



Determining Map, Data Assimilation and an Observable Regularity Criterion for the Three-Dimensional Boussinesq System

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Abstract

In this paper, we consider the analysis of a nudging based algorithm for data assimilation for the three-dimensional Boussinesq system, which we call the AOT system. A rigorous analysis of this algorithm for well-posed dissipative partial differential equations was first provided by Azouani, Olson and Titi (J Nonlinear Sci 24:277–304, 2014); thus justifying our terminology for the associated nudging system. We provide a sufficient condition, *based solely on the observed velocity data* obtained from a Leray-Hopf weak solution, for the global well-posedness, regularity and most crucially, the asymptotic tracking property of solutions of the associated (three-dimensional) nudging system. It is to be noted that neither regularity nor any knowledge of a uniform H^1 -norm bound is *a priori* assumed on the solution of the original three-dimensional Boussinesq system from which the observations are obtained. As a corollary of our result, we obtain a novel *observable regularity criterion* based on finitely many observational data. Our condition also guarantees the construction of the Lipschitz continuous *determining map*, which is known to play a crucial role in the construction of the so-called determining form and in statistical data assimilation.

Keywords Data assimilation · Determining modes · Determining volume elements · Determining map · Determining functionals · Three-dimensional Boussinesq system · Three-dimensional Navier-Stokes equations · Regularity criterion for three-dimensional Navier-Stokes equations

Mathematics Subject Classification Primary 35Q30 · 93C20; Secondary 35Q35 · 76B75

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1 Introduction

The task of forecasting using a physical or biological model is often hindered by a lack of adequate knowledge of the initial state and/or model parameters describing the system. In order to compensate for this, one may utilize available measurements of the system, collected on a *much coarser (spatial) scale than the desired resolution of the forecast*. An example of this occurs in weather prediction where one has almost continuously collected data from sparsely located weather stations. The objective of data assimilation and signal synchronization is to use this coarse scale observational measurements to fine tune our knowledge of the state and/or model to improve the accuracy of the forecasts [15].

Classically, data assimilation techniques are based on linear quadratic estimation, also known as the Kalman Filter. The Kalman Filter has the drawback of assuming that the underlying system and any corresponding observation models are linear. It also assumes that probability distribution of the measurement noise is Gaussian. For nonlinear models, this has been mitigated by practitioners via modifications, such as the Ensemble Kalman Filter (EnKF), Extended Kalman Filter (EKF) and the Unscented Kalman Filter [4, 13, 33, 45]. However, unlike the Kalman filter, these do not enjoy the optimality property and has other drawbacks, particularly for infinite dimensional chaotic dynamical systems governed by PDE's [36].

An alternative approach to data assimilation called *nudging*, which is often cheaper to implement, employs a feedback control paradigm via a *Newtonian relaxation scheme*. This is motivated by the existence of finite determining functionals (modes, nodes, volume elements) for dissipative systems [27, 29], and has recently been advocated in the context of dissipative partial differential equations (PDE) by Azoiani, Olson and Titi [6]. Although it has its antecedents, mainly in the context of finite-dimensional dynamical systems governed by ordinary differential equations and early work in meteorology [5, 12, 44], the rigorous analysis of this method in fluid dynamics, clarifying the central role of the observation/interpolation operators as well as the bounds on the system variables on the attractor, which in turn determines the requisite spatial resolution of the observed data, was first carried out in [6]. Due to this, we will henceforth refer to the system associated to the nudging algorithm as the *AOT system*. We provide below a schematic description of the AOT system based on the work of Azouani, Olson and Titi [6].

Assume that the observations are generated from a continuous dynamical system given by

$$\frac{d}{dt}u = F(u), u(0) = u_0.$$

The associated AOT system, based on the work of Azouani, Olson and Titi [6], is given by

$$\frac{d}{dt}w = F(w) - \mu I_h(w - u), w(0) = w_0 \text{ (arbitrary)}, \quad (1.1)$$

where I_h is a finite rank linear operator acting on the phase space, called *interpolant operator*, constructed *solely from observations* on u (e.g. low (Fourier) modes of u or values (or local averages) of u measured in a coarse spatial grid). Here h refers to the size of the spatial grid or, in case of the *modal interpolant*, the reciprocal of h stands for the number of observed modes. Thus a smaller h corresponds to an observation space with a larger dimension, i.e. it corresponds to a richer observation space. Moreover, $\mu > 0$ is the *relaxation/nudging parameter* an appropriate choice of which needs to be made for the algorithm to work, i.e. the system to be *globally (in time) well-posed and for its solution to possess the asymptotic tracking property*, namely, in a suitable norm,

$$\|w - u\| \longrightarrow 0 \text{ as } t \rightarrow \infty.$$

It turns out [6] that larger bounds on the system variables necessitates a smaller h , i.e. more observations.

Although first analyzed for the two-dimensional Navier-Stokes equations (2D NSE), the AOT system was later extended to include various other dissipative systems [1, 2, 21, 34], and to more general settings such as discrete-in-time and error-contaminated measurements and recovery of statistical solutions [8, 26]. This method has been shown to perform remarkably well in numerical simulations [3, 32] and has recently been successfully implemented for efficient dynamical downscaling of a global atmospheric circulation model [16]. Moreover, it was observed in [37] that a flexible variant of it implemented via deep learning neural networks is cheaper to implement (online) and has a superior performance compared to the commonly used EKF and EnKF in case of sparse observations. Recent applications also include its implementation in *reduced order modeling (ROM)* of turbulent flows to mitigate inaccuracies in ROM [46] and in estimating unknown flow parameters and turbulence configurations [10, 35].

Here, we consider two main problems: (i) global (in time) well-posedness of the three-dimensional AOT system corresponding to the three-dimensional Boussinesq system where the (velocity) observations are obtained from a Leray-Hopf weak solution of the three-dimensional Boussinesq system (ii) the asymptotic tracking property of the corresponding solution. The AOT system for the Boussinesq equations in the 2D case was addressed in [20], while the 3D case with *large, or infinite Prandtl* number or flow in porous media were addressed in [19, 21]. In all the cases mentioned before (with the exception of our recent work in [11]) where rigorous analysis of the AOT system is available [1, 2, 6, 19–21, 34], one crucially uses the fact that the associated dissipative systems are well-posed and regular. Perhaps more importantly, in each of these cases, uniform-in-time bound in a higher Sobolev norm (e.g. the \mathbb{H}^1 -norm) for the state variable(s) is available. The knowledge of such higher order Sobolev norm bounds (e.g. in terms of physical parameters such as the Grashoff for the NSE in [6] and Prandtl and Rayleigh for the Boussinesq in [19]) plays a crucial role in the analysis of the AOT system. More precisely, *these bounds are used in providing an upper bound on the spatial resolution h of the observations necessary for the AOT to be well-posed and globally stable and to possess the asymptotic tracking property*. Additionally, the value of the nudging parameter μ guaranteeing these properties also explicitly depends

on this uniform bound. However, Such higher order norm bounds are not available for a Leray-Hopf weak solution of the 3D Boussinesq system which might even be non-unique. In fact, the corresponding uniqueness problem for a Leray-Hopf weak solution for the 3D NSE or the Boussinesq system is still open.

In this work, we identify a condition on the *observed (velocity) data, u , which does not depend on any *a priori* assumption of regularity of the velocity, or the knowledge of a uniform \mathbb{H}^1 norm bound $M = \sup_{[0, T)} \|u\|_{\mathbb{H}^1}$, $T \leq \infty$, even if it is finite. A precise description of this condition on the observations is given in (1.7) below. We show that our condition on the observed data allows us to appropriately set the value of μ in (1.1), *based only on the quantities computed from the observed data*, and: (i) prove wellposedness of the AOT system (we note that unlike in the 2D case where one uses the 2D embedding inequalities, the global wellposedness of the AOT system crucially depends on our choice of μ which in turn depends on condition (1.7) that we identified *based solely on the observed data*) (ii) establish appropriate uniform bound on the state variables of (1.1) (iii) show that the solution of (1.1) tracks asymptotically the solution of the three-dimensional Boussinesq system, without any *a priori* assumption on the regularity of the solution of the Boussinesq system or any knowledge of a bound $M = \sup \|u\|_{\mathbb{H}^1}$ even if it is finite. Note that even if one knows that the Leray-Hopf weak solution under consideration is regular on an interval $[0, T)$, unlike the 2D case, no bound on it in terms of the system parameters, for instance the Raleigh and Prandtl numbers, is available. We emphasize *that our result applies quite generally to arbitrary Prandtl and Rayleigh numbers*. We also show that if $M = \sup_{[0, T)} \|u\|_{\mathbb{H}^1} < \infty$ for some $T \leq \infty$, then (1.7) holds. Thus, this can be viewed as a generalization to the 3D case of the results in [20].*

Quite remarkably, it turns out that condition (1.7), which guarantees the wellposedness and asymptotic tracking property for the AOT system, yields a regularity criterion. Since it involves only the observed part of the fluid velocity, we call it an *observable regularity criterion* which is different from all other regularity criterion known so far (e.g. the well-known Prodi-Serrin or the Beale-Kato-Majda regularity criteria [7, 40]). Those conditions involve the knowledge of the entire solution u , while ours is expressed in terms of finitely many observations on u , or equivalently, from information gleaned from a finite rank projection on the state space applied to u .

Finally, extending our earlier work in [9], we establish the existence of a *determining map* for the three-dimensional case. The determining map was first introduced for the 2D NSE in [22, 23] and its properties were studied in detail in [9]. It plays a pivotal role in the construction of the *determining form*, an ODE in an adequate trajectory space associated to a dissipative system. The dynamics of the determining form, is closely related to the dynamics of the dissipative system to which it is associated [24]. Recently, the determining map has also been used for data assimilation of statistical solutions [9] and in estimating unknown parameters (such as viscosity in the NSE) in a dissipative system [10]. We make crucial use of it in our work in the proof of the observable regularity criterion as well as in obtaining a result on the existence of determining modes and volume elements for the 3D Boussinesq system (Theorem 3.3). A more precise description of our main results now follows.

1.1 The 3D Boussinesq System

The Boussinesq system in thermohydraulics is commonly used to model thermal convection or the phenomenon of heat transfer by the motion of an incompressible, Newtonian fluid. It occurs in diverse fields such mantle convection to atmospheric motion [38]. The Bénard convection problem is a model of the Boussinesq convection system of an incompressible fluid layer, confined between two solid walls, which is heated from below in such a way that the lower wall maintains a temperature T_0 , while the upper one maintains a temperature $T_1 < T_0$. In this case, after some change of variables and proper scaling (by normalizing the distance between the walls and the temperature difference), the three-dimensional Boussinesq equations that govern the perturbation of the velocity(u) and temperature about the pure conduction steady state(θ) are

$$\frac{du}{dt} + \nu \Delta u + (u \cdot \nabla)u = \theta e_3 \quad (1.2)$$

$$\frac{d\theta}{dt} + \kappa \Delta \theta + (u \cdot \nabla)\theta - u \cdot e_3 = 0 \quad (1.3)$$

$$\nabla \cdot u = 0 \quad (1.4)$$

$$u(0, x) = u_0, \quad \theta(0, x) = \theta_0 \quad (1.5)$$

For boundary conditions, in the x_3 direction we have

$$u, \theta = 0, \text{ at } x_3 = 0 \text{ and } x_3 = 1$$

and in the x_1 and x_2 directions, for simplicity, we have the periodic boundary condition

$$u, \theta \text{ are periodic, of period } L \text{ in the } x_1 \text{ and } x_2 \text{ directions.}$$

Here, $x = (x_1, x_2, x_3)$ is a point in the domain, $u(t; x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the fluid velocity and $\theta = \theta(t, x)$ is the scaled fluctuation of the temperature around the pure scaled conduction steady-state background temperature profile $1 - x_3$. It is given by $\theta = T - (1 - x_3)$, where $T = T(t, x)$ is the scaled temperature of the fluid inside the domain Ω . κ and ν are the thermal diffusivity and kinematic viscosity, respectively. For more details on its derivation and physical interpretation can be found in [17].

1.2 A Sufficient Condition for Well-Posedness and the Asymptotic Tracking Property

For (u, θ) , the solution to the Boussinesq system given in (2.8)-(2.10), the observations on the velocity component u are used to define the quantity

$$M_{h,u}^2 = 32 \sup_{0 \leq t < T} \begin{cases} \|P_N(u)\|_{\mathbb{H}^1}^2 \sim \sum_{|k| \leq N} |\lambda_k|^2 |\hat{u}(k)|^2, \quad N \sim \frac{1}{h} & \text{(Modal Interpolant)} \\ Ch \sum_{\alpha} |\bar{u}_{\alpha}|^2, \quad \bar{u}_{\alpha} = \frac{1}{|Q_{\alpha}|} \int_{Q_{\alpha}} u & \text{(Volume Elements)} \end{cases} \quad (1.6)$$

where, $0 < T \leq \infty$ and λ_k are eigenvalues of the Stokes operator such that $\lambda_1 \leq \lambda_2 \leq \dots$ with P_N being the associated spectral projections. Also, in the above definition, $\{Q_{\alpha}\}$ denotes partition of the domain into finitely many cubes of side length h and the corresponding finite set $\{u_{\alpha}\}$ are called *volume element observations*. Due to the fact that any Leray-Hopf weak solution satisfies $\sup_{[0,\infty)} \|u\|_{L^2} < \infty$, we can show (see Remark 2.6 for a proof) that M_h thus defined is a *finite quantity*. Here, h_0 depends explicitly on the physical parameters such as viscosity and thermal diffusivity, i.e. the Raleigh and Prandtl numbers. If there exists $0 < h \leq h_0$ such that

$$h^2 M_{h,u}^4 \leq cv^4, \quad (v \text{ is the fluid viscosity and } c \text{ an adequate non-dimensional constant.}) \quad (1.7)$$

then a choice of μ exists such that the data assimilated solution (w, η) is *regular* (given in Definition 1) and converges exponentially to the actual solution (u, θ) . We also establish uniform in time bounds on $\|\eta\|_{L^{2p}(\Omega)}$ for all natural numbers p .

1.3 Connection to Regularity

A solution (u, θ) of the Boussinesq system is said to be regular on $[0, T)$, $T \leq \infty$ if $\sup_{[0,T)} \|u\|_{\mathbb{H}^1} < \infty$. Due to the well-known Sather-Serrin weak-strong uniqueness theorem [41], such a solution is unique in the class of weak solutions. It is the case that $M_{h,u}$ as defined in (1.6) satisfies

$$M_{h,u} \lesssim \sup_{[0,T)} \|u\|. \quad (1.8)$$

For the modal interpolant, this immediately follows from the inequality $\|P_N u\|_{\mathbb{H}^1} \lesssim \|u\|_{\mathbb{H}^1}$, while for the volume elements it follows from a similar estimate proven in Corollary 6.3. It then immediately follows that if $\sup_{[0,T)} \|u\|_{\mathbb{H}^1} < \infty$, $T \leq \infty$, then there exists an h satisfying (1.7); how small h needs to be depends on the bound $\sup_{[0,T)} \|u\|_{\mathbb{H}^1}$.

For two-dimensional systems such as the 2D NSE and 2D Boussinesq, it is well-known that $\sup_{[0,\infty)} \|u\|_{\mathbb{H}^1}$ is finite and can be estimated in terms of the system parameters such as the Grashoff for the NSE and Raleigh and Prandtl for the Boussinesq. Therefore, h , and consequently μ in (1.1), can be determined *a priori* in terms of the system parameters. In our case, the bound $\sup_{[0,T)} \|u\|_{\mathbb{H}^1}$ is not known as we do not observe the full solution. In fact, at this stage, it is *a priori* possible that

$\sup_{[0,T)} \|u\|_{\mathbb{H}^1} = \infty$. In other words, $\sup_{[0,T)} \|u\|_{\mathbb{H}^1} < \infty$ is only a sufficient condition for (1.7) to hold; but see next subsection. However, note that (1.7) depends only on finitely many observational data. In other words, (1.7) is a condition imposed on a finite rank projection of u in the phase space. Consequently, we can determine h and μ from observational data, without any reference to the unobserved part of the true solution. Thus our result can be viewed as a generalization of analogous results for wellposed systems.

1.4 A New Observable Regularity Criterion

It turns out that (1.7) is also a regularity condition, but on the weak attractor. As can be seen from (1.2), in the Boussinesq system, when there is no spatial temperature variation (for instance in the absence of a heating source, when there is no temperature difference between the top and bottom plates), or if the temperature variation is taken as given, then the Boussinesq system reduces to the NSE. For simplicity, and due to the fact that the existence and properties of the weak attractor for the 3D NSE (see Sect. 4 for details) has been established and studied in [28, 30], we confine ourselves to the case of the 3D NSE. However, the same result would also hold for the 3D Boussinesq system.

To describe our result for the 3D NSE on the weak attractor, let $h_0 > 0$ be defined as

$$h_0^{-2} = \max \left\{ \frac{1}{4c\lambda_1}, \frac{32c|f|^4}{v^8\lambda_1^2} \right\},$$

where f is the body force, ν is the viscosity and λ_1 is the smallest eigenvalue of the Stokes operator. Let $u(t)$, $t \in \mathbb{R}$ be a Leray-Hopf weak solution of the 3D Navier-Stokes on the weak global attractor \mathcal{A} . Let M_h be defined as in (1.6), except that the supremum is taken on the interval $(-\infty, T]$. Assume there exists $0 < h \leq h_0$ for which $h^2 M_h^4 \leq cv^4$ where c is an adequate, non-dimensional constant. Then $u(t)$ is regular on $(-\infty, T]$. We refer to this as *an observable regularity criterion* because it is purely based on the finitely many observations (modes or volume elements). This regularity criterion is completely different from other known ones for the NSE such as in [7, 40] which depend on control of all length scales of u . This leads us to the following open question:

Can one formulate an analogous regularity criterion based on finitely many nodal observations, that is, based on information on $\{u(x_i)\}_{i=1}^N$ where $\{x_i\}_{i=1}^N$ is an adequately chosen, finite set of points in the domain?

The organization of the paper is as follows. In Sect. 2, we discuss the well posedness and the tracking property of the data assimilated 3D Boussinesq system. In Sect. 3 and Sect. 4, we establish the properties of the forward determining map for the 3D Boussinesq system and the determining map for the 3D Navier-Stokes system respectively. In Sect. 5, we present our observable regularity criterion for the 3D NSE on the weak attractor. In Sect. 6, the Appendix, we prove some facts, mostly concerning

interpolant operators, that we need. To the best of our knowledge, these facts are new and may have independent interest.

2 Existence of Strong Solution and the Asymptotic Tracking Property

2.1 Notation and Preliminaries

Following [14, 42], here we briefly introduce the functional setting for (1.2)-(1.5). For $\alpha > 0$, $H^\alpha(\Omega)$ is the usual Sobolev space. We denote the inner product and norm of $L^2(\Omega)$ by (\cdot, \cdot) and $\|\cdot\|$ respectively and the inner product and norm of $H^1(\Omega)$ by $((\cdot, \cdot))$ and $\|(\cdot, \cdot)\|$ respectively. We define \mathcal{F} to be the set of $C^\infty(\Omega)$ functions defined in Ω , which are trigonometric polynomials in x_1 and x_2 with period L , and compactly supported in the x_3 -direction. We denote the space of vector valued functions on Ω that incorporates the divergence free condition by $\mathcal{V} = \{\phi \in \mathcal{F} \times \mathcal{F} \times \mathcal{F} | \nabla \cdot \phi = 0\}$. H_0 and H_1 are closures of \mathcal{V} and \mathcal{F} in $L^2(\Omega)$ respectively and V_0 and V_1 are closures of \mathcal{V} and \mathcal{F} in $H^1(\Omega)$ respectively.

H_0 and H_1 are endowed with the inner products

$$(u, v)_0 = \sum_{i=1}^3 \int_{\Omega} u_i(x) v_i(x) dx$$

and

$$(\phi, \psi)_1 = \int_{\Omega} \phi(x) \psi(x) dx$$

respectively, and the norms $\|u\|_0 = (u, u)_0^{1/2}$ and $\|\phi\|_1 = (\phi, \phi)_1^{1/2}$ respectively. V_0 and V_1 are endowed with the inner products

$$((u, v))_0 = \sum_{i,j=1}^3 \int_{\Omega} \partial_j u_i(x) \partial_j v_i(x) dx,$$

$$((\phi, \psi))_1 = \sum_{j=1}^3 \int_{\Omega} \partial_j \phi(x) \partial_j \psi(x) dx$$

respectively, and the associated norms $\|u\|_0 = ((u, u))_0^{1/2}$ and $\|\phi\|_1 = ((\phi, \phi))_1^{1/2}$ respectively. We also denote by P_σ the Leray-Hopf orthogonal projection operator from $L^2(\Omega)$ to H_0 .

Let $D(A_0) = V_0 \cap (H^2(\Omega))^3$, $D(A_1) = V_1 \cap H^2(\Omega)$ and $A_i : D(A_i) \rightarrow H_i$ be the unbounded linear operator defined by

$$(A_i u, v)_i = ((u, v))_i, \quad \text{for } i = 0, 1. \quad (2.1)$$

We recall that A_i , for $i = 0, 1$, is a positive self adjoint operator with a compact inverse. Moreover, there exists a complete orthonormal set of eigenfunctions $\phi_{j,i} \in H_i$, such that $A_i \phi_{j,i} = \lambda_{j,i} \phi_{j,i}$, where $0 < \lambda_{1,i} \leq \lambda_{2,i} \leq \lambda_{3,i} \leq \dots$ are the eigenvalues of A_i repeated according to multiplicity.

For $i = 0, 1$, we denote by H_n^i the space spanned by the first n eigenvectors of A_i and the orthogonal projection from H_i onto H_n^i is denoted by P_n^i . We also have the Poincare inequality

$$\lambda_1^{1/2} \|v\|_i \leq \|v\|_i, v \in V_i. \quad (2.2)$$

where $\lambda_1 = \min \{\lambda_{1,0}, \lambda_{1,1}\}$.

Let V'_i be the dual of V_i for $i = 0, 1$. We define the bilinear term $B_0 : V_0 \times V_0 \rightarrow V'_0$ by

$$\langle B_0(u, v), w \rangle_{V'_0, V_0} = (((u \cdot \nabla)v), w)_0$$

and $B_1 : V_0 \times V_1 \rightarrow V'_1$ by

$$\langle B_1(u, v), w \rangle_{V'_1, V_1} = (((u \cdot \nabla)v), w)_1.$$

The bilinear term B_i , for $i = 0, 1$, satisfies the orthogonality property

$$B_i(u, w, w) = 0 \quad \forall u \in V_0, w \in V_i. \quad (2.3)$$

We define the solution space $H = H_0 \times H_1$ equipped with the inner product

$$\langle s_1, s_2 \rangle = (u_1, u_2)_0 + (\theta_1, \theta_2)_1,$$

where $s_1 = (u_1, \theta_1)$ and $s_2 = (u_2, \theta_2)$.

We recall some well-known bounds on the bilinear term for velocity in the 3D case.

Proposition 2.1 *If $u, v \in V_0$ and $w \in H_0$, then*

$$|(B_0(u, v), w)_0| \leq c \|u\|_{L^6} \|\nabla v\|_{L^3} \|w\|_{L^2} \leq c \|u\|_0 \|v\|_0^{1/2} |Av|_0^{1/2} |w|_0 \quad (2.4)$$

Moreover if $u, v, w \in V_0$, then

$$|(B_0(u, v), w)_0| \leq c \|u\|_{L^4} \|\nabla v\|_{L^2} \|w\|_{L^4} \leq c |u|_0^{1/4} \|u\|_0^{3/4} \|v\|_0 |w|_0^{1/4} \|w\|_0^{3/4} \quad (2.5)$$

We also recall the Ladyzhenskaya's inequality for three dimensions :

$$\|w\|_{L^4} \leq C |w|_0^{\frac{1}{4}} \|w\|_0^{\frac{3}{4}} \quad (2.6)$$

Remark 2.1 The subscripts 0 and 1 in this work are used to denote norms, spaces and operators associated with the fluid velocity vector u and the scalar temperature field θ respectively. When used as subscripts for norms, they are not to be confused with the L^2 or the Sobolev H^1 norms.

Remark 2.2 From [25], we know that the weak solution is uniformly bounded in time in $H = H_0 \times H_1$. Hence, for u and θ as in (1.2)-(1.3), there exists $M_0(u_0, \theta_0)$, $M_1(u_0, \theta_0) \in \mathbb{R}$ such that

$$|u|_0 \leq M_0, \text{ and } |\theta|_1 \leq M_1. \quad (2.7)$$

We denote by P_σ the Leray-Hopf orthogonal projection operator from $L^2(\Omega)$ to H_0 . With the above notation, by applying P_σ to (1.2), we may express the 3-D Boussinesq equation in the following functional form:

$$\frac{du}{dt} + \nu A_0(u) + B_0(u, u) = P_\sigma(\theta e_3) \quad (2.8)$$

$$\frac{d\theta}{dt} + \kappa A_1(\theta) + B_1(u, \theta) - u \cdot e_3 = 0 \quad (2.9)$$

$$u(0, x) = u_0, \quad \theta(0, x) = \theta_0. \quad (2.10)$$

Henceforth, $a \lesssim b$ means $a \leq Cb$ and $a \gtrsim b$ means $a \geq Cb$. Also $a \sim b$ means both $a \lesssim b$ and $a \gtrsim b$ hold.

2.2 Interpolant Operators

A finite rank, bounded linear operator $I_h : L^2(\Omega) \rightarrow L^2(\Omega)$ is said to be a type-I interpolant operator if there exists a dimensionless constant $c > 0$ such that

$$|I_h(v)| \leq c|v| \quad \forall v \in L^2(\Omega) \text{ and } |I_h(v) - v| \leq ch\|v\| \quad \forall v \in H^1(\Omega). \quad (2.11)$$

We look at two main examples of type-I interpolants.

- **Modal interpolation:** In this case $I_h u = P_K^0(u)$ with $h \sim 1/\lambda_K^{1/2}$, where P_K^0 denotes the orthogonal projection onto the space spanned by the first K eigenvectors of the Stokes operator A_0 . Indeed, one can easily check that it satisfies (2.11):

$$|P_K(v)| \leq |v| \quad \forall v \in L^2(\Omega) \quad \text{and} \quad |P_K(v) - v| \lesssim \frac{1}{\lambda_K^{1/2}} \|v\| \quad \forall v \in H^1(\Omega). \quad (2.12)$$

where $\lambda_K = \min\{\lambda_K^0, \lambda_K^1\}$.

- **Volume interpolation:** In this case, Ω is partitioned into N smaller cuboids Q_α , where $\alpha \in \mathcal{J} = \{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 1 \leq j, k, l \leq \sqrt[3]{N}\}$. Each cuboid is of

diameter $h = \sqrt{2L^2 + 1}/\sqrt[3]{N}$. The interpolation operator is defined as follows:

$$I_h(v) = \sum_{\alpha \in \mathcal{J}} \bar{v}_\alpha \chi_{Q_\alpha}(x), \quad v \in L^2(\Omega) \quad (2.13)$$

where

$$\bar{v}_\alpha = \frac{1}{|Q_\alpha|} \int_{Q_\alpha} v(x) dx, \quad (2.14)$$

and $|Q_\alpha|$ is the volume of Q_α . For $v \in V_0$, we define $I_h(v) = (I_h(v_1), I_h(v_2), I_h(v_3))$, where $v = (v_1, v_2, v_3)$.

2.3 Well-Posedness

Definition 1 (u, θ) is said to be a weak solution to (2.8)-(2.10) if for all $T > 0$,

- $u \in L^\infty(0, T; H_0) \cap L^2(0, T; V_0)$ and $\theta \in L^\infty(0, T; H_1) \cap L^2(0, T; V_1)$
- u, θ satisfy, $\forall v \in V_0$ and $\forall \zeta \in V_1$, a.e.t

$$\begin{aligned} \frac{d}{dt} (u, v)_0 + v((u, v))_0 + (B_0(u, u), v) &= (\theta e_3, v)_0 \\ \frac{d}{dt} (\theta, \zeta)_1 + \kappa((\theta, \zeta))_1 + (B_1(u, \theta), \zeta)_1 - (u \cdot e_3, \zeta)_1 &= 0. \end{aligned} \quad (2.15)$$

A Leray-Hopf weak solution additionally satisfies, a.e. s , and for all $t \geq s$, the energy inequality

$$|u(t)|^2 + \int_s^t v \|u(\sigma)\|^2 d\sigma \leq |u(s)|^2 + \int_s^t (\theta e_3(\sigma), u(\sigma)) d\sigma. \quad (2.16)$$

$$|\theta(t)|^2 + \int_s^t \kappa \|\theta(\sigma)\|^2 d\sigma \leq |\theta(s)|^2 + \int_s^t (u \cdot e_3(\sigma), \theta(\sigma)) d\sigma. \quad (2.17)$$

A weak solution is said to be a strong/regular solution if it also belongs to

$$\left[L^\infty(0, T; V) \cap L^2(0, T; D(A)) \right] \times \left[L^\infty(0, T; V_1) \cap L^2(0, T; D(A_1)) \right].$$

Remark 2.3 Similar to the case of 3D Navier-Stokes, we have existence but not uniqueness of weak solution [43]. Also, given initial data in $V_0 \times V_1$, a unique strong solution exists for some $[0, T]$ and a strong solution is unique in the larger class of Leray-Hopf weak solutions.

We will now be proving estimates and results for the weak solution to the data assimilated 3-D NSE given by (2.18)-(2.20). This will be done by first working with the Galerkin approximation of the system given by (2.21)-(2.23) and then using density arguments to obtain results for the original system.

2.4 Existence of Weak Solution

In this section, we prove the existence of a weak solution to the data assimilated Boussinesq equation. Our data assimilated algorithm is given by the solution (w, η) of

$$\frac{dw}{dt} + \nu A_0(w) + B_0(w, w) = P_\sigma(\eta e_3) + \mu(P_\sigma(I_h(u) - I_h(w))) \quad (2.18)$$

$$\frac{d\eta}{dt} + \kappa A_1(\eta) + B_1(w, \eta) - w \cdot e_3 = 0 \quad (2.19)$$

$$w(x, 0) = 0, \eta(x, 0) = 0. \quad (2.20)$$

The Galerkin approximation (w_n, η_n) of (w, η) satisfies

$$\frac{dw_n}{dt} + \nu A_0(w_n) + P_n B_0(w_n, w_n) = (\eta_n e_3) + \mu(P_n(I_h(u) - I_h(w_n))) \quad (2.21)$$

$$\frac{d\eta_n}{dt} + \kappa A_1(\eta_n) + P_n B_1(w_n, \eta_n) - w_n \cdot e_3 = 0 \quad (2.22)$$

$$w_n(x, 0) = 0, \eta_n(x, 0) = 0 \quad (2.23)$$

where, by abuse of notation, P_n denotes the projection onto the space spanned by the first n eigenvectors of A_i for $i = 0, 1$.

Remark 2.4 The initial condition for the data assimilation algorithm is arbitrarily chosen to be zero. The idea is to show that the solution (w, η) of the data assimilation algorithm asymptotically converges to the solution (u, θ) of the original system. Hence, regardless of the choice of the initial data, we will be able to approximate the actual solution with a prescribed precision as long as we let the algorithm run for a sufficiently long time. As shown in Theorem 2.9, the run time however depends on how close the initial guess of the algorithm is to the actual initial condition.

Theorem 2.2 Let (u, θ) be a Leray-Hopf weak solution to (2.8)–(2.10) for all $t \geq 0$ and I_h be any type 1 interpolant. Let $h_0 > 0$ be such that

$$h_0^2 = \frac{\nu \kappa \lambda_1}{16c}.$$

Then, provided $h \leq h_0$ and μ is chosen satisfying

$$\frac{\nu}{2ch_0^2} \leq \mu \leq \frac{\nu}{2ch^2} \left(\frac{1}{\lambda_1} \sim L^2 \right), \quad (2.24)$$

there exists a weak solution (w, η) of (2.18) such that for any $T > 0$,

$$w \in L^\infty(0, T; H_0) \cap L^2(0, T; V_0), \text{ and } \eta \in L^\infty(0, T; H_1) \cap L^2(0, T; V_1).$$

Proof Taking the inner product of (2.21) with w_n and (2.22) with η_n and adding, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(|w_n|_0^2 + |\eta_n|_1^2 \right) + \nu \|w_n\|_0^2 + \kappa \|\eta_n\|_1^2 \\ & \leq 2|\eta_n|_1 |w_n|_0 + \mu (I_h(u) - I_h(w_n), w_n) \\ & \leq 2|\eta_n|_1 |w_n|_0 + \mu (w_n - I_h(w_n), w_n) \\ & \quad - \mu |w_n|_0^2 + \mu (I_h(u), w_n) \end{aligned} \quad (2.25)$$

We bound each of the terms on the RHS below.

First, using Young's inequality, (2.24) and (2.2), we have

$$2|\eta_n|_1 |w_n|_0 \leq \frac{2|w_n|_0^2}{\kappa \lambda_1} + \frac{\kappa \lambda_1 |\eta_n|_1^2}{2} \leq \frac{\mu |w_n|_0^2}{4} + \frac{\kappa \|\eta_n\|_1^2}{2}. \quad (2.26)$$

Next, using (2.11), Cauchy-Schwartz, Young's inequality and the second inequality in (2.24), we obtain

$$\begin{aligned} & \mu |(w_n - I_h(w_n), w_n)|_0 \\ & \leq \mu |w_n - I_h(w_n)|_0 |w_n|_0 \leq \mu ch^2 \|w_n\|_0^2 + \frac{\mu}{4} |w_n|_0^2 \\ & \leq \frac{\nu}{2} \|w_n\|_0^2 + \frac{\mu}{4} |w_n|_0^2, \end{aligned} \quad (2.27)$$

Lastly, using Cauchy-Schwartz and Young's inequality, we have

$$\mu |(I_h(u), w_n)|_0 \leq \mu |I_h(u)|_0^2 + \frac{\mu}{4} |w_n|_0^2. \quad (2.28)$$

Combining all the estimates, we obtain

$$\frac{d}{dt} \left(|w_n|_0^2 + |\eta_n|_1^2 \right) + \nu \|w_n\|_0^2 + \kappa \|\eta_n\|_1^2 + \frac{\mu}{2} |w_n|_0^2 \leq 2\mu |I_h(u)|_0^2. \quad (2.29)$$

Splitting the above inequality and using (2.2), we have

$$\frac{d}{dt} |w_n|_0^2 + \frac{\mu}{2} |w_n|_0^2 \leq 2\mu |I_h(u)|_0^2 \quad \text{and} \quad \frac{d}{dt} |\eta_n|_1^2 + \kappa \lambda_1 |\eta_n|_1^2 \leq 2\mu |I_h(u)|_0^2.$$

Applying Gronwall, (2.7), (2.11) and (2.23), we obtain

$$|w_n|_0^2 \leq 4M_0^2, \quad \text{and} \quad |\eta_n|_1^2 \leq \frac{2\mu M_0^2}{\kappa \lambda_1}. \quad (2.30)$$

Dropping the first and the last term on LHS of (2.29) and integrating on the interval $[s, s + 1]$, we obtain

$$\int_s^{s+1} \left(\nu \|w_n\|_0^2 + \kappa \|\eta_n\|_1^2 \right) \leq 2\mu M_0^2. \quad (2.31)$$

The remainder of the proof is similar to the proof of existence of weak solutions of the 3D NSE ([42], Theorem 1.2, page 164). \square

Remark 2.5 The solution of the AOT system (1.1) that we have obtained is also unique. Later, from Theorem 3.1, we obtain uniqueness of solution on the time interval $[0, \infty)$. The same argument can be used to show uniqueness here on the time interval $[0, T]$.

2.5 Time Independent Bound on Data Assimilated Temperature

The following theorem will establish a time independent bound on $\|\eta\|_{L^{2p}}$, which will be a stepping stone in proving convergence of the data assimilated solution to the actual solution.

Theorem 2.3 *Let $p \in \mathbb{N}$ and η and η_n be as in (2.19) and (2.23) respectively. Assume that the hypotheses of Theorem 2.2 hold. Then $\|\eta\|_{L^{2p}}$ is uniformly bounded in time and satisfies*

$$\|\eta\|_{L^{2p}} \leq \frac{CpM_0}{((2p-1)\lambda_1)^{\frac{1}{2p}}} := S_{p,u} = S_p, \quad (2.32)$$

where M_0 is as in (2.7).

Proof Taking the inner product of (2.22) with η_n^{2p-1} , we obtain

$$\left(\frac{d\eta_n}{dt}, \eta_n^{2p-1} \right)_1 - \kappa \left(\Delta \eta_n, \eta_n^{2p-1} \right)_1 + \left((w_n \cdot \nabla) \eta_n, \eta_n^{2p-1} \right)_1 = (w_n \cdot e_3, \eta_n^{2p-1})_1 \quad (2.33)$$

We estimate each term below.

$$\begin{aligned} \frac{1}{2p} \frac{d\eta_n^{2p}}{dt} &= \eta_n^{2p-1} \frac{d\eta_n}{dt} \\ \Rightarrow \left(\frac{d\eta_n}{dt}, \eta_n^{2p-1} \right)_1 &= \frac{1}{2p} \frac{d}{dt} \int_{\Omega} \eta_n^{2p} dx = \frac{1}{2p} \frac{d}{dt} \|\eta_n\|_{L^{2p}}^{2p} \end{aligned} \quad (2.34)$$

Integrating by part, we obtain

$$\begin{aligned} -\kappa(\Delta \eta_n, \eta_n^{2p-1})_1 &= (2p-1)\kappa \left(\nabla \eta_n, \eta_n^{2p-2} \nabla \eta_n \right)_1 \\ &= (2p-1)\kappa \left(\eta_n^{p-1} \nabla \eta, \eta_n^{p-1} \nabla \eta_n \right)_1 \\ &= \frac{2p-1}{p^2} |\nabla (\eta_n^p)|_1^2 \end{aligned} \quad (2.35)$$

$$\begin{aligned} \left((w_n \cdot \nabla) \eta_n, \eta_n^{2p-1} \right)_1 &= - \left((w_n \cdot \nabla) \eta_n^{2p-1}, \eta_n \right)_1 \\ &= -(2p-1) \left(\eta_n^{2p-2} (w_n \cdot \nabla) \eta_n, \eta_n \right)_1 \\ &= -\frac{2p-1}{p^2} \left((w_n \cdot \nabla \eta_n), \eta_n^{2p-1} \right)_1 \end{aligned} \quad (2.36)$$

Therefore, $\left((w \cdot \nabla \eta_n, \eta_n^{2p-1}) \right)_1 = 0$.

From Holder's inequality, we have

$$|(w_n \cdot e_3, \eta_n^{2p-1})_1| \leq \|w_n\|_{L^{q_1}} \cdot \left\| (\eta_n^p)^{2-\frac{1}{p}} \right\|_{L^{q_2}} \quad (2.37)$$

where $q_1 = \frac{6p}{4p+1}$ and $q_2 = \frac{6p}{2p-1}$. Note that $q_1 \leq 2 \forall p \in \mathbb{N}$ and since Ω is a bounded domain of finite measure, we obtain

$$\begin{aligned} |(w_n \cdot e_3, \eta_n^{2p-1})_1| &\leq C|w_n|_0 \cdot \left\| (\eta_n^p)^{2-\frac{1}{p}} \right\|_{L^{q_2}} \leq C|w_n|_0 \cdot \left(\int_{\Omega} (\eta_n^p)^6 \right)^{1/q_2} \\ &\leq C|w_n|_0 \cdot \left(\|\eta_n^p\|_{L^6} \right)^{\frac{2p-1}{p}} \end{aligned} \quad (2.38)$$

Applying the Sobolev embedding $L^6(\Omega) \hookrightarrow H^1$, Young's inequality and (2.30), we obtain

$$\begin{aligned} |(w_n \cdot e_3, \eta_n^{2p-1})_1| &\leq C|w_n|_0 \cdot \left(|\nabla (\eta_n^p)|_1 \right)^{\frac{2p-1}{p}} \\ &\leq Cp^{2p-2} |w_n|_0^{2p} + \left(\frac{2p-1}{2p^2} \right) |\nabla (\eta_n^p)|_1^2 \\ &\leq Cp^{2p-2} M_0^{2p} + \left(\frac{2p-1}{2p^2} \right) |\nabla (\eta_n^p)|_1^2, \end{aligned} \quad (2.39)$$

where M_0 is as in (2.7). Combining all the estimates we obtain

$$\frac{1}{2p} \frac{d}{dt} \|\eta_n\|_{L^{2p}}^{2p} + \left(\frac{2p-1}{2p^2} \right) |\nabla (\eta_n^p)|_1^2 \leq Cp^{2p-2} M_0^{2p} \quad (2.40)$$

Substituting $\zeta_n = \eta_n^p$, we obtain

$$\frac{1}{2p} \frac{d}{dt} |\zeta_n|_1^2 + \left(\frac{2p-1}{2p^2} \right) |\nabla \zeta_n|_1^2 \leq Cp^{2p-2} M_0^{2p} \quad (2.41)$$

Since $\zeta_n = \eta_n^p$, it shares the same boundary conditions as η_n , and hence (2.2) is applicable to the above equation, giving us

$$\frac{d}{dt} |\zeta_n|_1^2 + \left(\frac{(2p-1)\lambda_1}{p} \right) |\zeta_n|_1^2 \leq Cp^{2p-1} M_0^{2p} \quad (2.42)$$

Now, applying Gronwall, we obtain

$$\|\eta_n\|_{L^{2p}}^{2p} = |\zeta_n|_1^2 \leq C \left(\frac{p^{2p}}{(2p-1)\lambda_1} \right) M_0^{2p} \quad (2.43)$$

Therefore, we have

$$\|\eta_n\|_{L^{2p}} \leq \frac{CpM_0}{((2p-1)\lambda_1)^{\frac{1}{2p}}} := S_{p,u} = S_p \quad (2.44)$$

Hence η_n is a bounded sequence in $L^\infty(0, T; L^{2p}(\Omega))$ and there exists a subsequence η_{n_k} and η^* in $L^\infty(0, T; L^{2p}(\Omega))$ such that η_{n_k} converges to η^* in $L^\infty(0, T; L^{2p}(\Omega))$ in the weak star topology. We also know that η_n , and hence $\eta_{n,k}$, converges to η in $L^\infty(0, T; L^2(\Omega))$ in the weak star topology. Now since $p \geq 1$ and Ω is bounded, $L^{2p}(\Omega) \subset L^2(\Omega)$ and $L^2(\Omega) \subset L^q(\Omega)$, where $2p$ and q are Holder conjugates. Hence

$$L^\infty(0, T; L^{2p}(\Omega)) \text{ is a subspace of } L^\infty(0, T; L^2(\Omega)), \quad (2.45)$$

resulting in the sequence η_{n_k} to have two weak star limits in $L^1(0, T; L^2(\Omega))$. therefore we must have $\eta^* = \eta$ and

$$\|\eta\|_{L^{2p}} \leq \frac{CpM_0}{((2p-1)\lambda_1)^{\frac{1}{2p}}} := S_{p,u} = S_p$$

□

2.6 Global Existence of Strong Solution

Another key result we will need in order to show convergence of the data assimilated solution to the actual solution is the regularity of the data assimilated velocity w . In order to do so, we will have to impose conditions on our data and look at the term $\|I_h(u)\|_0$. For the case of modal interpolation, in addition to satisfying (2.11), I_h also satisfies

$$\|I_h(v)\| \leq \|P_N(u)\| \leq c\|v\| \quad \forall v \in H^1(\Omega), \quad N \sim \frac{1}{h} \quad (2.46)$$

The piece-wise constant volume interpolant as defined in (2.13) does not satisfy (2.46) due to the lack of regularity of the characteristic function. In order to establish an inequality similar to (2.46) and also an inequality explicitly in terms of the data for volume interpolation, we define a smoothed volume interpolant operator \tilde{I}_h that satisfies

$$\|\tilde{I}_h(v)\|_0^2 \leq Ch \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq C\|v\|_0^2 \quad \forall v \in V_0, \quad (2.47)$$

where \bar{v}_α is as in (2.14). The proof of (2.47) and the justification as to why \tilde{I}_h is a type-I interpolant is provided in the appendix.

Hence, we can now define a modified type-I interpolant which encompasses the usual modal interpolant and the smoothed volume interpolant.

Next, for a general Leray-Hopf weak solution u of (2.8)-(2.10), we define the quantity $M_{h,u}(= M_{h,u,T})$ as

$$M_{h,u}^2 = 32 \sup_{0 \leq t \leq T} \begin{cases} \|P_N(u)\|^2 \sim \sum_{|k| \leq N} |\lambda_{k,0}|^2 |\hat{u}(k)|^2, \quad N \sim \frac{1}{h} \quad (\text{Modal}) \\ Ch \sum_{\alpha} |\bar{u}_\alpha|^2, \quad \bar{u}_\alpha = \frac{1}{|Q_\alpha|} \int_{Q_\alpha} u \quad (\text{Volume}) \end{cases} \quad (2.48)$$

where, $\lambda_{k,0}$ is the k^{th} smallest eigenvalue of A_0 corresponding to the eigenvector $\phi_{k,0}$ and $P_N(u) = \sum_{k=1}^N \hat{u}(k) \phi_{k,0}$.

Remark 2.6 Observe that $M_{h,u}$, as defined in (2.48), depends only on the data and is always finite. For the modal case, from (2.48), we may write

$$\|P_N(u)\|^2 \leq \lambda_N |u|_0 \leq \lambda_N M_0 \quad (\text{which is finite}). \quad (2.49)$$

In the volume interpolation case, from (2.13), we have

$$\begin{aligned} \sum_{\alpha \in \mathcal{J}} |\bar{u}_\alpha|^2 &\leq \sum_{\alpha \in \mathcal{J}} \left(\left| \frac{1}{|Q_\alpha|} \int_{Q_\alpha} u(x) dx \right| \right)^2 \\ &\leq \frac{N^2}{|\Omega|^2} \sum_{\alpha \in \mathcal{J}} \|u\|_{L^1(Q_\alpha)}^2 \leq \frac{N}{|\Omega|} \sum_{\alpha \in \mathcal{J}} \|u\|_{L^2(Q_\alpha)}^2 \\ &\leq \frac{N}{|\Omega|} |u|_0^2 \leq \frac{N}{|\Omega|} M_0^2 \quad (\text{which is finite}). \end{aligned} \quad (2.50)$$

Next, we have the following theorem that establishes the regularity of w .

Theorem 2.4 Let \tilde{I}_h be a modified general type-I interpolant, $M_{h,u}$ be as in (2.48) and $0 < T \leq \infty$. Let $h_0 > 0$ be given by

$$h_0^{-2} = \frac{c}{v} \max \left\{ \frac{1}{\kappa \lambda_1}, \frac{1}{\kappa} \left(1 + \frac{1}{\lambda_1^2} \right) \right\}. \quad (2.51)$$

Assume that for some $0 < h \leq h_0$

$$h^2 M_{h,u}^4 \leq \frac{v^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{v^3} \leq \frac{v}{ch^2}. \quad (2.52)$$

Let μ be chosen such that

$$\max \left\{ \frac{v}{ch_0^2}, \frac{cM_{h,u}^4}{v^3} \right\} \leq \mu \leq \frac{v}{ch^2}. \quad (2.53)$$

Then, the data assimilated fluid velocity is regular and satisfies

$$\|w\|_0 \leq M_{h,u}. \quad (2.54)$$

Proof Taking the inner product of (2.21) with $A_0(w_n)$ and (2.22) with η_n and adding, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|w_n\|_0^2 + |\eta_n|_1^2 \right) + \nu |A_0 w_n|_0^2 + \kappa \|\eta_n\|_1^2 \\ &= -(B_0(w_n, w_n), A_0 w_n)_0 + \mu (w_n - \tilde{I}_h w_n, A_0 w_n)_0 \\ & \quad - \mu \|w_n\|_0^2 + \mu (\tilde{I}_h u, A_0 w_n)_0 + (\eta_n, A_0(w_n) \cdot e_3)_1 + (\eta_n, w_n \cdot e_3)_1 \end{aligned}$$

First, applying (2.4) and Young's inequality, we have

$$|(B_0(w_n, w_n), A_0 w_n)|_0 \leq c \|w_n\|_0^{3/2} |A_0(w_n)|_0^{3/2} \leq \frac{c}{v^3} \|w_n\|_0^6 + \frac{v}{4} |A_0 w_n|_0^2.$$

Also, from (2.46), (2.47) and (2.48), we see that, for a modified type-I interpolant \tilde{I}_h , we may write

$$32 \|\tilde{I}_h u(t)\|_0 \leq M_{h,u}, \quad t \in [0, T]. \quad (2.55)$$

Similar to the proof of Theorem 2.2, applying Cauchy-Schwarz, Young's inequality, (2.53) and (2.2) to the remaining terms, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|w_n\|_0^2 + |\eta_n|_1^2 \right) + \frac{\nu}{2} |A_0 w_n|_0^2 + \left(\kappa - \frac{1}{\mu} - \frac{2}{\mu \lambda_1^2} \right) \|\eta_n\|_1^2 \\ &+ \left(\frac{\mu}{8} - \frac{c}{v^3} \|w_n\|_0^4 \right) \|w_n\|_0^2 \leq \mu \sup_{[0, T]} \|\tilde{I}_h u(t)\|_0^2. \end{aligned}$$

Let $[0, T_1]$ be the maximal interval on which $\|w_n(t)\| \leq M_{h,u}$ for $t \in [0, T_1]$, where $M_{h,u}$ as in (2.47). Note that $T_1 > 0$ exists because we have $w_n(0) = 0$. Assume that $T_1 < T$. Then by continuity, we must have $\|w_n(T_1)\| = M_{h,u}$. Applying (2.7),(2.30), the first inequality in (2.53) and dropping all terms except the first and the last term on the LHS, we obtain

$$\frac{d}{dt} \|w_n\|_0^2 + \frac{\mu}{8} \|w_n\|_0^2 \leq 2\mu \sup_{[0, T_1]} \|\tilde{I}_h u(t)\|_0^2.$$

Applying Gronwall's inequality and (2.55), we obtain

$$\|w_n(t)\|_0^2 \leq 16 \sup_{[0, T]} \|\tilde{I}_h u(t)\|_0^2 \leq \frac{1}{2} M_{h,u}^2 \quad \forall t \in [0, T_1].$$

This contradicts the fact that $\|w_n(T_1)\| = M_{h,u}$. Therefore $T_1 \geq T$ and consequently, $\|w_n(t)\| \leq M_{h,u}$ for all $t \in [0, T]$. Passing to the limit as $n \rightarrow \infty$, we obtain the desired conclusion for w . \square

2.7 Synchronization for General Type-I Interpolant

We will now show that the data assimilated solution approaches the actual solution(synchronization) for the general type-I interpolant. To show synchronization, in addition to (2.16) and (2.17), we will need to establish an energy *equality* for the data assimilation equation. To do so, we borrow the following definition and result from [42].

Definition 2 Let us assume that X_0, X, X_1 , are Hilbert spaces with

$$X_0 \subset X \subset X_1, \quad (2.56)$$

the injections being continuous and

$$\text{the injection of } X_0 \text{ into } X \text{ is compact.} \quad (2.57)$$

For given $\gamma > 0$, we define the space

$$\mathcal{H}^\gamma(0, T; X_0, X_1) = \left\{ v \in L^2(0, T; X_0), D_t^\gamma v \in L^2(0, T; X_1) \right\},$$

where, $D_t^\gamma v$ is the derivative in t of order γ of v which is the inverse Fourier transform of $(2i\pi\tau)^\gamma \hat{v}$ or

$$\widehat{D_t^\gamma v(t)} = 2\pi\tau \hat{v}(\tau).$$

We also state the following compactness result from [42].

Theorem 2.5 *Let us assume that X_0, X, X_1 , are Hilbert spaces that satisfy (2.56) and (2.57). Then the injection of $\mathcal{H}^\gamma(0, T; X_0, X_1)$ into $L^2(0, T; X)$ is compact.*

Another result we need is the regularity of η on a bounded interval of time, which is shown below.

Lemma 2.6 *Let η and η_n be as in (2.19) and (2.23) respectively, \tilde{I}_h be a modified general type-I interpolant, $M_{h,u}$ be as in (2.48) and $0 < T < \infty$. Let $h_0 > 0$ be given by*

$$h_0^{-2} = \frac{c}{v} \max \left\{ \frac{1}{\kappa}, \frac{1}{\kappa \lambda_1}, \frac{1}{\kappa} \left(1 + \frac{1}{\lambda_1^2} \right) \right\}. \quad (2.58)$$

Assume that for some $0 < h \leq h_0$

$$h^2 M_{h,u}^4 \leq \frac{v^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{v^3} \leq \frac{v}{ch^2}. \quad (2.59)$$

Let μ be chosen such that

$$\max \left\{ \frac{v}{ch_0^2}, \frac{cM_{h,u}^4}{v^3} \right\} \leq \mu \leq \frac{v}{ch^2}. \quad (2.60)$$

Then η is regular and satisfies

$$\|\eta(t)\|_1 \leq \frac{2\mu M_0^2}{\beta} e^{\beta T}, \quad \forall t \in [0, T], \quad \text{with } \beta = \kappa + C\kappa^2 M_h^2$$

and M_0 is as in (2.7).

Proof Taking the inner product of (2.21) with w_n and (2.22) with $A_1 \eta_n$ and adding, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(|w_n|_0^2 + \|\eta_n\|_1^2 \right) + \nu \|w_n\|_0^2 + \kappa |A_1 \eta_n|_1^2 + (B_1(w_n, \eta_n), A_1 \eta_n) \\ & \leq |\eta_n|_1 |w_n|_0 + |A_1 \eta_n|_1 |w_n|_0 + \mu (I_h(u) - I_h(w_n), w_n) \\ & \leq |\eta_n|_1 |w_n|_0 + |A_1 \eta_n|_1 |w_n|_0 + \mu (w_n - I_h(w_n), w_n) - \mu |w_n|_0^2 + \mu (I_h(u), w_n) \end{aligned} \quad (2.61)$$

We bound each of the terms on the RHS below.

First, using Young's inequality, (2.66) and (2.2), we have

$$\begin{aligned} |\eta_n|_1 |w_n|_0 & \leq \frac{\mu |w_n|_0^2}{4} + \frac{\kappa \|\eta_n\|_1^2}{2} \quad \text{and} \\ |A_1 \eta_n|_1 |w_n|_0 & \leq \frac{\mu |w_n|_0^2}{8} + \frac{\kappa}{4} |A_1 \eta_n|_1^2 \end{aligned} \quad (2.62)$$

Next, using (2.4), we see that

$$|(B_1(w_n, \eta_n), A_1 \eta_n)| \leq C \|w_n\|_0 |A_1 \eta_n|_1^{3/2} \|\eta_n\|_1^{1/2} \leq C \kappa^2 M_h^2 \|\eta_n\|^2 + \frac{\kappa}{4} |A_1 \eta_n|_1^2.$$

Combining the above estimates with (2.7), (2.27) and (2.28), we obtain

$$\frac{d\|\eta_n\|_1^2}{dt} - \left(\kappa + C \kappa^2 M_h^2 \right) \|\eta_n\|_1^2 \leq 2\mu |I_h(u)|_0^2 \leq 2\mu M_0^2. \quad (2.63)$$

Now applying Gronwall, we obtain

$$\|\eta_n(t)\|_1 \leq \frac{2\mu M_0^2}{\beta} e^{\beta t} \leq \frac{2\mu M_0^2}{\beta} e^{\beta T}, \quad \text{where } \beta = \kappa + C \kappa^2 M_h^2.$$

Passing to the limit, we obtain the statement of the theorem. \square

The energy equality is shown in the following lemma.

Lemma 2.7 *Let (w, η) be the weak solution to (2.18)–(2.20), \tilde{I}_h be a modified general type-I interpolant, $M_{h,u}$ be as in (2.48) and $0 < T < \infty$. Let $h_0 > 0$ be given by*

$$h_0^{-2} = \frac{c}{v} \max \left\{ \frac{1}{\kappa}, \frac{1}{\kappa \lambda_1}, \frac{1}{\kappa} \left(1 + \frac{1}{\lambda_1^2} \right) \right\}. \quad (2.64)$$

Assume that for some $0 < h \leq h_0$

$$h^2 M_{h,u}^4 \leq \frac{v^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{v^3} \leq \frac{v}{ch^2}. \quad (2.65)$$

Let μ be chosen such that

$$\max \left\{ \frac{v}{ch_0^2}, \frac{M_{h,u}^4}{v^3} \right\} \leq \mu \leq \frac{v}{ch^2}. \quad (2.66)$$

Then

$$\begin{aligned} |w(t)|_0^2 + 2v \int_0^t \|w(s)\|_0^2 ds &= |w(0)|_0^2 + 2\mu \int_0^t (I_h(u(s) - w(s)), w(s))_0 ds \\ &\quad + 2 \int_0^t (w \cdot e_3(s), \eta(s))_1 ds. \end{aligned} \quad (2.67)$$

$$|\eta(t)|_1^2 + 2\kappa \int_0^t \|\eta(s)\|_1^2 ds = |\eta(0)|_1^2 + 2 \int_0^t (w \cdot e_3(s), \eta(s))_1 ds. \quad (2.68)$$

Proof Taking the inner product of (2.21) with w_n and (2.22) with η_n and then integrating over time on the interval $[0, t]$, we obtain

$$\begin{aligned} |w_n(t)|_0^2 + 2\nu \int_0^t \|w_n(s)\|_0^2 ds &= |w_n(0)|_0^2 + 2\mu \int_0^t (I_h(u(s) - w_n(s)), w_n(s))_0 ds \\ &\quad + 2 \int_0^t (w_n \cdot e_3(s), \eta_n(s))_1 ds. \end{aligned} \quad (2.69)$$

$$|\eta_n(t)|_1^2 + 2\kappa \int_0^t \|\eta_n(s)\|_1^2 ds = |\eta_n(0)|_1^2 + 2 \int_0^t (w_n \cdot e_3(s), \eta_n(s))_1 ds. \quad (2.70)$$

Now, let

$$C_+^\infty([0, T]) = \{f \in C([0, T]) | f \text{ is smooth and } f(x) > 0 \forall x \in [0, T]\}.$$

Multiplying (2.69) with $\phi \in C_+^\infty([0, T])$, integrating over $[0, T]$, we obtain

$$\begin{aligned} &\int_0^T |w_n(t)|_0^2 \phi(t) dt + \int_0^T 2\nu \phi(t) \int_0^t \|w_n(s)\|_0^2 ds dt \\ &= \int_0^T |w_n(0)|_0^2 \phi(t) dt + \int_0^T 2\mu \phi(t) \int_0^t (I_h(w(s) - w_n(s)), w_n(s))_0 ds dt \\ &\quad + \int_0^T 2\mu \phi(t) \int_0^t (I_h(u(s) - w(s)), w_n(s))_0 ds dt \\ &\quad + 2 \int_0^T \phi(t) \int_0^t (w_n \cdot e_3(s), \eta_n(s))_1 ds dt. \end{aligned} \quad (2.71)$$

Observe that from Theorem (2.4), we know that

$$w_n \rightharpoonup w \text{ in } L^2(0, T; D(A_0)) \quad \text{and} \quad w_n \overset{*}{\rightharpoonup} w \text{ in } L^\infty(0, T; V_0).$$

Also, from (2.4), we obtain

$$|B_0(w, w)|_0 \leq \|w\|_0^{3/2} |Aw|_0^{1/2}. \quad (2.72)$$

Applying the estimates (2.72), (2.54) and (2.31) to (2.18), we may conclude that

$$\frac{dw}{dt} \in L^2(0, T; H_0).$$

Therefore, using Theorem 2.5, we obtain

$$w_n \rightarrow w \text{ in } L^2(0, T; V_0). \quad (2.73)$$

Hence, passing through the limit using (2.73), we obtain

$$\begin{aligned} \int_0^T |w(t)|_0^2 \phi(t) dt + \int_0^T 2\nu \phi(t) \int_0^t \|w(s)\|_0^2 ds dt &= \int_0^T |w_n(0)|_0^2 \phi(t) dt \\ &+ \int_0^T 2\mu \phi(t) \int_0^t (I_h(u(s) - w(s)), w(s))_0 ds dt \\ &+ 2 \int_0^T \phi(t) \int_0^t (w \cdot e_3(s), \eta(s))_1 ds dt. \end{aligned} \quad (2.74)$$

for all $\phi \in C_+^\infty([0, T])$. This amounts to saying (2.67) holds for almost everywhere $t \in [0, T]$.

Due to Theorem 2.6, we may repeat the exact same argument for η to obtain (2.68). \square

Using the energy equality, We now establish the following lemma needed to show synchronization.

Lemma 2.8 *Let (u, θ) be the general Leray-Hopf weak solution of (2.8)–(2.10) and (w, η) be the weak solution to (2.18)–(2.20). Let $\tilde{w} = w - u$, $\tilde{\eta} = \eta - \theta$ and $0 < T < \infty$. Then,*

$$\begin{aligned} |\tilde{w}(t)|_0^2 + 2\nu \int_0^t \|\tilde{w}(s)\|_0^2 ds &\leq |\tilde{w}(0)|_0^2 - 2\mu \int_0^t (I_h(\tilde{w}(s)), \tilde{w}(s))_0 ds \\ &+ 2 \int_0^t B_0(\tilde{w}(s), \tilde{w}(s)), w(s))_0 ds \\ &+ 2 \int_0^t (\tilde{w}(s) \cdot e_3, \tilde{\eta}(s))_1 ds. \end{aligned} \quad (2.75)$$

$$\begin{aligned} |\tilde{\eta}(t)|_1^2 + 2\kappa \int_0^t \|\tilde{\eta}(s)\|_1^2 ds &\leq |\tilde{\eta}(0)|_1^2 + 2 \int_0^t B_1(\tilde{w}(s), \tilde{\eta}(s)), \eta(s))_1 ds \\ &+ 2 \int_0^t (\tilde{w}(s) \cdot e_3, \tilde{\eta}(s))_1 ds \end{aligned} \quad (2.76)$$

Proof Recall that we denote by ϕ_j^0 the eigenfunction corresponding to λ_j^0 , $j = 1, 2, \dots$, with $\lambda_1^0 \leq \lambda_2^0 \leq \dots$ being the eigenvalues of A_0 . Let H_n^0 denote the span of $\phi_1^0, \phi_2^0, \dots, \phi_n^0$ and P_n^0 denote the projection onto the space H_n^0 . Also, let $P_n^0(u) = u_n$ and $P_n^0(w) = w_n$. It is important to note that u_n and w_n in this lemma are projections of u and w onto a finite dimensional subspace and must not be confused with the Galerkin projections of u and w . We observe that since u_n finite dimensional, $\frac{\partial u_n}{\partial t}$ exists in the classical sense. We hence have

$$\begin{aligned} \frac{d}{dt}(w_n, u_n)_0 &= \left(\frac{\partial w_n}{\partial t}, u_n \right)_0 + \left(w_n, \frac{\partial u_n}{\partial t} \right)_0 \\ &= (-B_0(w, w) - \nu A_0(w) + \mu(I_h(u - w) + \eta e_3), u_n)_0 \\ &\quad + (w_n, f - B_0(u, u) - \nu A_0(u) + \theta e_3)_0. \end{aligned} \quad (2.77)$$

Using the fact that P_n^0 commutes with A_0 , we obtain

$$(A_0(u), w_n)_0 = (A_0(u), P_n(w))_0 = (P_n(u), A_0(w))_0 = (u_n, A_0(w))_0. \quad (2.78)$$

Integrating on the interval $[s, t]$, $0 \leq s < t \leq T$, applying (2.78) and passing through the limit, we see that for almost all s and t , $0 < s < t < T$:

$$\begin{aligned} & (w(t), u(t)) - (w(s), u(s)) \\ &= \int_s^t ((\eta(\sigma)e_3, u(\sigma))_0 + (\theta(\sigma)e_3, w(\sigma))_0) d\sigma \\ & \quad - 2\nu \int_s^t (A_0(w)(\sigma), u(\sigma))_0 d\sigma + 2\mu \int_s^t (I_h(u(\sigma) - w(\sigma)), u(\sigma))_0 d\sigma \\ & \quad - \int_s^t \{(B_0(w(\sigma), w(\sigma)), u(\sigma))_0 + (B_0(u(\sigma), u(\sigma)), w(\sigma))_0\} d\sigma \end{aligned} \quad (2.79)$$

Since u is weakly continuous in H_0 and w is strongly continuous in H_0 , the function $t \rightarrow (u(t), w(t))$ is continuous and the relation (2.79) holds for all s and t , $0 \leq s < t \leq T$. Moreover, we observe that

$$(B_0(w, w), u)_0 + (B_0(u, u), w)_0 = (B_0(w - u, w), u)_0 = (B_0(w - u, w - u), u)_0.$$

Incorporating this in (2.79) and letting $s = 0$, we obtain

$$\begin{aligned} & (w(t), u(t)) + 2\nu \int_0^t ((w, u))_0 \\ &= (w(0), u(0))_0 + \int_0^t ((\eta(\sigma)e_3, u(\sigma))_0 + (\theta(\sigma)e_3, w(\sigma))_0) d\sigma \\ & \quad - \int_0^t (B_0(\tilde{w}(s), \tilde{w}(s)), w(s)) ds - 2\mu \int_0^t I_h(\tilde{w}(s), u) ds \end{aligned} \quad (2.80)$$

Adding (2.16) and (2.67) and subtracting two times (2.80), we obtain (2.75). Repeating the same arguments for

$$\frac{d}{dt}(\eta_n, \theta_n)_1 = \left(\frac{\partial \eta_n}{\partial t}, \theta_n \right)_1 + \left(\eta_n, \frac{\partial \theta_n}{\partial t} \right)_1,$$

we obtain (2.68). \square

With the above lemma in place, we will now prove synchronization in the following theorem.

Theorem 2.9 *Let I_h be the general type-I interpolant, $0 < T < \infty$ and $M_{h,u}$ as in (2.48). Also, let $h_0 > 0$ be defined as*

$$h_0^{-2} = \max \left\{ \frac{c}{\nu \kappa \lambda_1}, \frac{c}{\nu \kappa} \left(1 + \frac{1}{\lambda_1^2} \right), \frac{CS_2^8}{\nu^4 \kappa^4} \right\}, \quad (2.81)$$

where S_2 is as in (2.32) with $p = 2$. We define $\tilde{w} = w - u$ and $\tilde{\eta} = \eta - \theta$. Assume that for some $h \leq h_0$,

$$h^2 M_{h,u}^4 \leq \frac{v^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{v^3} \leq \frac{v}{ch^2}. \quad (2.82)$$

Let μ be chosen such that

$$\max \left\{ \frac{v}{ch_0^2}, \frac{M_{h,u}^4}{v^3} \right\} \leq \mu \leq \frac{v}{ch^2}. \quad (2.83)$$

Then,

$$\left(|\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 \right) \leq \left(|\tilde{w}(0)|_0^2 + |\tilde{\eta}(0)|_1^2 \right) e^{-\alpha t} \quad \forall t \in [0, T] \quad (2.84)$$

where $\alpha = \min \left\{ \frac{\mu}{4}, \frac{\kappa \lambda_1}{2} \right\}$. In particular, if in the statement of Theorem 2.4, $T = \infty$, then

$$\lim_{t \rightarrow \infty} \left(|\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 \right) = 0. \quad (2.85)$$

Proof Adding equations (2.67) and (2.68), we obtain

$$\begin{aligned} & |\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 \\ & + 2\nu \int_0^t \|\tilde{w}(s)\|_0^2 ds + 2\kappa \int_0^t \|\tilde{\eta}(s)\|_1^2 ds \leq |\tilde{w}(0)|_0^2 + |\tilde{\eta}(0)|_1^2 - 2\mu |\tilde{w}|^2 \\ & + 2 \int_0^t (B_0(\tilde{w}(s), \tilde{w}(s)), w(s))_0 ds + 2 \int_0^t (B_1(\tilde{w}(s), \tilde{\eta}(s)), \eta(s))_1 ds \\ & + 2\mu \int_0^t (\tilde{w} - I_h(\tilde{w}(s), \tilde{w}(s)))_0 ds + 4 \int_0^t (\tilde{w}(s) \cdot e_3, \tilde{\eta}(s))_1 ds \end{aligned} \quad (2.86)$$

We bound each term on the RHS.

First, applying (2.5), (2.32), Cauchy-Schwartz and Young's inequality, we obtain

$$\begin{aligned} |B_0(\tilde{w}, \tilde{w}), w)_0| & \leq |(B_0(\tilde{w}, w), \tilde{w})|_0 \leq c |\tilde{w}|_0^{1/2} \|\tilde{w}\|_0^{3/2} \|w\|_0 \\ & \leq \frac{c}{v^3} \|w\|_0^4 |\tilde{w}|_0^2 + \frac{v}{2} \|\tilde{w}\|_0^2. \\ |(B_1(\tilde{w}, \tilde{\eta}), \eta)|_1 & \leq |\tilde{w}|_0^{1/4} \|\tilde{w}\|_0^{3/4} \|\tilde{\eta}\|_1 \|\eta\|_{L^4} \leq \frac{v}{4} \|\tilde{w}\|_0^2 + \frac{\kappa}{2} \|\tilde{\eta}\|_1^2 + \frac{C}{v^3 \kappa^4} S_2^8 |\tilde{w}|_0^2 \end{aligned} \quad (2.87)$$

Similar to the proof of Theorem 2.2, we apply Cauchy-Schwarz, Young's inequality, (2.53) and (2.2) to estimate the remaining terms. Combining the estimates and applying (2.2), (2.81), (2.83) and the fact that $\|w\|_0 \leq M_{h,u}$, we obtain

$$|\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 + \frac{\mu}{4} \int_0^t |\tilde{w}(s)|_0^2 ds + \frac{\kappa \lambda_1}{2} \int_0^t |\tilde{\eta}(s)|_1^2 ds \leq |\tilde{w}(0)|_0^2 + |\tilde{\eta}(0)|_1^2. \quad (2.88)$$

Changing the interval of integration from $[0, t]$ to $[t - 1/\alpha, t]$, applying Lemma 6.6 with $y = |\tilde{w}(t)|^2 + |\tilde{\eta}(t)|^2$, we obtain

$$\begin{aligned} & (|\tilde{w}(t)|^2 + |\tilde{\eta}(t)|^2) \\ & \leq (|\tilde{w}(0)|_0^2 + |\tilde{\eta}(0)|_1^2) e^{-\alpha(t - \frac{1}{\alpha})} \quad \forall t \in [0, T], \quad \text{where } \alpha = \min \left\{ \frac{\mu}{4}, \frac{\kappa \lambda_1}{2} \right\} \end{aligned} \quad (2.89)$$

□

Remark 2.7 The proof of Theorem 2.9, when applied to the NSE, provides an alternate proof of the well-known Sather-Serrin weak-strong uniqueness result [41] for the 3D NSE which avoids the use of time mollification/regularization.

3 Forward Determining Map

We now describe the determining map introduced in [22], which played a crucial role in obtaining the determining form for evolution equations. Using the ideas in [9], we construct a similar map for the 3D Boussinesq and in the next section, for the 3D Navier-Stokes equations. This allows us to obtain a result concerning the existence of determining modes and volume elements for the 3D Boussinesq system (Theorem 3.3) as well as the aforementioned observable regularity criterion on the weak attractor for the 3D NSE.

We begin by introducing spaces that contain the domain and ranges of the map W^+ and by introducing the evolution equation which yields the definition of W^+ .

We denote by $L_b^2(\mathbb{R}^+; D(A_0))$ the functions in $L^2(\mathbb{R}^+; D(A_0))$ which are translation bounded, i.e.,

$$\sup_{s \geq 0} \int_s^{s+1} |A_0(u(r))|_0^2 dr < \infty. \quad (3.1)$$

Similarly, $L_b^2(\mathbb{R}^+; V)$ denotes the functions in $L^2(\mathbb{R}^+; V)$ which satisfy

$$\sup_{s \geq 0} \int_s^{s+1} \|u(r)\|_0^2 dr < \infty.$$

Let

$$Y_+ = (C_b(\mathbb{R}_+; V_0) \cap L_b^2(\mathbb{R}_+; D(A_0))) \times C_b(\mathbb{R}_+; H_1)$$

and

$$Z_+ = \left(C_b(\mathbb{R}_+; H_0) \cap L_b^2(\mathbb{R}_+; V_0) \right) \times C_b(\mathbb{R}_+; H_1)$$

where, $C_b(I; B)$ denotes the space of all bounded and continuous functions over the interval I with values in the Banach space B . Y_+ and Z_+ are Banach spaces with norms

$$\|(u, \theta)\|_{Y_+} = \left\{ \sup_{s \in \mathbb{R}_+} \|u(s)\|_0^2 + \sup_{s \geq 0} \int_s^{s+1} |A_0(u(r))|_0^2 dr + \sup_{s \in \mathbb{R}_+} |\theta(s)|_1^2 \right\}^{\frac{1}{2}} \quad (3.2)$$

and

$$\|u\|_{Z_+} = \left\{ \sup_{s \in \mathbb{R}_+} |u(s)|_0^2 + \sup_{s \geq 0} \int_s^{s+1} \|u(r)\|_0^2 dr + \sup_{s \in \mathbb{R}_+} |\theta(s)|_1^2 \right\}^{\frac{1}{2}} \quad (3.3)$$

respectively. Moreover, let X be the Banach space

$$X_+ = C_b(\mathbb{R}^+; (H^1(\Omega))^3)$$

equipped with the norm

$$\|v\|_{X_+} = \sup_{s \geq 0} \|v(s)\|_0. \quad (3.4)$$

The observed spatial coarse-mesh data is denoted by $v(t)$. For the purpose of data assimilation, we consider the case $v \in X_+$ with $\|v\|_{X_+} \leq \rho \sim h^{-1/2}$ for some $\rho > 0$. We use $B_{X_+}(\rho)$ to denote the closed ball in X_+ of radius ρ centered at 0. We also define the Banach space

$$P_+ = C_b(\mathbb{R}_+; H_0) \times C_b(\mathbb{R}_+; H_1)$$

with the norm

$$\|(u, \theta)\|_{P_+} = \left\{ |u|_0^2 + |\theta|_1^2 \right\}^{\frac{1}{2}} \quad (3.5)$$

For $\sigma \in I \subset \mathbb{R}^+$ and a Banach space B , we define the time translation $\tau_\sigma : C(I; B) \rightarrow C(I; B)$ as

$$\tau_\sigma(u(t)) = u(t + \sigma). \quad (3.6)$$

Now given $v \in B_{X_+}(\rho)$, we consider the following initial-value problem:

$$\frac{dw}{dt} + \nu A_0(w) + B_0(w, w) = P_\sigma(\eta e_3) + \mu(P_\sigma(v - I_h(w))) \quad (3.7)$$

$$\frac{d\eta}{dt} + \kappa A_1(\eta) + B_1(w, \eta) - w \cdot e_3 = 0 \quad (3.8)$$

$$w(x, 0) = 0, \eta(x, 0) = 0. \quad (3.9)$$

The corresponding Galerkin approximation is given by

$$\frac{dw_n}{dt} + \nu A_0(w) + B_0(w_n, w_n) = P_n(\eta_n e_3) + \mu(P_n(v - I_h(w))) \quad (3.10)$$

$$\frac{d\eta_n}{dt} + \kappa A_1(\eta_n) + B_1(w, \eta_n) - w_n \cdot e_3 = 0 \quad (3.11)$$

$$w_n(x, 0) = 0, \eta_n(x, 0) = 0. \quad (3.12)$$

We can now define the map W_+ .

Definition 3 Let $\rho > 0$. We define the forward determining map $W_+ : B_{X_+}(\rho) \rightarrow Y_+$ as

$$W_+(v) = (w, \eta). \quad (3.13)$$

Theorem 3.1 Let I_h be a general type-I interpolant, $v \in B_{X_+}(\rho)$ for some $\rho > 0$ ($\rho \sim 1/\sqrt{h}$), $0 < T \leq \infty$ and

$$M_{h,u}^2 = 32 \sup_{[0,T]} \|v\|_0^2. \quad (3.14)$$

Also, let $h_0 > 0$ be defined as

$$h_0^{-2} = \max \left\{ \frac{c}{\nu \kappa \lambda_1}, \frac{c}{\nu \kappa} \left(1 + \frac{1}{\lambda_1^2} \right), \frac{CS_2^8}{\nu^4 \kappa^4} \right\}, \quad (3.15)$$

where c, h are as in (2.11) and S_2 is as in (2.32) with $p = 2$. Assume that for some $h \leq h_0$,

$$h^2 M_{h,u}^4 \leq \frac{\nu^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{\nu^3} \leq \frac{\nu}{ch^2}. \quad (3.16)$$

Let μ be chosen such that

$$\max \left\{ \frac{\nu}{ch_0^2}, \frac{M_{h,u}^4}{\nu^3} \right\} \leq \mu \leq \frac{\nu}{ch^2}. \quad (3.17)$$

Then, there exists a weak solution (w, η) to (3.7)–(3.9). Moreover, the following statements hold true

- (a) $\|w\|_0 \leq M_{h,u}$
- (b) $W_+ : B_{X_+}(\rho) \rightarrow Z_+$ is Lipschitz continuous.

- (c) Let (u, θ) be a weak solution (2.8)–(2.10). Assume that $|I_h(u)(s) - v(s)|_0 \rightarrow 0$ as $s \rightarrow \infty$. Then $\|W_+(v)(s) - (u(s), \theta(s))\|_{P_+} \rightarrow 0$ as $s \rightarrow \infty$
- (d) Let $v_1, v_2 \in B_{X_+}(\rho)$. Then, $W_+(v_1) = W_+(v_2)$ iff $P_\sigma(v_1 - v_2) = P_\sigma(\bar{v}) = 0$, where $\bar{v} = v_1 - v_2$.
- (e) For every $\sigma \in \mathbb{R}^+$,

$$W_+ \circ \tau_\sigma(v) = \tau_\sigma \circ W_+(v). \quad (3.18)$$

Proof The existence of a weak solution (w, η) and the proof of (a) is obtained by repeating the proof of Theorem 2.2 and Theorem 2.4, after replacing $I_h(u)$ with v .

To prove (b), we define $\bar{w}_n = w_n^1 - w_n^2$, $\bar{\eta}_n = \eta_n^1 - \eta_n^2$ and $\bar{v}_n = P_n(\bar{v}_1 - \bar{v}_2)$, where (w_n^1, η_n^1) and (w_n^2, η_n^2) are Galerkin approximations of (w^1, η^1) and (w^2, η^2) respectively and $(w^1, \eta^1) = W_+(v_1)$ and $(w^2, \eta^2) = W_+(v_2)$.

\bar{w}_n and $\bar{\eta}_n$ satisfy the equations

$$\begin{aligned} \frac{d\bar{w}_n}{dt} + \nu A_0(\bar{w}_n) + B_0(\bar{w}_n, w_n^2) + B_0(w_n^1, \bar{w}_n) \\ = \bar{\eta}_n e_3 \mu P_n(\bar{v}_n - I_h(\bar{w}_n)) \\ = \mu \bar{v}_n + \mu P_n(\bar{w}_n - I_h(\bar{w}_n)) - \mu \bar{w}_n \end{aligned} \quad (3.19)$$

$$\frac{d\bar{\eta}_n}{dt} + \kappa A_1(\bar{\eta}_n) + B_1(\bar{w}_n, \eta_n^2) + B_1(w_n^1, \bar{\eta}_n) - \bar{w}_n \cdot e_3 = 0 \quad (3.20)$$

Taking the inner product of \bar{w}_n and $\bar{\eta}_n$ with (3.19) and (3.20) respectively, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\bar{w}_n|_0^2 + |\bar{\eta}_n|_1^2) + \nu \|\bar{w}_n\|_0^2 + \kappa \|\bar{\eta}_n\|_1^2 \\ + \mu |\bar{w}_n|_0^2 \leq |(B_0(\bar{w}_n, w_n^2), \bar{w}_n)|_0 + |(B_1(\bar{w}_n, \eta_n^2), \bar{\eta}_n)|_1 \\ + 2|\bar{w}_n|_0 |\bar{\eta}_n|_1 + |\mu(\bar{w}_n - I_h(\bar{w}_n))|_0 |\bar{w}_n|_0 + \mu |\bar{v}_n|_0 |\bar{w}_n|_0. \end{aligned} \quad (3.21)$$

Applying Young's inequality, we obtain

$$\mu |\bar{v}_n|_0 |\bar{w}_n|_0 \leq \mu |\bar{v}_n|_0^2 + \frac{\mu}{4} |\bar{w}_n|_0^2, \quad (3.22)$$

Bounding the other terms on the RHS of (3.21) exactly as in Theorem 2.9, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\bar{w}_n|_0^2 + |\bar{\eta}_n|_1^2) + \nu \|\bar{w}_n\|_0 + \left(\frac{\mu}{4} - \frac{c}{\nu^3} \|w_n\|_0^4 - \frac{CS_2^8}{\nu^3 \kappa^4} \right) |\bar{w}_n|_0^2 \\ + \left(\frac{\kappa \lambda_1}{2} - \frac{4}{\mu} \right) |\bar{\eta}_n|_1^2 \leq \mu |\bar{v}_n|_0^2. \end{aligned} \quad (3.23)$$

We already have $\|w\|_0 \leq M_{h,u}$. Now applying (3.17) and (3.15), we have

$$\frac{d}{dt} (|\bar{w}_n|_0^2 + |\bar{\eta}_n|_1^2) + \frac{\mu}{4} |\bar{w}_n|_0^2 + \frac{\kappa \lambda_1}{2} |\bar{\eta}_n|_1^2 \leq 2\mu |\bar{v}_n|_0^2. \quad (3.24)$$

Next, applying Gronwall (for $t, \sigma \in [0, T]$ with $t > \sigma$) separately to $|\bar{w}_n|^2$ and $|\bar{\eta}_n|^2$, we obtain

$$\begin{aligned} |\bar{w}_n(t)|_0^2 &\leq |\bar{w}_n(\sigma)|_0^2 e^{-(\mu/4)(t-\sigma)} + 8 \sup_{t \in [0, T]} |\bar{v}_n|_0^2, \\ |\bar{\eta}_n(t)|_1^2 &\leq |\bar{\eta}_n(\sigma)|_1^2 e^{-\left(\frac{\kappa\lambda_1}{2}\right)(t-\sigma)} + \frac{4\mu}{\kappa\lambda_1} \sup_{t \in \mathbb{R}_+} |\bar{v}_n|_0^2. \end{aligned} \quad (3.25)$$

\bar{w}_n and $\bar{\eta}_n$ converge respectively to \bar{w} and $\bar{\eta}$ weakly. Also $\bar{v}_n \rightarrow \bar{v}$ in $B_{x+}(\rho)$. Therefore, letting $n \rightarrow \infty$, setting $\sigma = 0$ and noting that $\bar{w}(0) = \bar{\eta}(0) = 0$, we obtain

$$|\bar{w}(t)|_0^2 \leq 8 \sup_{t \in \mathbb{R}_+} |\bar{v}|_0^2 \quad \text{and} \quad |\bar{\eta}(t)|_1^2 \leq \frac{4\mu}{\kappa\lambda_1} \sup_{t \in \mathbb{R}_+} |\bar{v}|_0^2. \quad (3.26)$$

Using (3.23), we may write

$$v \|\bar{w}_n\|_0 \leq \mu \sup_{t \in \mathbb{R}_+} |\bar{v}_n(t)|_0^2. \quad (3.27)$$

Letting $n \rightarrow \infty$ and integrating both sides on the interval $[t, t+1]$, we obtain

$$\sup_{t \in \mathbb{R}_+} \int_t^{t+1} v \|\bar{w}(s)\|_0 ds \leq \mu \sup_{t \in \mathbb{R}_+} |\bar{v}(t)|_0^2. \quad (3.28)$$

(3.26) and (3.28) together prove (b).

To prove (c), we define $\tilde{w} = w - u$ and $\tilde{\eta} = \eta - \theta$, where $(w, \eta) = W_+(v)$. Replacing $I_h(u)$ with v in (2.67) and repeating the arguments of Lemma (2.8), we obtain a modified version of (2.75), given by

$$\begin{aligned} |\tilde{w}(t)|_0^2 + 2v \int_\sigma^t \|\tilde{w}(s)\|_0^2 ds &\leq |\tilde{w}(0)|_0^2 + 2 \int_\sigma^t B_0(\tilde{w}(s), \tilde{w}(s)), w(s))_0 ds \\ &\quad + 2\mu \int_\sigma^t (v(s) - I_h w(s), \tilde{w}(s))_0 ds \\ &\quad + 2 \int_\sigma^t (\tilde{w}(s) \cdot e_3, \tilde{\eta}(s))_1 ds. \\ &\leq |\tilde{w}(0)|_0^2 + 2 \int_\sigma^t B_0(\tilde{w}(s), \tilde{w}(s)), w(s))_0 ds \\ &\quad + 2\mu \int_\sigma^t (v(s) - I_h u(s), \tilde{w}(s))_0 ds \\ &\quad + 2\mu \int_\sigma^t (\tilde{w} - I_h \tilde{w}(s), \tilde{w}(s))_0 ds \\ &\quad - \mu \int_\sigma^t \|\tilde{w}\|_0 ds + 2 \int_\sigma^t (\tilde{w}(s) \cdot e_3, \tilde{\eta}(s))_1 ds. \end{aligned} \quad (3.29)$$

Applying Cauchy-Schwarz and Young's inequality, we obtain

$$2\mu |(v(s) - I_h u(s), \tilde{w}(s))_0| \leq 2\mu |v(s) - I_h u(s)|_0^2 + \frac{\mu}{2} |\tilde{w}(s)|_0^2. \quad (3.30)$$

Also,

$$|v|_0^2 \leq \frac{1}{\lambda_1} \|v\|_0^2 \leq \frac{1}{\lambda_1} \rho^2. \quad (3.31)$$

and $|I_h u|_0 \leq |u|_0 \leq M_0$. Hence it makes sense to talk about the term

$$\sup_{0 \leq s \leq \infty} |I_h(u)(s) - v(s)|_0.$$

Proceeding exactly as in Theorem 2.9, we obtain

$$\begin{aligned} |\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 + \frac{\mu}{4} \int_{\sigma}^t |\tilde{w}(s)|_0^2 ds + \frac{\kappa \lambda_1}{2} \int_{\sigma}^t |\tilde{\eta}(s)|_1^2 ds &\leq |\tilde{w}(\sigma)|_0^2 + |\tilde{\eta}(\sigma)|_1^2 \\ &+ 2\mu \sup_{\sigma \leq s \leq t} |I_h(u)(s) - v(s)|_0. \end{aligned} \quad (3.32)$$

Next, letting

$$\alpha = \min \left\{ \frac{\mu}{4}, \frac{\kappa \lambda_1}{2} \right\}, \sigma = t/2$$

and applying Corollary 6.7, (2.30) and (2.7), we obtain

$$\left(|\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 \right) \leq M e^{-\alpha(t/2)} + 4\mu \sup_{t/2 \leq s \leq t} |I_h(u)(s) - v(s)|_0, \quad (3.33)$$

where

$$M = \max \left\{ 4M_0^2, \frac{2\mu M_0^2}{\kappa \lambda_1} \right\}.$$

Taking the \limsup as $t \rightarrow \infty$ and using the hypothesis that $|I_h(u)(s) - v(s)|_0 \rightarrow 0$ as $s \rightarrow \infty$, we see that

$$\lim_{t \rightarrow \infty} \left(|\tilde{w}(t)|_0^2 + |\tilde{\eta}(t)|_1^2 \right) = 0, \quad (3.34)$$

proving (c).

To prove (d), we note that $\bar{w} = w_1 - w_2 = 0$ and $\bar{\eta} = \eta_1 - \eta_2 = 0$, where $(w_1, \eta_1) = W_+(v_1)$ and $(w_2, \eta_2) = W_+(v_2)$. Since w_1 and w_2 are regular, the term \bar{w} is differentiable a.e on \mathbb{R}^+ . From (3.7), we may write

$$\frac{d\bar{w}}{dt} + \nu A_0(\bar{w}) + B_0(\bar{w}, w_1) + B_0(w_2, \bar{w}) - \bar{\eta}e_3 = \mu P_\sigma(\bar{v} - I_h(\bar{w})) \quad (3.35)$$

Letting $\bar{w} = 0$ and $\bar{\eta} = 0$, we obtain $P_\sigma(\bar{v}) = 0$. If $P_\sigma(\bar{v}) = 0$, we obtain $W_+(v_1) = W_+(v_2)$ from the Lipschitz continuity of W_+ .

To show (e), we observe that $\tau_\sigma \circ W_+(v)$ is a solution of (3.7)–(3.9) corresponding to $\tau_\sigma(v)$. From the Lipschitz property of the map W_+ , we have uniqueness of solution. Hence

$$W_+ \circ \tau_\sigma(v) = \tau_\sigma \circ W_+(v).$$

□

Corollary 3.2 *Let I_h be a general type-I interpolant and u be a weak solution (2.8)–(2.10). Let the hypothesis of Theorem 3.1 hold. Then $\|W_+(I_h(u))(s) - (u(s), \theta(s))\|_{P_+} \rightarrow 0$ as $s \rightarrow \infty$*

Proof Applying part (c) of Theorem 3.1 with $v = I_h(u)$, we obtain the statement of the corollary. □

Remark 3.1 The fact that W^+ is Lipschitz continuous means that the way we reconstruct our solution from the data is "stable". Lipschitz continuity in turn implies uniqueness of solutions as well. Also, Corollary 3.2 says that the solution obtained from type-I interpolation data (for appropriate h) asymptotically approaches the actual solution.

Theorem 3.3 *Let (u_1, θ_1) and (u_2, θ_2) be two restricted Leray-Hopf weak solutions and M_{h,u_i} (for $i = 1, 2$) be as in (2.48). Also, let $h_0 > 0$ be defined as*

$$h_0^{-2} = \max \left\{ \frac{c}{\nu \kappa \lambda_1}, \frac{c}{\nu \kappa} \left(1 + \frac{1}{\lambda_1^2} \right), \frac{CS_2^8}{\nu^4 \kappa^4} \right\}, \quad (3.36)$$

where c, h are as in (2.11) and S_2 is as in (2.32) with $p = 2$. Assume that on $[0, \infty)$, for $i = 1, 2$, for some $h \leq h_0$,

$$h^2 M_{h,u_i}^4 \leq \frac{\nu^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u_i}^4}{\nu^3} \leq \frac{\nu}{ch^2}. \quad (3.37)$$

Then, if

$$\lim_{t \rightarrow \infty} |I_h(u_1)(t) - I_h(u_2)(t)|_0 = 0$$

then

$$\lim_{t \rightarrow \infty} |u_1(t) - u_2(t)| = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} |\theta_1(t) - \theta_2(t)| = 0.$$

Proof Repeating the proof of part (c) of Theorem 3.1 with $(u, \theta) = (u_1, \theta_1)$, $(w, \eta) = (u_2, \theta_2)$ and $v = I_h u_2$, we obtain the statement of the theorem. \square

Remark 3.2 The above theorem shows that $h \leq h_0$ satisfying (3.37) is asymptotically determining.

4 Determining Map on the Weak Attractor

In this section, only for simplicity, we will be looking at solutions to the data assimilated 3D NSE on the weak attractor. We discuss well-posedness of the data assimilated equation for time ranging over all real numbers as well as the question of uniqueness and regularity for 3-D NSE on the weak attractor when the low modes are known. The three-dimensional incompressible NSE with time independent forcing (assumed for simplicity) is given by

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \quad (4.1)$$

Concerning the boundary conditions, we either consider the 3D NSE on a domain $\Omega \mathbb{R}^3$ with $\partial\Omega \in C^2$ and homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ or on the torus \mathbb{T}^3 with space-periodic boundary condition with period L in each of the spatial variables. Applying the Leray-Hopf projection P_σ on divergence-free vector fields (see [14]) to (4.1), we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} + B_0(u, u) + A_0(u) &= f. \\ \nabla \cdot u &= 0 \end{aligned} \quad (4.2)$$

where, by abuse of notation, we denote $P_\sigma(f)$ by f . Applying P_n^0 to (4.2), we obtain the Galerkin approximation to the system, given by

$$\begin{aligned} \frac{\partial u_n}{\partial t} + B_0(u_n, u_n) + A_0(u_n) &= P_n^0 f \\ \nabla \cdot u_n &= 0 \end{aligned} \quad (4.3)$$

4.1 Well-Posedness

We begin by providing definitions of weak and strong solutions [14, 42].

Definition 4 A (Leray-Hopf) weak solution on a time interval $I = [0, T] \subset \mathbb{R}$ is defined as a function $u = u(t)$ on I with values in H_0 and satisfying the following properties:

- $u \in L^\infty(0, T; H_0) \cap L^2(0, T; V_0) \cap C(0, T; V_0')$

- $\frac{du}{dt} \in L^{4/3}(0, T; V'_0)$
- u satisfies the functional equation (4.2) in the distribution sense on I , with values in V'_0 ;
- For almost all $t' \in I$, u satisfies the following energy inequality

$$|u(t)|_0^2 + 2\nu \int_{t'}^t \|u(s)\|_0^2 ds \leq |u(t')|_0^2 + 2 \int_{t'}^t (f(s), u(s))_0 ds. \quad (4.4)$$

for all $t \in I$, with $t' > t$.

Definition 5 A weak solution is said to be a strong/regular solution if it also belongs to $L^\infty(0, T; V_0) \cap L^2(0, T; D(A_0))$.

We have the following previously established results on existence and uniqueness.

Theorem 4.1 Let $f \in L^2(0, T; V'_0)$. Then, there exists a weak solution u of (4.2), satisfying all the properties given in Definition 4.

Theorem 4.2 Let v be a strong solution to (4.2). Then, there doesn't exist any other weak solution u of (4.2).

Remark 4.1 For 3-D NSE, we have existence of weak solution(Theorem 4.1), but not uniqueness. Moreover, we have uniqueness of strong solutions(Theorem 4.2), but not existence.

4.2 Weak Attractor

Despite the lack of a well-posedness result for the three-dimensional Navier- Stokes equations, it is still a natural question to ask what the dynamics and the asymptotic behaviors of their weak solutions are, despite the possibility that they are not unique with respect to the initial condition. In particular, it is natural to ask whether there exists some sort of global attractor in this case. Due to the lack of a well-defined semigroup associated with the solutions of the system, the classical theory of dynamical system does not apply directly. Nevertheless, it is still possible to adapt a number of results from the classical theory to this situation.

One of the first and main results in this direction was given in [30], and subsequently in [39], in which an object called the weak global attractor was defined. Its topological properties were further studied in [28].

Definition 6 The Weak attractor for the NSE, \mathbb{A} , denotes the set of $u_0 \in H_0$ for which there exists a weak solution $u(t)$ of (4.2), for $t \in \mathbb{R}$, such that

- $u \in L^\infty(\mathbb{R}; H_0)$
- $u(0) = u_0$.

Remark 4.2 The weak global attractor for the 3-D Navier-Stokes operator has the following properties:

- For every weak solution u of (4.2) on the time interval $(0, \infty)$, we have

$$u(t) \rightarrow \mathbb{A} \text{ weakly in } H_0, \text{ as } t \rightarrow \infty.$$

- \mathbb{A} is weakly compact in H_0 .
- \mathbb{A} is invariant in the sense that if $u_0 \in \mathbb{A}$ and u is a global weak solution uniformly bounded in H_0 with $u(0) = u_0$ then $u(t) \in \mathbb{A}$ for all $t \in \mathbb{R}$.

4.3 Determining Map

We begin by introducing spaces that contain the domain and ranges of the determining map W and by introducing the evolution equation which yields the definition of W .

We denote by $L_b^2(\mathbb{R}; D(A_0))$ the functions in $L^2(\mathbb{R}; D(A_0))$ which are translation bounded, i.e.,

$$\sup_{s \in \mathbb{R}} \int_s^{s+1} |A_0(u(r))|_0^2 dr < \infty. \quad (4.5)$$

Similarly, $L_b^2(\mathbb{R}; V)$ denotes the functions in $L^2(\mathbb{R}; V)$ which satisfy

$$\sup_{s \in \mathbb{R}} \int_s^{s+1} \|u(r)\|_0^2 dr < \infty.$$

Let

$$Y = C_b(\mathbb{R}; V_0) \cap L_b^2(\mathbb{R}; D(A_0)) \text{ and } Z = C_b(\mathbb{R}; H_0) \cap L_b^2(\mathbb{R}; V_0).$$

where, $C_b(I; B)$ denotes the space of all bounded and continuous functions over the interval I with values in the Banach space B . Y and Z are Banach spaces with norms

$$\|u\|_Y = \left\{ \|u\|_0^2 + \sup_{s \in \mathbb{R}} \int_s^{s+1} |A_0(u(r))|_0^2 dr \right\}^{\frac{1}{2}}. \quad (4.6)$$

and

$$\|u\|_Z = \left\{ |u|_0^2 + \sup_{s \in \mathbb{R}} \int_s^{s+1} \|u(r)\|_0^2 dr \right\}^{\frac{1}{2}} \quad (4.7)$$

respectively.

For $\tau \in I \subset \mathbb{R}$ and a Banach space B , we define the time translation $\tau_\sigma : C(I; B) \rightarrow C(I; B)$ as

$$\tau_\sigma(u(t)) = u(t + \sigma). \quad (4.8)$$

Moreover, let X be the Banach space

$$X = C_b(\mathbb{R}; (\dot{H}^1(\Omega))^3) \cap V_0$$

equipped with the norm

$$\|v\|_X = \sup_{s \in \mathbb{R}} \|v(s)\|_0. \quad (4.9)$$

The observed spatial coarse-mesh data is denoted by $v(t)$. For the purpose of data assimilation, we consider the case $v \in X$ with $\|v\|_X \leq \rho$ for some $\rho > 0$. We use $B_X(\rho)$ to denote the closed ball in X of radius ρ centered at 0.

The data assimilation algorithm is given by the solution w of the equation

$$\begin{aligned} \frac{\partial w}{\partial t} + B_0(w, w) + \nu A_0(w) &= f + \mu(v - I_h(w)) \\ \nabla \cdot w &= 0 \end{aligned} \quad (4.10)$$

Observe that the above system is not an initial value problem but an evolution equation for all $t \in \mathbb{R}$. The Galerkin approximation of (4.10) is obtained by applying P_n and is given by

$$\begin{aligned} \frac{\partial w_n}{\partial t} + B_0(w_n, w_n) + \nu A_0(w_n) &= P_n f + \mu(P_n(v) - I_h(w_n)) \\ \nabla \cdot w_n &= 0 \end{aligned} \quad (4.11)$$

where u_n is as in (4.3).

Theorem 4.3 *Let u be the solution to (2.8)–(2.9) on the weak attractor for $t \in \mathbb{R}$, $v \in B_X(\rho)$ for some $\rho > 0$ and I_h be any type 1 interpolant. Define*

$$M_h^2 = 8 \left(\frac{|f|^2}{\nu^2 \lambda_1} + \rho^2 \right). \quad (4.12)$$

Also, let $h_0 > 0$ be defined as

$$h_0^2 = \frac{1}{4c\lambda_1}. \quad (4.13)$$

Assume that for some $h \leq h_0$,

$$h^2 M_{h,u}^4 \leq \frac{\nu^4}{c} \quad \text{or equivalently} \quad \frac{M_{h,u}^4}{\nu^3} \leq \frac{\nu}{ch^2}. \quad (4.14)$$

Let μ be chosen such that

$$\max \left\{ \frac{M_h^4}{\nu^3}, \frac{\nu}{ch_0^2} \right\} \leq \mu \leq \frac{\nu}{ch^2}. \quad (4.15)$$

Then there exists a unique global solution (w, η) of (2.18)-(2.19) such that

$$w \in L^\infty(\mathbb{R}; V_0) \cap L^2(\mathbb{R}; D(A_0)). \quad (4.16)$$

Moreover, the following bounds hold

$$(1) \|w\|_0 \leq M_h.$$

$$(2) \int_t^{t+1} |A_0 w(t)|_0^2 dt \leq \frac{4}{\nu} \left(\frac{1}{\nu} |f|_0^2 + \mu \rho^2 \right)$$

Also, consider $v_1, v_2 \in B_X(\rho)$ and let w_1 and w_2 be solutions to (4.10) corresponding to inputs v_1 and v_2 respectively. Denote $\tilde{w} = w_1 - w_2$ and $\tilde{v} = v_1 - v_2$. Then

$$(3) |\tilde{w}(t)|_0^2 \leq 4 \|\tilde{v}\|_X^2$$

$$(4) \int_s^{s+1} \|\tilde{w}(t)\|_0^2 dt \leq \frac{4\mu}{\nu} \|\tilde{v}\|_X^2$$

Proof The proof existence of global solution satisfying (4.16) is obtained by showing that the solution, w_n to the Galerkin system (4.11) with $w(-N) = 0$ satisfies (1) and (2) on the time interval $[-N, \infty)$ (the proof for which follows exactly that of Theorem 2.4) and extracting a subsequence via the diagonal process and then passing to the limit. We proceed as in Theorem 2.4. We will first look at equation (4.11) on the time interval $[-N, \infty]$ with the initial condition $w_n(-N) = 0$.

Taking the inner product of (4.11) with $A_0 w_n$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w_n\|_0^2 + \nu |A_0 w_n|_0^2 \\ &= -(B_0(w_n, w_n), A_0(w_n))_0 + \mu(w_n - I_h(w_n), A_0(w_n))_0 \\ & \quad - \mu \|w_n\|_0^2 + \mu(P_n(v), A_0(w_n))_0 + (f, A_0(w_n)) \end{aligned} \quad (4.17)$$

Applying (2.1) and Cauchy-Schwarz, we obtain

$$\begin{aligned} \mu(P_n(v), A_0(w_n))_0 &\leq \mu \|P_n(v)\|_0^2 + \frac{\mu}{4} \|w_n\|_0^2 \\ |(f, A_0 w_n)_0| &\leq \frac{1}{\nu} |f|_0^2 + \frac{\nu}{4} |A_0 w_n|_0^2. \end{aligned}$$

Bounding the remaining terms as in Theorem 2.4 and combining estimates, we obtain

$$\frac{d}{dt} \|w_n\|_0^2 + \left(\mu - \frac{c}{\nu^3} \|w_n\|_0^4 \right) \|w_n\|_0^2 + \frac{\nu}{2} |A_0 w_n|_0^2 \leq \frac{2}{\nu} |f|_0^2 + 2\mu \|P_n(v)\|_0^2 \quad (4.18)$$

Let $[-N, T_1]$ be the maximal interval on which $\|w_n(t)\| \leq M_h$ for $t \in [-N, T_1]$, where M_h as in (4.15). Note that $T_1 > -N$ exists because we have $w_n(-N) = 0$. Assume that $T_1 < \infty$. Then by continuity, we must have $\|w_n(T_1)\| = M_h$. Applying (2.7), (2.30), the first inequality in (2.53) and dropping all terms except the first and the last term on the LHS of (4.18), we obtain

$$\frac{d}{dt} \|w_n\|_0^2 + \frac{\mu}{2} \|w_n\|_0^2 \leq \frac{2}{\nu} |f|_0^2 + 2\mu \sup_{t \in \mathbb{R}} \|P_n(v)\|_0^2. \quad (4.19)$$

Applying Gronwall, we obtain, for all $t \in [-N, T_1]$,

$$\begin{aligned}\|w_n\|_0^2 &\leq 4 \left(\frac{|f|^2}{v^2 \lambda_1} + \|(v)\|_X^2 \right) \left(1 - e^{-\frac{\mu}{8}(n+t)} \right) \leq 4 \left(\frac{|f|^2}{v^2 \lambda_1} + \rho^2 \right) \\ &\leq \frac{1}{2} M_h^2.\end{aligned}$$

This contradicts the fact that $\|w_n(T_1)\| = M_h$. Therefore $\|w_n(t)\| \leq M_h$ for all $t \in [-n, \infty)$. Dropping all terms except the last term on the LHS of (4.18) and integrating both sides over the interval $[t, t+1]$, it follows that

$$\int_t^{t+1} |A_0 w_n|^2 \leq \frac{4}{v} \left(\frac{1}{v} |f|_0^2 + \mu \rho^2 \right) \quad (4.20)$$

Therefore, we have a sequence of solutions w_n that satisfy (1) and (2) on the interval $[-N, \infty)$. One can now extract a convergent subsequence via the diagonal process and pass through the limit to show that the limit w is a solution to (4.10) and satisfies (4.16), (1) and (2).

We now proceed to prove (3) and (4). Let $w_{1,n}$ and $w_{2,n}$ satisfy (4.11) with $v = v_1$ and $v = v_2$ respectively. Then $\tilde{w}_n = w_{1,n} - w_{2,n}$ and $\tilde{v}_n = P_n(\tilde{v})$ satisfy

$$\begin{aligned}\frac{d\tilde{w}_n}{dt} + v A_0(\tilde{w}_n) + B_0(\tilde{w}_n, w_{1,n}) + B_0(w_{2,n}, \tilde{w}_n) \\ = \mu P_n(\tilde{v} - I_h(\tilde{w})) = \mu \tilde{v}_n + \mu(\tilde{w}_n - I_h(\tilde{w}_n)) - \mu \tilde{w}_n\end{aligned} \quad (4.21)$$

Taking the inner product of (4.21) with \tilde{w} , we obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} |\tilde{w}_n|_0^2 + v \|\tilde{w}_n\|_0^2 + \mu |\tilde{w}_n|_0^2 \\ \leq |B_0(\tilde{w}_n, w_{1,n}, \tilde{w}_n)|_0 + |\mu(\tilde{w}_n - I_h(\tilde{w}_n))|_0 |\tilde{w}_n|_0 + \mu |\tilde{v}_n|_0 |\tilde{w}_n|_0.\end{aligned} \quad (4.22)$$

We bound each term on the RHS.

First, applying (2.5), Cauchy-Schwartz and Young's inequality, we obtain

$$\begin{aligned}|(B_0(\tilde{w}_n, w_{1,n}), \tilde{w}_n)|_0 &\leq c |\tilde{w}_n|_0^{1/2} \|\tilde{w}_n\|_0^{3/2} \|w_{1,n}\|_0 \\ &\leq \frac{c}{v^3} \|w_{1,n}\|_0^4 |\tilde{w}_n|_0^2 + \frac{v}{2} \|\tilde{w}_n\|_0^2.\end{aligned} \quad (4.23)$$

Applying (2.11), Cauchy-Schwartz, Young's inequality and (4.15) as in Theorem 2.2 to bound the remaining terms, we obtain

$$\frac{d}{dt} |\tilde{w}_n|_0^2 + \frac{v}{2} \|\tilde{w}_n\|_0^2 + \left(\mu - \frac{c}{v^3} \|w_{1,n}\|_0^4 \right) |\tilde{w}_n|_0^2 \leq 2\mu |\tilde{v}_n|_0. \quad (4.24)$$

From proof of (1), we know that $\|w_{1,n}(t)\|_0 \leq M_h \quad \forall t \in \mathbb{R}$. Hence, applying (4.15) and integrating on the interval $[\sigma, t]$, we obtain

$$|\tilde{w}_n(t)|_0^2 \leq |\tilde{w}_n(\sigma)|^2 e^{\frac{\mu}{2}(\sigma-t)} + 4\|\tilde{v}_n\|_X^2 \left(1 - e^{\frac{\mu}{2}(\sigma-t)}\right). \quad (4.25)$$

Letting $\sigma \rightarrow -\infty$, we obtain

$$|\tilde{w}_n(t)|_0^2 \leq 4\|\tilde{v}_n\|_X^2 \quad (4.26)$$

Keeping only the second term on the LHS of (4.24) and integrating on the interval $[s, s+1]$, we obtain

$$\int_s^{s+1} \|\tilde{w}_n(t)\|_0^2 dt \leq \frac{4\mu}{\nu} \|\tilde{v}_n\|_X^2 \quad (4.27)$$

Taking the limit $n \rightarrow \infty$ in (4.26) and (4.27), we obtain (3) and (4). \square

Definition 7 Consider $\rho > 0$, $\mu > 0$ and $h > 0$ satisfying the hypothesis of Theorem 4.3. Then, the determining map $W : B_X(\rho) \rightarrow Y$ is given by $W(v) = w$.

Remark 4.3 Observe that $Y \subset Z$. Hence, from (3) and (4) in Theorem 4.3, we may conclude that $W : B_X(\rho) \rightarrow Z$ is Lipschitz continuous. Lipschitz continuity in turn implies uniqueness of solutions.

Corollary 4.4 The determining map W defined in Definition 7, in addition to being Lipschitz continuous has the following properties

- (1) Let $v_1, v_2 \in B_X(\rho)$, Then, $W(v_1) = W(v_2)$ iff $P_\sigma(v_1 - v_2) = P_\sigma(\bar{v}) = 0$.
- (2) For every $\sigma \in \mathbb{R}$,

$$W \circ \tau_\sigma(v) = \tau_\sigma \circ W(v). \quad (4.28)$$

Proof To prove (4.4), we note that $\bar{w} = w_1 - w_2 = 0$, where $w_1 = W(v_1)$ and $w_2 = W(v_2)$. Since w_1 and w_2 are regular, the term \bar{w} is differentiable a.e on \mathbb{R} . From (4.10), we may write

$$\frac{\partial \bar{w}}{\partial t} + \nu A_0(\bar{w}) + B_0(\bar{w}, w_1) + B_0(w_2, \bar{w}) = \mu P_\sigma(\bar{v} - I_h(\bar{w})) \quad (4.29)$$

Letting $\bar{w} = 0$, we obtain $P_\sigma(\bar{v}) = 0$. If $P_\sigma(\bar{v}) = 0$, We obtain $W(v_1) = W(v_2)$ from the Lipschitz continuity of W .

To show (4.4), we observe that $\tau_\sigma \circ W(v)$ is a solution of (4.10) corresponding to $\tau_\sigma(v)$. From the Lipschitz property of the map W , we have uniqueness of solution. Hence

$$W \circ \tau_\sigma(v) = \tau_\sigma \circ W(v).$$

\square

5 An Observable Regularity Criterion on the Weak Attractor

In this section we establish that the Leray-Hopf weak solution u to (4.2) on weak attractor is in fact a strong solution. We restrict our attention to the 3D NSE only for simplicity. Similar result can be established using the same method for the 3D Boussinesq system as well.

The regularity criterion for the 3D NSE on the weak attractor is as follows:

Theorem 5.1 *Let u be a Leray-Hopf weak solution of (4.2) on the weak attractor. Let $h_0 > 0$ be defined as*

$$h_0^{-2} = \max \left\{ \frac{1}{c\lambda_1}, \frac{c|f|^4}{\nu^8\lambda_1^2} \right\} \quad (f \text{ is the body force}).$$

Assume there exists $0 < h \leq h_0$ for which

$$h^2 M_{h,u}^4 \leq \frac{\nu^4}{c}.$$

Then $u(t)$ is regular for all $t \in \mathbb{R}$.

We will use the following three lemmas to prove Proposition 5.5, which will help us obtain our regularity criterion stated above. The first lemma is obtained by repeating the arguments of Lemma 2.8.

Lemma 5.2 *Let u be the general Leray-Hopf weak solution of (4.1) and w be the strong solution to (4.10) as given in [11]. Let $\tilde{w} = w - u$. Then,*

$$\begin{aligned} |\tilde{w}(t)|_0^2 + 2\nu \int_0^t \|\tilde{w}(s)\|_0^2 ds &\leq |\tilde{w}(0)|_0^2 + 2 \int_0^t b(\tilde{w}(s), \tilde{w}(s), w(s)) ds \\ &\quad - 2\mu \int_0^t (I_h(\tilde{w}(s), \tilde{w}(s)))_0 ds \end{aligned} \quad (5.1)$$

Bounding each term in (5.1) as in Theorem 4.3 and applying Lemma 6.6, we obtain the second lemma.

Lemma 5.3 *Let u be a general Leray-Hopf weak solution of (4.1) and w be a strong solution to (4.10) as given in [11]. Also, let $\tilde{w} = w - u$. Then,*

$$|\tilde{w}(t)|_0^2 \leq 4M^2 e^{-\mu\left(t - \frac{1}{\mu}\right)}. \quad (5.2)$$

where

$$M = \max \left\{ \sup_{0 \leq t \leq \frac{1}{\mu}} |w(t)|_0, \sup_{0 \leq t \leq \frac{1}{\mu}} |u(t)|_0 \right\}. \quad (5.3)$$

Lemma 5.4 *Let u be a Leray-Hopf weak solution of (4.1) on the weak attractor and w be the strong solution of (4.10) as given in the Theorem 4.3. Let $\tilde{w} = w - u$ and $\tilde{w}_\tau(t) = \tilde{w}(t - \tau - 1/\mu)$. Then*

$$|\tilde{w}_\tau(t)|_0^2 \leq 4M^2 e^{-\mu(t-\frac{1}{\mu})}. \quad (5.4)$$

where M is as in (5.3).

Proof Dropping the middle term on the LHS of (??), we obtain

$$|\tilde{w}(t)|_0^2 + \mu \int_0^t |\tilde{w}(s)|_0^2 ds \leq |\tilde{w}(0)|_0^2 \quad (5.5)$$

Changing the interval of integration from $[0, t]$ to $[t - \tau - 2/\mu, t - \tau - 1/\mu]$, we obtain

$$|\tilde{w}(t - \tau - 1/\mu)|_0^2 + \mu \int_{t-\tau-2/\mu}^{t-\tau-1/\mu} |\tilde{w}(s)|_0^2 ds \leq |\tilde{w}(t - \tau - 2/\mu)|_0^2 \quad (5.6)$$

Applying the definition of \tilde{w}_τ , (5.6) can be rewritten as

$$|\tilde{w}_\tau(t)|_0^2 + \mu \int_{t-1/\mu}^t |\tilde{w}_\tau(s)|_0^2 ds \leq |\tilde{w}_\tau(t - 1/\mu)|_0^2. \quad (5.7)$$

Proceeding exactly as in Lemma 5.3, we obtain the statement of the lemma. \square

Proposition 5.5 *Let u be a general Leray-Hopf weak solution of (4.1) and w be a strong solution to (4.10) as given in Theorem 4.3. Also, let $\tilde{w} = w - u$ and $\tilde{w}_\tau(t) = \tilde{w}(t - \tau - 1/\mu)$. Then, $|\tilde{w}(t)|_0 = 0 \forall t \in \mathbb{R}$.*

Proof It is enough to show that for any $t_0 \in \mathbb{R}$, $|\tilde{w}(t_0)|_0^2 \leq \epsilon$. Given $t_0 \in \mathbb{R}$, let $\tau \in \mathbb{R}$ be such that

$$\left(4M^2 e^{-\mu\tau}\right) e^{-\mu(t_0)} \leq \epsilon, \quad (5.8)$$

where M is as in (5.3). Then, from Lemma 5.4, we have

$$|\tilde{w}_\tau(t_0 + \tau + 1/\mu)|_0^2 \leq \left(4M^2 e^{-\mu\tau}\right) e^{-\mu(t_0)}. \quad (5.9)$$

Applying (5.8) and the definition of \tilde{w}_τ , we obtain

$$|\tilde{w}(t_0)|_0^2 \leq \epsilon. \quad (5.10)$$

\square

From the above theorem we see that on the weak attractor, $u = w$, where u is a general Leray-Hopf weak solution to (4.1) and w be a strong solution to (4.10). This would in turn mean that u on the weak attractor can be shown to be regular whenever we can construct such a w , giving us a regularity criterion summarized in Theorem 5.1.

5.1 Weakened Regularity Condition

In Theorem 4.3, the condition we imposed on the data to show regularity of w can be weakened. In this section, we will show that the weakened condition on the data is sufficient to obtain the same regularity result as that in Theorem 4.3(1).

We define $K_h := K_{h,\tau_0}$, for $\tau_0 > 0$ and Holder conjugates $p \geq 3$ and $q \geq 1$, as

$$K_h = \sup_{t \in [-\infty, T]} \left(\int_t^{t+\tau_0} \|I_h u\|^{2p} ds \right)^{1/2p}. \quad (5.11)$$

The weakened condition is obtained in terms of K_h instead of ρ . The criterion is as follows:

Theorem 5.6 *Let u be a Leray-Hopf weak solution of (4.2) on the weak attractor and K_h be defined as in (5.11). Let $h_0 > 0$ be defined as*

$$h_0^{-2} = \max \left\{ c\lambda_1, \frac{c|f|^4}{\nu^5 \lambda^2} \right\}.$$

Assume there exists $\tau_0 > 0$ and $0 < h \leq h_0$ for which $K_h < \infty$ and

$$h^2 \max \left\{ \frac{\nu}{ch_0^2}, \left(\frac{CK_h^4}{q^{\frac{2}{q}}} \right)^{\frac{p}{p-2}} \right\} \leq \frac{\nu}{c}.$$

Then $u(t)$ is regular for all $t \in \mathbb{R}$.

To obtain the above result, we first prove the following proposition.

Proposition 5.7 *Let \tilde{I}_h be a modified general type-I interpolant, K_h be defined as in (5.11) and $0 \leq T \leq \infty$. Let $h_0 > 0$ be given by*

$$h_0^{-2} = \max \left\{ c\lambda_1, \frac{c|f|^4}{\nu^8 \lambda^2} \right\} \quad (5.12)$$

Assume there exists $\tau_0 > 0$ and $0 < h \leq h_0$ such that $K_h < \infty$ and

$$h^2 \left(\frac{CK_h^4}{q^{\frac{1}{q}}} \right)^{\frac{p}{p-2}} \leq \frac{\nu}{c}. \quad \text{or equivalently} \quad \left(\frac{CK_h^4}{q^{\frac{1}{q}}} \right)^{\frac{p}{p-2}} \leq \frac{\nu}{ch^2}. \quad (5.13)$$

Let μ be chosen such that

$$\max \left\{ \frac{v}{ch_0^2}, \left(\frac{CK_h^4}{q^{\frac{1}{q}}} \right)^{\frac{p}{p-2}} \right\} \leq \mu \leq \frac{v}{4ch^2}. \quad (5.14)$$

Then, the data assimilated fluid velocity is regular and satisfies

$$\|w\|_0 \leq M_h, \quad (5.15)$$

where

$$M_h^2 = \frac{8|f|^2}{\lambda_1 v^2} + \frac{CK_h^2 \mu^{1/p}}{q^{1/q}} \left(\frac{1}{1 - e^{-\frac{v\lambda_1 p}{4}\tau_0}} \right)^{1/p}. \quad (5.16)$$

Proof Taking the inner product of the 3-D NSE with $A_0 w_n$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|w_n\|_0^2 + v |A_0 w_n|_0^2 \\ &= -(B_0(w_n, w_n), A_0(w_n))_0 + \mu(w_n - I_h(w_n), A_0(w_n))_0 \\ & \quad - \mu \|w_n\|_0^2 + \mu(I_h u, A_0(w_n))_0 + (f, A_0(w_n)) \end{aligned} \quad (5.17)$$

Applying Cauchy-Schwartz and Young's inequality, we obtain

$$\mu(I_h u, A_0(w_n))_0 \leq \mu \|I_h u\|_0^2 + \frac{\mu}{4} \|w_n\|_0^2.$$

$$|(f, A_0 w_n)_0| \leq |f|_0 |A_0 w_n|_0 \leq \frac{1}{v} |f|_0^2 + \frac{v}{4} |A_0 w_n|_0^2$$

Bounding the remaining terms similar to Theorem 2.4 and inserting the above estimate into (5.17), we obtain

$$\frac{d}{dt} \|w_n\|_0^2 + \left(\mu - \frac{c}{v^3} \|w_n\|_0^4 \right) \|w_n\|_0^2 + \frac{v}{2} |A_0 w_n|_0^2 \leq \frac{2}{v} |f|_0^2 + 2\mu \|I_h u\|_0^2 \quad (5.18)$$

Let $[0, T_1]$ be the maximal interval on which $\|w_n(t)\| \leq M_h$ for $t \in [0, T_1]$. Note that $T_1 > 0$ exists because we have $w_n(0) = 0$. Assume that $T_1 < T$. Then by continuity, we must have $\|w_n(T_1)\| = M_h$. Applying (5.14) and (5.16) to (5.18) and dropping the last term on the LHS of (5.18), we obtain

$$\frac{d}{dt} \|w_n\|_0^2 + \frac{\mu}{2} \|w_n\|_0^2 \leq \frac{2}{v} |f|_0^2 + 2\mu \|P_n(v)\|_0^2. \quad (5.19)$$

Applying Gronwall's and Holder's inequalities, for any $t \in [0, T_1]$ we obtain

$$\begin{aligned}\|w_n(t)\|^2 &\leq \frac{4|f|^2}{\mu\nu} + 2\mu \int_0^t e^{-\frac{\mu}{2}(t-s)} \|I_h u\|^2 ds \\ &\leq \frac{4|f|^2}{\mu\nu} + 2\mu \left(\int_0^t e^{-\frac{\mu q}{4}(t-s)} ds \right)^{1/q} \left(\int_0^t e^{-\frac{\mu p}{4}(t-s)} \|I_h u\|^{2p} ds \right)^{1/p} \\ &\leq \frac{4|f|^2}{\mu\nu} + \frac{C\mu^{1/p}}{q^{1/q}} \left(\int_0^t e^{-\frac{\mu p}{4}(t-s)} \|I_h u\|^{2p} ds \right)^{1/p}\end{aligned}\quad (5.20)$$

Where $1 \leq p, q \leq \infty$ are Holder conjugates. We now try to bound the second term on the RHS of the above inequality.

Case 1: $t \geq \tau_0$.

Let $k = \left\lfloor \frac{t}{\tau_0} \right\rfloor$ be the largest integer such that $k\tau_0 \leq t$. Therefore,

$$t = k\tau_0 + \epsilon, \text{ where } 0 \leq \epsilon \leq \tau_0. \quad (5.21)$$

Let $\alpha = \frac{p\mu}{4}$. Then, from (5.11) and (5.21), we may write

$$\begin{aligned}\int_0^t e^{-\frac{\mu p}{4}(t-s)} \|I_h u\|^{2p} ds &\leq \sum_{j=1}^k \int_{(j-1)\tau_0}^{j\tau_0} e^{-\alpha(t-s)} \|I_h(u)\|^{2p} ds \\ &\quad + \int_{n\tau_0}^t e^{-\alpha(t-s)} \|I_h(u)\|^{2p} ds \\ &\leq e^{-\alpha\epsilon} \sum_{j=1}^k e^{-\alpha(n-j)\tau} \int_{(j-1)\tau_0}^{j\tau_0} \|I_h(u)\|^{2p} ds \\ &\quad + \int_{n\tau_0}^{n\tau_0+\epsilon} \|I_h(u)\|^{2p} ds \\ &\leq e^{-\alpha\epsilon} K_h^{2p} \sum_{m=0}^{k-1} e^{m\alpha\tau_0} + K_h^{2p} \leq \frac{2K_h^{2p}}{1 - e^{\alpha\tau_0}}.\end{aligned}\quad (5.22)$$

Case 2: $t \leq \tau_0$.

From (5.11), we see that

$$\int_0^t e^{-\alpha(t-s)} \|I_h u\|^2 ds \leq \int_0^t \|I_h u\|^2 ds \leq \int_0^{\tau_0} \|I_h u\|^2 ds \leq K_h^{2p} \leq \frac{2K_h^{2p}}{1 - e^{\alpha\tau_0}} \quad (5.23)$$

Applying (5.22), (5.23) and (5.16) to (5.20), we obtain that for any $t \in [0, T_1]$

$$\begin{aligned} \|w_n(t)\|^2 &\leq \frac{4|f|^2}{\mu v} + \frac{CK_h^2 \mu^{1/p}}{q^{1/q}} \left(\frac{2}{1 - e^{-\frac{\mu p}{4} \tau_0}} \right)^{1/p} \leq \frac{4|f|^2}{\lambda_1 v^2} \\ &\quad + \frac{CK_h^2 \mu^{1/p}}{q^{1/q}} \left(\frac{2}{1 - e^{-\frac{v \lambda_1 p}{4} \tau_0}} \right)^{1/p} \leq \frac{1}{2} M_h^2. \end{aligned} \quad (5.24)$$

This contradicts the fact that $\|w_n(T_1)\| = M_h$. Therefore $T_1 \geq T$ and consequently, $\|w_n(t)\| \leq M_h$ for all $t \in [0, T]$. Passing to the limit as $n \rightarrow \infty$, we obtain the desired conclusion for w . \square

Similar to Theorem 5.1, using Lemma 5.2 - Lemma 5.4, we obtain the regularity criterion given in Theorem 5.6

Remark 5.1 Note that in Theorem 5.6, the definition of M_h is given by (5.16), which is not the same as (2.48). Also, the regularity criterion given by Theorem 5.6 is in the spirit of the criterion given by Corollary 5.2 in [31]. However, our condition solely depends on the observed data.

6 Appendix

We now present the proofs of a few novel results that have been used to prove theorems in this paper. Although some of the ideas of the proofs are borrowed from [6], to the best of our knowledge, these estimates concerning the interpolant operators are new and may have independent interest.

Recall that for volume interpolation, we divided our domain into smaller sub domains (cuboids Q_α) of diameter h and indexed by the set \mathcal{J} . Let us define the set $\mathcal{E} \subset \mathcal{J}$ as

$$\mathcal{E} = \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{J} : \alpha_3 = 1 \text{ or } \alpha_3 = \sqrt[3]{N} \right\}. \quad (6.1)$$

\mathcal{E} represents the collection of sub domains touching the top and bottom boundaries ($x_3 = 0$ and $x_3 = 1$). For sub domains Q_α with $\alpha \in \mathcal{E}$, we introduce a modification $Q_{\epsilon, \alpha}$ for $0 < \epsilon < h$, given by

$$Q_{\epsilon, \alpha} = \begin{cases} ((\alpha_1 - 1) * h_L, \alpha_1 * h_L) \times ((\alpha_2 - 1) * h_L, \alpha_2 * h_L) \times (\epsilon, h_1) & , \text{ for } \alpha_3 = 1 \\ ((\alpha_1 - 1) * h_L, \alpha_1 * h_L) \times ((\alpha_2 - 1) * h_L, \alpha_2 * h_L) \times (1 - h_1, 1 - \epsilon) & , \text{ for } \alpha_3 = \sqrt[3]{N}, \end{cases} \quad (6.2)$$

where $h_x = x / \sqrt[3]{N}$. We now define the \tilde{I}_h as

$$\tilde{I}_h(v)(x) = \sum_{\alpha \in \mathcal{J}} \tilde{v}_\alpha \phi_\alpha(x), \quad v \in H^1(\Omega) \quad (6.3)$$

where

$$\phi_\alpha = \rho_\epsilon * \psi_{Q_\alpha}, \quad \rho_\epsilon(x) = \epsilon^{-3} \rho(x/\epsilon), \quad \bar{v}_\alpha = \frac{1}{|Q_\alpha|} \int_{Q_\alpha} v(x) dx. \quad (6.4)$$

$$\psi_{Q_\alpha}(x) = \begin{cases} \chi_{Q_\alpha} & , \text{ for } \alpha \notin \mathcal{E} \\ \chi_{Q_{\epsilon,\alpha}} & , \text{ for } \alpha \in \mathcal{E} \end{cases} \quad (6.5)$$

$$\rho(x) = \begin{cases} K_0 \exp\left(\frac{-1}{1-x^2}\right) & , \text{ for } |x| < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (6.6)$$

and

$$(K_0)^{-1} = \int_{|x|<1} \exp\left(\frac{-1}{1-x^2}\right) dx.$$

We set $\epsilon = h/10$.

Remark 6.1 $\|\tilde{I}_h\|_0$ is well defined since the characteristic function of each sub domain has been mollified. The characteristic function of sub domains touching the top and bottom boundaries have been modified ($Q_{\epsilon,\alpha}$) so that it's support is an “ ϵ distance” away from the top and the bottom boundaries. This is done so that after mollification, the modified characteristic function respects the Dirichlet boundary condition at $x_3 = 0, 1$.

Lemma 6.1 Let ϕ_α and ρ be defined as in (6.4) and (6.6) respectively. Then, for $i = 1, 2$ and 3

$$|\partial_{x_i} \phi_\alpha|^2 \leq Ch \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2$$

Proof Recall that $\phi_\alpha = \rho_\epsilon * \psi_{Q_\alpha}$. Applying Young's inequality for convolutions, we obtain

$$\begin{aligned} |\partial_{x_i} (\rho_\epsilon * \psi_{Q_\alpha})|^2 &= |(\partial_{x_i} \rho_\epsilon) * \psi_{Q_\alpha}|^2 \\ &\leq \|\partial_{x_i} \rho_\epsilon\|_{L^1(\Omega)}^2 |\psi_{Q_\alpha}|^2 \end{aligned} \quad (6.7)$$

Now, we look at each term on the RHS of (6.7). Differentiating the second Eq. in (6.4) with x_i and using the fact that $\epsilon = h/10$, we obtain

$$\begin{aligned} \|\partial_{x_i} \rho_\epsilon\|_{L^1(\Omega)}^2 &= \left(\int_{|x| \leq \epsilon} \left| \epsilon^{-4} \partial_{x_i} \rho(x/\epsilon) \right| dx \right)^2 \leq C \epsilon^{-2} \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \\ &\leq Ch^{-2} \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2. \end{aligned} \quad (6.8)$$

From the definition of χ_{Q_α} , we readily obtain

$$|\psi_{Q_\alpha}|^2 \leq \int_{Q_\alpha} 1^2 dx = \frac{|\Omega|}{N} \leq h^3 \quad (6.9)$$

Combining (6.7), (6.8) and (6.9) and noting that $|\Omega|/N \leq h^3$, we obtain

$$|\partial_{x_i} \phi_\alpha|^2 \leq \frac{C|\Omega|h^{-2}}{N} \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \leq Ch \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \quad (6.10)$$

□

We now prove the following theorem. The proof technique was borrowed from [6], where it was used to prove a similar statement for the two dimensional case.

Theorem 6.2 *Let $\mathcal{K} = \{-1, 0, 1\}^3$ and \tilde{I}_h be as in (6.3) and $K_\rho = \left(\sum_{i=1}^3 \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \right)^{1/2}$. Then ,*

$$|\tilde{I}_h(v)|_{H^1(\Omega)}^2 \leq Ch K_\rho^2 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \quad \forall v \in H^1(\Omega). \quad (6.11)$$

Proof We set $\epsilon = h/10$. Hence, it follows immediately that $|\phi_\alpha \phi_\beta| = 0$ for $\alpha - \beta \notin \mathcal{K}$, where $\mathcal{K} = \{-1, 0, 1\}^3$. Differentiating (6.3) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\partial_{x_i} \tilde{I}_h(v)|^2 &\leq \int_\Omega \sum_{\gamma \in \mathcal{K}} \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha \partial_{x_i} \phi_\alpha(x)| |\bar{v}_{\alpha+\gamma} \partial_{x_i} \phi_{\alpha+\gamma}(x)| dx \\ &\leq \int_\Omega \sum_{\gamma \in \mathcal{K}} \left(\sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 |\partial_{x_i} \phi_\alpha(x)|^2 \right)^{1/2} \left(\sum_{\alpha \in \mathcal{J}} |\bar{v}_{\alpha+\gamma}|^2 |\partial_{x_i} \phi_{\alpha+\gamma}(x)|^2 \right)^{1/2} dx \\ &\leq \int_\Omega \sum_{\gamma \in \mathcal{K}} \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 |\partial_{x_i} \phi_\alpha(x)|^2 dx \leq 27 \int_\Omega \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 |\partial_{x_i} \phi_\alpha(x)|^2 dx \\ &\leq 27 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 |\partial_{x_i} \phi_\alpha|^2 \end{aligned} \quad (6.12)$$

Applying (6.10) to (6.12), we may write

$$|\partial_{x_i} \tilde{I}_h(v)|^2 \leq \frac{C|\Omega|h^{-2}}{N} \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq Ch \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \quad (6.13)$$

Now, summing over i , we obtain

$$|\tilde{I}_h(v)|_{H^1(\Omega)}^2 \leq \frac{C|\Omega|h^{-2}}{N} K_\rho^2 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq Ch K_\rho^2 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \quad \forall v \in H^1(\Omega). \quad (6.14)$$

□

Corollary 6.3 Let \tilde{I}_h be as in (6.3) and $K_\rho = \left(\sum_{i=1}^3 \|\partial_{x_i} \rho\|_{L^\infty(\Omega)}^2 \right)^{1/2}$. Then

$$\|\tilde{I}_h(v)\| \leq CK_\rho \|v\| \quad \forall v \in H^1(\Omega) \text{ and } \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \|v\|^2. \quad (6.15)$$

Proof We first look at the term $\sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2$ in (6.13). Using the definition of \bar{v}_α given in (2.14), Holder's inequality and Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\begin{aligned} \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 &\leq \sum_{\alpha \in \mathcal{J}} \left(\left| \frac{1}{|Q_\alpha|} \int_{Q_\alpha} v(x) dx \right| \right)^2 \leq \frac{N^2}{|\Omega|^2} \sum_{\alpha \in \mathcal{J}} \|v\|_{L^1(Q_\alpha)}^2 \\ &\leq \frac{N^2}{|\Omega|^2} \sum_{\alpha \in \mathcal{J}} |Q_\alpha|^{5/3} \|v\|_{L^6(Q_\alpha)}^2 \\ &\leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \sum_{\alpha \in \mathcal{J}} \|v\|_{H^1(Q_\alpha)}^2 \leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^3 \left\| (\partial_{x_i} v) \chi_{Q_\alpha} \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \sum_{i=1}^3 \left(\sum_{\alpha \in \mathcal{J}} (\partial_{x_i} v) \chi_{Q_\alpha}, \sum_{\alpha \in \mathcal{J}} (\partial_{x_i} v) \chi_{Q_\alpha} \right) \end{aligned} \quad (6.16)$$

Next, using the fact that for $\alpha, \beta \in \mathcal{J}$,

$$\chi_{Q_\alpha} \chi_{Q_\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \chi_{Q_\alpha} & \text{if } \alpha = \beta, \end{cases} \quad (6.17)$$

we can simplify the expression on the RHS of (6.16) to obtain

$$\sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \sum_{i=1}^3 \|\partial_{x_i} v\|_{L^2(\Omega)}^2 \leq \frac{CN^{1/3}}{|\Omega|^{1/3}} \|v\|^2. \quad (6.18)$$

Applying (6.18) to the first inequality in (6.14) and summing over i , we obtain

$$\|\tilde{I}_h(v)\| \leq CK_\rho \|v\| \quad \forall v \in H^1(\Omega). \quad (6.19)$$

□

Next, we briefly try to see why \tilde{I}_h is a type-I interpolant. We first look at a lemma, which is a modified version of a similar result in [18].

Lemma 6.4 Let $U = \{(p, q, r) \in \mathbb{R}^3 : |p - a| < 0.5h, |q - b| < 0.5h, |r - c| < 0.5h, \text{ for } a, b, c \in \mathbb{R}\}$ be a cube of side length $h > 0$ and center $m = (a, b, c)$

and $u \in W^{1,p}(U)$. Assume $1 \leq p \leq \infty$. Then there exists a C , depending on only on p , such that

$$\|u - (u)_U\|_p \leq Ch \|\nabla u\|_p,$$

where

$$(u)_U = \frac{1}{|U|} \int_U u(x) dx$$

Proof From Theorem 2.2, it follows that the statement is true for a cube V of side length one given by $V = \{(p, q, r) \in \mathbb{R}^3 : |p| < 0.5, |q| < 0.5, |r| < 0.5\}$. For $y \in V$, we define $v \in W^{1,p}(V)$ by

$$v(y) = u(hy + m).$$

From Theorem 2.2 we obtain

$$\|v - (v)_V\|_p \leq C \|\nabla v\|_p.$$

Changing variables, we obtain the statement of the theorem. \square

Theorem 6.5 Let \tilde{I}_h be the smoothed volume interpolant as defined in (6.3). Then \tilde{I}_h is a type-I interpolant.

Proof Recall that $|\phi_\alpha \phi_\beta| = 0$ for $\alpha - \beta \notin \mathcal{K}$, where $\mathcal{K} = \{-1, 0, 1\}^3$. Repeating the arguments in (6.12) and applying Young's inequality for convolutions, we obtain

$$|\tilde{I}_h(v)|^2 \leq 27 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 |\phi_\alpha|^2 \leq 27 \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \|\rho_\epsilon\|_{L^1(\Omega)}^2 |\psi_{Q_\alpha}|^2 \quad (6.20)$$

Observe that

$$\begin{aligned} \|\rho_\epsilon\|_{L^1(\Omega)}^2 &= \left(\int_{|x| \leq \epsilon} |\epsilon^{-3} \rho(x/\epsilon)| dx \right)^2 \\ &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \end{aligned} \quad (6.21)$$

Applying (6.21) and the middle inequality of (6.9) to (6.20), we obtain

$$\begin{aligned} |\tilde{I}_h(v)|^2 &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \frac{|\Omega|}{N} \sum_{\alpha \in \mathcal{J}} |\bar{v}_\alpha|^2 \leq C \|\rho\|_{L^\infty(\Omega)}^2 \frac{|\Omega|}{N} \sum_{\alpha \in \mathcal{J}} \left(\left| \frac{1}{|Q_\alpha|} \int_{Q_\alpha} v(x) dx \right| \right)^2 \\ &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \frac{N}{|\Omega|} \sum_{\alpha \in \mathcal{J}} \|v\|_{L^1(Q_\alpha)}^2 \leq C \|\rho\|_{L^\infty(\Omega)}^2 \sum_{\alpha \in \mathcal{J}} \|v\|_{L^2(Q_\alpha)}^2 \\ &\leq C \|\rho\|_{L^\infty(\Omega)}^2 |v|^2 \end{aligned} \quad (6.22)$$

We now look at $|v - \tilde{I}_h(v)|$.

$$|v - \tilde{I}_h(v)|^2 \leq \left| \sum_{\alpha \in \mathcal{J}} (v - \bar{v}_\alpha) \phi_\alpha \right|^2 \quad (6.23)$$

After repeating the arguments in (6.20) and (6.21), we obtain

$$\begin{aligned} |v - \tilde{I}_h(v)|^2 &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \sum_{\alpha \in \mathcal{J}} |(v - \bar{v}_\alpha) \phi_\alpha|^2 \\ &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \frac{|\Omega|}{N} \sum_{\alpha \in \mathcal{J}} |v - \bar{v}_\alpha|^2 \|\phi_\alpha\|_{L^\infty(\Omega)}^2 \\ &\leq C \|\rho\|_{L^\infty(\Omega)}^2 \frac{N}{|\Omega|} \sum_{\alpha \in \mathcal{J}} |v - \bar{v}_\alpha|^2 \|\rho_\epsilon\|_{L^\infty(\Omega)}^2 |\psi|_{L^1(\Omega)}^2 \\ &\leq Ch^{-6} \frac{|\Omega|^2}{N^2} \|\rho\|_{L^\infty(\Omega)}^4 \sum_{\alpha \in \mathcal{J}} |v - \bar{v}_\alpha|^2 \end{aligned} \quad (6.24)$$

Lastly, applying Lemma 6.4 and noting that $|\Omega|/N \leq h^3$, we obtain

$$|v - \tilde{I}_h(v)|^2 \leq Ch^2 \|\rho\|_{L^\infty(\Omega)}^4 \sum_{\alpha \in \mathcal{J}} \|v\|_{H^1(Q_\alpha)}^2 \leq Ch^2 \|\rho\|_{L^\infty(\Omega)}^4 \|v\|_{H^1(\Omega)}^2 \quad (6.25)$$

Therefore, from (6.22) and (6.25), we see that \tilde{I}_h is a type-I interpolant. \square

We now state and prove the following lemma due to E.S Titi (via private communication). ‘

Lemma 6.6 *Let $y : \mathbb{R} \rightarrow \mathbb{R}$ be a real valued function, with $y(t) \geq 0 \ \forall t \in \mathbb{R}$. Let $K \geq 0$. If y satisfies, for $0 \leq s \leq t$ and a constant $\mu > 0$, the inequality*

$$y(t) + \mu \int_s^t y(\tau) d\tau \leq y(s) + K, \quad (6.26)$$

then

$$y(t) \leq \mu e^{-\mu(t-\frac{1}{\mu})} \int_0^{\frac{1}{\mu}} y(\tau) d\tau + 2K, \quad \forall t \geq 1/\mu. \quad (6.27)$$

Proof Let

$$\psi(t) = \int_{t-\frac{1}{\mu}}^t y(\tau) d\tau. \quad (6.28)$$

Choosing $s = (t - 1/\mu)$, we can then rewrite (6.26) as

$$\frac{\partial \psi(t)}{\partial t} + \mu \psi(t) \leq K. \quad (6.29)$$

Thus, from standard Gronwall over the interval $[\sigma, t]$ inequality, we obtain

$$\psi(t) \leq \psi(\sigma) e^{-\mu(t-\sigma)} + \frac{K}{\mu}. \quad (6.30)$$

Choosing $\sigma = 1/\mu$, we obtain

$$\psi(t) \leq \psi(1/\mu) e^{-\mu(t-\frac{1}{\mu})} + \frac{K}{\mu}. \quad (6.31)$$

Now, keeping only the first term on the LHS of (6.26), integrating with respect to s over the interval $[t - \frac{1}{\mu}, t]$ and applying (6.31), we see that

$$\begin{aligned} \left(\frac{1}{\mu}\right) y(t) &\leq \int_{t-\frac{1}{\mu}}^t y(\tau) d\tau + \frac{K}{\mu} \leq \psi(1/\mu) e^{-\mu(t-\frac{1}{\mu})} + \frac{2K}{\mu} \\ &\leq e^{-\mu(t-\frac{1}{\mu})} \int_0^{\frac{1}{\mu}} y(\tau) d\tau + \frac{2K}{\mu}. \end{aligned} \quad (6.32)$$

Multiplying (6.32) by a factor of μ , we obtain (6.27). □

If, in (6.30), we choose $\sigma = t/2$, we obtain the following Corollary

Corollary 6.7 *Let $y : \mathbb{R} \rightarrow \mathbb{R}$ be a real valued function, with $y(t) \geq 0 \forall t \in \mathbb{R}$. Let $K \geq 0$. If y satisfies, for $0 \leq s \leq t$ and a constant $\mu > 0$, the inequality*

$$y(t) + \mu \int_s^t y(\tau) d\tau \leq y(s) + K, \quad (6.33)$$

then

$$y(t) \leq \mu e^{-\mu(t/2)} \int_{t/2-1/\mu}^{t/2} y(\tau) d\tau + 2K, \quad \forall t \geq 1/\mu + 2K. \quad (6.34)$$

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