$ of the semi-infinite wedge. In this problem x is the distance from the leading edge on the wedge surface and y is the distance from the wedge surface [Figure 1.4.1].

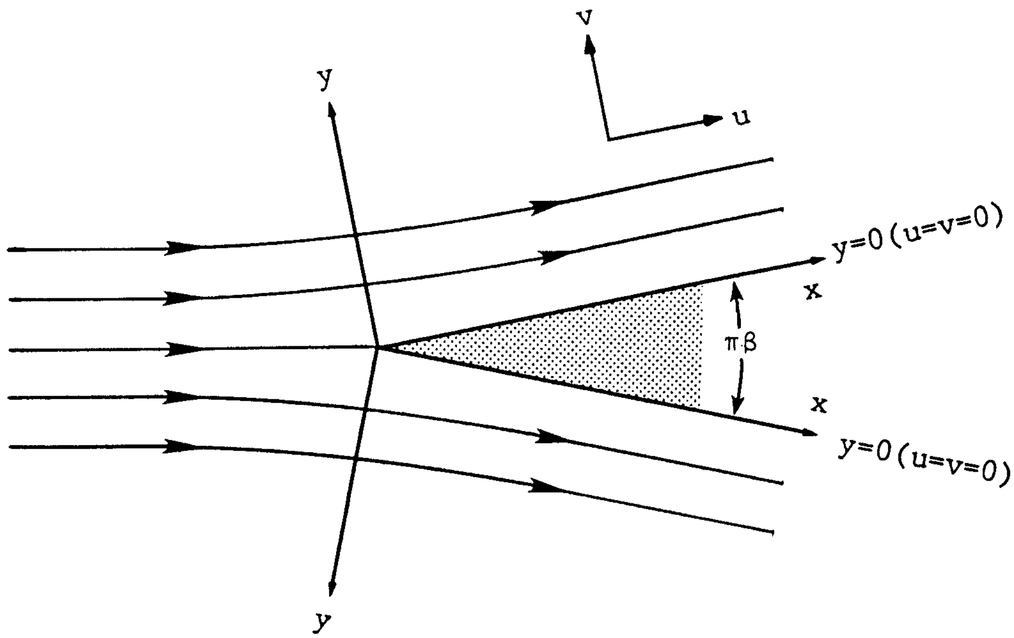

Figure 1.4.1

As for the Prandtl boundary layer equations (1.49a,b) introduce a stream function $\psi(x,y)$. Use scaling invariance to reduce the given problem to a BVP for an ordinary differential equation. Choose coordinates so that the Blasius equation arises if $\ell = 0$.

5. The following BVP for a nonlinear diffusion equation arises from a biphasic continuum model of soft tissue [Holmes (1984)]:

$$\frac{\partial^2 u}{\partial x^2} - K\left(\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = 0, \quad 0 < x < \infty,\ 0 < t < \infty,$$

where K is a function of $\frac{\partial u}{\partial x}$, with boundary conditions $\frac{\partial u}{\partial x}(0,t) = -1$, $u(\infty,t) = u(x,0) = 0$. Reduce this problem to a BVP for an ordinary differential equation.

6. Use invariance under scalings of the variables to solve the Rayleigh flow problem (1.76a–d).

7. Consider again the source problem for heat conduction in terms of the dimensionless form arising from dynamical units

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \delta(x),$$

$$\lim_{x\to\pm\infty} u(x,t) = 0.$$

The use of scaling invariance with respect to the variables (1.80a–c) leads to the similarity form for the solution $u = \frac{1}{\sqrt{t}}G(\frac{x}{\sqrt{t}})$.

(a) Show that the problem is invariant under the one-parameter (β) family of transformations

$$x^* = x - \beta t,\ t^* = t,\ u^* = ue^{\frac{1}{2}\beta x - \frac{1}{4}\beta^2 t}, \qquad (1.93)$$

for any constant β, $-\infty < \beta < \infty$.

(b) Check that t and $ue^{x^2/4t}$ are invariants of these transformations.

(c) Show that these transformations lead to the similarity form

$$u(x,t) = e^{-x^2/4t}H(t). \qquad (1.94)$$

Hence show that invariance under scalings (1.80a–c) and the transformations (1.93) lead to the well-known fundamental solution

$$u = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}.$$

1.5 Discussion

Dimensional analysis is necessary for ascertaining fundamental dimensions and consequent essential quantities which arise in a real problem in order to design proper model experiments. If a given problem can be described in terms of a boundary value problem (BVP) for a system of partial differential equations then dimensional analysis may lead to a reduction in the number of independent variables. Moreover if such a reduction exists, it can always be accomplished by considering the invariance properties of the BVP under scaling transformations applied only to its variables.

As will be seen in Chapter 4 the invariance properties of partial differential equations (or more particularly BVP's) under scalings of variables can be generalized to the study of the invariance properties of partial differential equations under arbitrary one-parameter Lie groups of point transformations of their variables. Moreover for a given differential equation such transformations are found algorithmically. [For example one can easily deduce transformations (1.93) and (1.94).] This follows from the properties of such transformations, most importantly their characterization by infinitesimal generators [see Chapter 2].

References on dimensional analysis specific to various fields include: de Jong (1967) [economics]; Sedov (1959), Birkhoff (1950) and Barenblatt (1979) [mechanics, elasticity, and hydrodynamics]; Venikov (1969) [electrical engineering]; Taylor (1974) [mechanical engineering]; Becker (1976) [chemical engineering]; Kurth (1972) [astrophysics]; Murota (1985) [systems analysis].

Examples of dimensional analysis and scaling invariance applied to BVP's appear in Sedov (1959), Birkhoff (1950), Barenblatt (1979), Dresner (1983), Hansen (1964), and Seshadri and Na (1985). Examples which use scalings to convert BVP's for ordinary differential equations to initial value problems appear in Klamkin (1962), Na (1967, 1979), Dresner (1983), and Seshadri and Na (1985).