2^{nd} Order Linear Ordinary Differential Equations

Solutions for equations of the following general form:

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = h(x)$$

Reduction of Order

If terms are missing from the general second-order differential equation, it is sometimes possible to reduce the equation to a first-order ordinary differential equation. Second-order differential equations can be solved by reduction of order for two cases.

Dependent Variable (y) is Missing

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} = h(x)$$

The procedures is to define a new variable p as:

$$\frac{dy}{dx} = p$$

which can be differentiated again with respect to x to give:

$$\frac{d^2 y}{dx^2} = \frac{dp}{dx}$$

These are substituted into the differential equation to give:

$$\frac{dp}{dx} + a_1(x)\, p = h(x)$$

which can then be solved by integrating factors to give (see handout on solution methods for 1^{st} order differential equations):

$$p = \frac{1}{F(x)}\int h(x)\,F(x)\,dx + I_1$$

$$\text{where } F(x) = \exp\left[\int a_1(x)\,dx\right]$$

The solution y is found by substituting for $p = dy\,/\,dx$ and integrating again with respect to x.

$$\boxed{\begin{gathered} y = \int \left[\frac{1}{F(x)} \int h(x) F(x) dx \right] dx + \int I_1 dx + I_2 \\ \text{where} \quad F(x) = \exp\left[\int a_1(x) dx \right] \end{gathered}}$$

where I_1 and I_2 are constants of integration. (Note that throughout this document constants of integration will be indicated by this notation.)

Independent Variable (x) is Missing

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2\, y = 0$$

The procedures is to define a new variable p as:

$$\frac{dy}{dx} = p$$

which can be differentiated again with respect to x to give:

$$\frac{d^2 y}{dx^2} = \frac{dp}{dx}$$

but remember that p can be written as a function of y which is a function of x. That is $p = f(y(x))$. This can be differentiated by chain rule to give (remembering that $p = dy / dx$):

$$\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} p$$

These relationships for p are substituted into the differential equation to give:

$$p \frac{dp}{dy} + a_1\, p + a_2\, y = 0$$

The 2nd order differential equation of y with respect to x has now been converted into a 1st order differential equation of p with respect to y. This equation is nonlinear (because p multiplies dp / dx) and can only be solved analytically if it is possible to separate and integrate. Once p is determined as a function of y (e.g., $p = f(y)$), then y can be found by integrating $f(y)$ with respect to x to give:

$$\boxed{\int \frac{dy}{f(y)} = x + I_2}$$

The first constant of integration will be contained in the function $f(y)$.

Variation of Parameters

This method can be used anytime you already know one solution, $y_1(x)$, to the homogeneous form of the general differential equation given below.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = h(x)$$

The complete solution is found by substituting $y = u(x)y_1(x)$ into the above differential equation. The differentials of y are as follows:

$$y' = uy_1' + u'y_1$$

$$y'' = uy_1'' + 2u'y_1' + u''y_1$$

which when substituted into the differential equation gives:

$$u''y_1 + u'\left(2y_1' + a_1(x)y_1\right) + u\left(y_1'' + a_1(x)y_1' + a_2(x)y_1\right) = h(x)$$

Since y_1 is a solution to the homogeneous form of the differential equation shown at the top of the page, $y_1'' + a_1(x)y_1' + a_2(x)y_1 = 0$, and the above equation reduces to give the following differential equation in u:

$$u''y_1 + u'\left(2y_1' + a_1(x)y_1\right)_1 = h(x)$$

The function u can be found from this differential equation by reducing order and then solving by integrating factors. The complete solution y can be found by multiplying u by y_1 to give the general solution:

$$y = uy_1 = y_1\int\left[\frac{1}{y_1^2\hat{F}}\int h(x)y_1\hat{F}dx + \frac{I_2}{y_1^2\hat{F}}\right]dx + I_1y_1$$

where

$$\hat{F} = \exp\left[\int a_1 dx\right] \quad \text{when} \quad a_1 \neq 0$$

$$\hat{F} = 1 \quad \text{when} \quad a_1 = 0$$

From which you can identify the second solution and the particular solution as follows:

$$y_2 = y_1\int\frac{dx}{y_1^2\hat{F}}$$

$$y_p = y_1\int\frac{1}{y_1^2\hat{F}}\left[\int h(x)y_1\hat{F}dx\right]dx$$

<u>Constant Coefficient Ordinary Differential Equations</u>

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where a, b and c are constants

The form of the differential equation suggests solutions of $y = e^{rx}$.

(At this point this form is deduced by understanding the properties of differentiating e^{rx}. Later, we will develop a general approach for determining that this is the form of the solution.)

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$(ar^2 + br + c)e^{rx} = 0$$

$$\text{Characteristic Equation:} \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: Real & Unequal Roots ($b^2 - 4ac > 0$)

(a) If $r_1 \neq r_2$ and $r_1 \neq -r_2$, then

$$\boxed{y = I_1 e^{r_1 x} + I_2 e^{r_2 x}}$$

(b) If $r_1 = -r_2 = r$, then

$$\boxed{y = I_1 e^{rx} + I_2 e^{-rx}}$$

or

$$\boxed{y = A\sinh(rx) + B\cosh(rx)}$$

$$\text{where } I_1 = \frac{A+B}{2} \text{ and } I_2 = \frac{B-A}{2}$$

Case 2: Complex Roots ($b^2 - 4ac < 0$)

$r = \alpha \pm i\beta$, complex conjugate where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

$$y = e^{\alpha x}\left(I_1 e^{i\beta x} + I_2 e^{-i\beta x}\right)$$

But $e^{ix} = \cos x + i\sin x$

$$y = e^{\alpha x}\left(I_1\left[\cos(\beta x) + i\sin(\beta x)\right] + I_2\left[\cos(\beta x) - i\sin(\beta x)\right]\right)$$

Let $I_1 = A_1 + iB_1$ and $I_2 = A_2 + iB_2$ and $A = A_1 + A_2$ and $B = B_2 - B_1$

$$\boxed{y = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)}$$

Case 3: Real and Equal Roots ($b^2 - 4ac = 0$)

$$r_1 = r_2 = r = -\frac{b}{2a}$$

The characteristic equation gives only one solution,

$$y_1 = e^{rx}$$

The second solution can be found by variation of parameters to give:

$$y_2 = y_1 \int \frac{dx}{y_1^2 \hat{F}} \quad \text{where } \hat{F} = \exp\left[\int \frac{b}{a}\,dx\right] = \exp\left(\frac{bx}{a}\right)$$

$$y_2 = e^{rx}\int \frac{dx}{e^{-bx/a}\,e^{bx/a}} = e^{rx}\int dx = xe^{rx}$$

$$\boxed{y = I_1 e^{rx} + I_2 x e^{rx}}$$

Equidimensional Ordinary Differential Equation

$$\frac{d^2 y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{b}{x^2} y = 0$$

where a and b are constants

The form of the differential equation suggests solutions of $y = x^r$

(At this point this form is deduced by understanding the properties of differentiating x^r. Later, we will develop a general approach for determining that this is the form of the solution.)

$$y' = rx^{r-1}$$

$$y'' = r(r-1)x^{r-2}$$

$$\left(r(r-1) + ar + b\right)x^{r-2} = 0$$

Characteristic Equation: $r^2 + (a-1)r + b = 0$

$$r = \frac{1 - a \pm \sqrt{(1-a)^2 - 4b}}{2}$$

Case 1: Real & Unequal Roots $(1-a)^2 - 4b > 0$

$$\boxed{y = I_1 x^{r_1} + I_2 x^{r_2}}$$

Case 2: Complex Roots $(1-a)^2 - 4b < 0$

$$r = \alpha \pm \mathrm{i}\beta$$

where $\alpha = \dfrac{1-a}{2}$ and $\beta = \dfrac{\sqrt{4b^2 - (1-a)^2}}{2}$

$$y = I_1 x^\alpha x^{\mathrm{i}\beta} + I_2 x^\alpha x^{-\mathrm{i}\beta} = I_1 x^\alpha e^{\mathrm{i}\beta \ln x} + I_2 x^\alpha e^{-\mathrm{i}\beta \ln x}$$

But $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$

Let $I_1 = A_1 + \mathrm{i}B_1$ and $I_2 = A_2 + \mathrm{i}B_2$ and $A = A_1 + A_2$ and $B = B_2 - B_1$

$$\boxed{y = Ax^\alpha \cos(\beta \ln x) + Bx^\alpha \sin(\beta \ln x)}$$

Case 3: Real and Equal Roots $(1-a)^2 - 4b = 0$

$$r_1 = r_2 = r = \frac{1-a}{2}$$

The characteristic equation gives only one solution,

$$y_1 = x^r$$

The second solution can be found by variation of parameters to give:

$$y_2 = y_1 \int \frac{dx}{y_1^2 \hat{F}} \quad \text{where } \hat{F} = \exp\left[\int \frac{a}{x} dx\right] = \exp(a \ln x) = x^a$$

$$y_2 = x^r \int \frac{dx}{x^{2(1-a)/2} x^a} == x^r \int \frac{dx}{x^{(1-a)} x^a} = x^r \int \frac{dx}{x} = x^r \ln x$$

$$\boxed{y = I_1 x^r + I_2 x^r \ln x}$$

Special Equations

Several second-order ordinary differential equations arise so often that they have been given names. Some of these are listed below.

Harmonic Equation

The following differential equation commonly arises for problems written in a rectangular coordinate system.

$$\frac{d^2 y}{dx^2} + b^2 y = 0$$

where b^2 is not a function of x or y

This differential equation is a constant coefficient equation with the solution:

$$\boxed{y = I_1 \sin(b\,x) + I_2 \cos(b\,x)}$$

Modified Harmonic Equation

Like the harmonic equation, this equation commonly arises for problems written in a rectangular coordinate system.

$$\frac{d^2 y}{dx^2} - b^2 y = 0$$

where b^2 is not a function of x or y

This differential equation is a constant coefficient equation with the solution:

$$\boxed{y = I_1 \sinh(b\,x) + I_2 \cosh(b\,x)}$$

This equation can also be written in terms of the exponential as:

$$\boxed{y = I_1 \exp(b\,x) + I_2 \exp(-b\,x)}$$

As a general rule, it is usually convenient to use the sinh/cosh form of the solution for problems with finite boundaries and to use the exponential form of the solution for problems with one or more infinite boundaries.

Bessel's Equation

The following differential equation commonly arises for problems written in a cylindrical coordinate system.

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(b^2 x^2 - p^2\right) y = 0$$

where b^2 and p^2 are constants. This differential equation has the solution

$$\boxed{y = A\,J_p(b\,x) + B\,J_{-p}(b\,x)}$$

where A and B are constants of integration, and J_p is the Bessel function of the first kind and order p. If p is an integer or if $p = 0$, then the differential equation is:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(b^2\,x^2 - n^2\right)y = 0$$

where n is an integer or zero. The solution to this equation is:

$$\boxed{y = A\,J_n(b\,x) + B\,Y_n(b\,x)}$$

where Y_n is the Bessel function of the second kind and order n.

The Bessel functions of the first and second kind are similar to the sine and cosine functions (i.e., solutions to the harmonic equation). In particular, like the sine and cosine functions, Bessel functions of the first and second kind are periodic for real arguments.

Modified Bessel's Equation

Like Bessel's equation, this equation commonly arises for problems written in cylindrical a coordinate system.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(b^2\,x^2 + p^2\right)y = 0$$

where b^2 and p^2 are constants. This differential equation has the solution

$$\boxed{y = A\,I_p(b\,x) + B\,I_{-p}(b\,x)}$$

where A and B are constants of integration, and I_p is the modified Bessel function of the first kind and order p. If p is an integer or if $p = 0$, then the differential equation is:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(b^2\,x^2 + n^2\right)y = 0$$

where n is an integer or zero. The solution to this equation is:

$$y = A\,I_n(b\,x) + B\,K_n(b\,x)$$

where K_n is the modified Bessel function of the second kind and order n.

Modified Bessel functions of the first and second kind are similar to the hyperbolic sine and hyperbolic cosine functions (i.e., solutions to the modified harmonic equation). Most importantly, modified Bessel functions of the first and second kind are not periodic functions.

Special Functions

Error Function

A number of physical problems of interest to chemical engineers will produce equations in which

$$\frac{dy}{dx} = e^{-x^2}$$

which has the solution

$$y = \int \exp\left(-x^2\right) dx + I$$

where *I* is a constant of integration. To clarify the exact operation that is intended by the above equation, it is better to write the solution as:

$$y = \int_0^x \exp\left(-s^2\right) ds + I$$

The integral of the $\exp(-s^2)$ must be determined numerically, except when x is infinity, in which case

$$\int_0^\infty \exp\left(-s^2\right) ds = \frac{\sqrt{\pi}}{2}$$

Because problems with this type of solution arise frequently, it was convenient to define a function that represents this integral. The name of this function is the *Error Function* and it is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-s^2\right) ds$$

By defining it in this way, $erf(x) = 0$ when $x = 0$ and $erf(x) = 1$ when $x \to \infty$. Using the error function, the solution to the differential equation at the top of this page is:

$$y = \frac{\sqrt{\pi}}{2} erf(x) + I$$

Complementary Error Function

The complementary error function $erfc(x)$ is defined as:

$$erfc(x) = 1 - erf(x)$$