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SPACE AND TIME ANALYTICITY FOR INVISCID EQUATIONS OF FLUID DYNAMICS

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Abstract. In this article, we examine the temporal regularity of inviscid fluid flow on a torus as viewed in Eulerian variables. We consider several models for inviscid flow, and in each case show that when the initial condition is Gevrey regular (a stronger condition than real analytic), there exists a complex neighbourhood, $\mathcal{R}$, of the initial time over which there is a unique holomorphic (in time) solution of the complexified version of the governing system, which remains Gevrey regular (in space) at each complex time in $\mathcal{R}$. In addition, we obtain *explicit estimates* on the region $\mathcal{R}$, and therefore on the persistence time of solutions in the analytic Gevrey class.

Our proof technique is based on the seminal work of Foias and Temam (1989), where they introduced the so-called analytic Gevrey class technique for the Navier–Stokes equations. Our arguments are general enough to apply to various models for inviscid flow. In particular, we demonstrate the full analysis with the Euler equations, and extend our results to the inviscid forms of the surface quasi-geostrophic equation (SQG), the Boussinesq equations and the magnetohydrodynamic equations (MHD), as well as to scalar equations with an analytic nonlinearity.

1. Introduction

It is well-known that solutions to a large class of dissipative equations are analytic in space and time [2, 9, 10, 24–26, 29, 41, 42]. In fluid dynamics, the space analyticity radius has a physical interpretation: it denotes a length scale below which the viscous effects dominate and the Fourier spectrum decays exponentially, while above it, the inertial effects dominate [21]. This fact concerning exponential decay can be used to show that the Galerkin approximation converges exponentially to the exact solution [20]. Other applications of analyticity radius occur in establishing sharp temporal decay rates of solutions in higher Sobolev norms [8,42], establishing geometric regularity criteria for solutions, and in measuring the spatial complexity of fluid flows [28,32]. Likewise, time analyticity also has several important applications, including establishing backward uniqueness of trajectories [14], parameterization of turbulent flows by finitely many space-time points [33], and numerical determination of the attractor [27].

Space and time analyticity of inviscid equations, particularly the Euler equations, has received considerable attention recently (as well as in the past). Space analyticity for Euler, in the Eulerian variables, was considered for instance in [3,4,34,35,38], while in [1,22,37] real analyticity in the time (and space) variable is established using harmonic analysis tools. In the above mentioned works, the initial data are

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taken to be analytic in the space variable. By contrast, in a recent work [17], it is shown that the Lagrangian trajectories are real analytic (in time), even though the initial velocity fields are slightly more regular than Lipschitz in the space variable. Similar results also appear elsewhere; see for instance in [46, 48, 50] and the references therein. Additionally, the contrast between the analytic properties in the Eulerian and Lagrangian variables has been considered recently in [15].

In this paper, we show that solutions of the Euler, as well as the inviscid versions of the SQG, Boussinesq, MHD, and similar equations with an analytic nonlinearity, when given analytic initial data, extend as holomorphic solutions of the complexified versions of the equations taking values in a suitable analytic Gevrey class of functions of the space variable. Belonging to an analytic Gevrey class is sufficient for a function to be analytic, and so this immediately establishes that the solutions extend as analytic functions of time and space. In contrast to, for instance, the results in [1, 22], where the authors show the *real time-analyticity* in a region defined *implicitly* by the flow map generated by the solutions, we obtain *holomorphic* extensions and obtain *explicit* estimates on the domain of (time) analyticity. Our approach follows [26], in which the desired results are obtained for the Navier–Stokes equations. We also make use of the ideas introduced in [38] and [34].

Note that unlike their "real" counterparts, the complexified inviscid models are not known to conserve "energy", due to the fact that the complexified nonlinear terms do not in general possess equivalent cancellation properties. However, as in [38], we observe a mild dissipation due to working in an analytic Gevrey class setting, which is enough for the local existence of the complexified versions of these inviscid models.

The paper is organized as follows. In Section 2 we introduce the notation and definitions we will use and state some of the associated classical results that we will use. In the following sections, we apply our analysis first on the Euler equations, followed by the inviscid surface quasi-geostrophic equations, the inviscid Boussinesq equations, the inviscid magnetohydrodynamic equations and an equation with an analytic nonlinearity.

2. Preliminaries

We will consider several different evolutionary equations, and in each case, we will study the flow on a d-dimensional torus; i.e. we take the space domain to be $\Omega = [0, l]^d, d \in \mathbb{N}$, supplemented with periodic boundary conditions (with period l). For notational simplicity, fix

$$l = 2\pi, \text{ and therefore, } \kappa_0 := \frac{2\pi}{l} = 1.$$

We denote the inner product on $L^2(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d,\ \int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty \right\}$ by $\langle \cdot\,, \cdot \rangle$ and the corresponding L^2-norm by $\|\cdot\|$. As usual, the Euclidean length of a vector in $\mathbb{R}^d(\text{ or } \mathbb{C}^d)$ is denoted by $|\cdot|$.

For a function $\boldsymbol{u} : \Omega \to \mathbb{R}^d(\text{ or } \mathbb{C}^d)$, its Fourier coefficients are defined by

$$\widehat{\boldsymbol{u}}(\boldsymbol{k}) = \frac{1}{(2\pi)^d} \int_\Omega \boldsymbol{u}(\boldsymbol{x}) e^{-i\kappa_0 \boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{x} \ (\boldsymbol{k} \in \mathbb{Z}^d).$$

By the Parseval identity,

$$\|\boldsymbol{u}\|^2 = (2\pi)^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2.$$

As usual, we define the Sobolev spaces $H^s(\Omega)$ for $s \in \mathbb{R}$ by

$$H^s(\Omega) = \{\boldsymbol{u} \in L^2(\Omega) : \|\boldsymbol{u}\|_{H^s(\Omega)} := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+|\boldsymbol{k}|^2)^s |\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2 < \infty\}.$$

For each model we consider, if the space average of the initial data is zero, then the space average of the solution will remain zero at all future times. Therefore, we will always make the assumption that the elements of the phase space have space average zero. In terms of the Fourier coefficients this amounts to the condition $\widehat{\boldsymbol{u}}(\boldsymbol{0}) = \boldsymbol{0}$ (which is then preserved under the evolution).

We denote

$$\dot{L}^2(\Omega) = \left\{\boldsymbol{u} \in L^2(\Omega) : \int_\Omega \boldsymbol{u}(\boldsymbol{x})\, d\boldsymbol{x} = 0, \text{ or equivalently, } \widehat{\boldsymbol{u}(\boldsymbol{k})} = 0\right\}.$$

As we are considering incompressible flows, we will define the phase space, H, as

$$H = \left\{\boldsymbol{u} \in \dot{L}^2(\Omega),\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0}\right\},$$

where the derivative is understood in the distributional sense. H can alternatively be characterized by

$$H = \{\boldsymbol{u} \in L^2(\Omega),\ \widehat{\boldsymbol{u}}(\boldsymbol{0}) = \boldsymbol{0},\ \boldsymbol{k}\cdot\widehat{\boldsymbol{u}}(\boldsymbol{k}) = \boldsymbol{0},\ \widehat{\boldsymbol{u}}(-\boldsymbol{k}) = \overline{\widehat{\boldsymbol{u}}(\boldsymbol{k})}\}.$$

Note that the space $(-\Delta)(H \cap H^2(\Omega)) \subset H$. The Stoke's operator, A, with domain $\mathcal{D}(A) = H \cap H^2(\Omega)$, is defined to be

$$A = (-\Delta)|_{\mathcal{D}(A)}.$$

We will now briefly review some standard results. The Stoke's operator A is positive and self adjoint with a compact inverse. It therefore admits a unique, positive square root, denoted $A^{1/2}$, with domain V characterized by

$$V = \{\boldsymbol{u} \in H : \|A^{1/2}\boldsymbol{u}\|^2 = (2\pi)^d \sum_{\boldsymbol{k}\in\dot{\mathbb{Z}}^d} |\boldsymbol{k}|^2 |\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2 < \infty\},$$

where $\dot{\mathbb{Z}}^d = \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. The set of eigenvectors, $\{\boldsymbol{e}_i\}_{i=1}^\infty$, of A form an orthonormal basis of H, and the corresponding eigenvalues, $1 = \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$, are elements of $\{|\boldsymbol{k}|^2 : \boldsymbol{k} \in \dot{\mathbb{Z}}^d\}$. We denote $H_N = \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$.

The dual of V can be characterized as

$$V' = \{\boldsymbol{v} \in \mathcal{D} : \widehat{\boldsymbol{v}}(\boldsymbol{k}) = \overline{\widehat{\boldsymbol{v}}(-\boldsymbol{k})},\ \widehat{\boldsymbol{v}}(\boldsymbol{0}) = \boldsymbol{0},\ \sum_{\boldsymbol{k}\in\dot{\mathbb{Z}}^d} \frac{|\widehat{\boldsymbol{v}}(\boldsymbol{k})|^2}{|\boldsymbol{k}|^2} < \infty\},$$

where $\mathcal{D}$ denotes the space of distributions. The duality bracket between V and V' is given by

$${}_V\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V'} = \sum_{\boldsymbol{k}\in\dot{\mathbb{Z}}^d} \widehat{\boldsymbol{u}}(\boldsymbol{k}) \cdot \overline{\widehat{\boldsymbol{v}}(\boldsymbol{k})}, \quad \boldsymbol{u} \in V, \boldsymbol{v} \in V'.$$

When $a \le Cb$ for some constant C which is independent of a and b and may depend only on l or d, we will write $a \lesssim b$. When $a \lesssim b$ and $b \lesssim a$, we write $a \sim b$.

Note that for any $s \in \mathbb{R}$,

$$\|A^{s/2}\boldsymbol{u}\|^2 = (2\pi)^d \sum_{\boldsymbol{k}\in\dot{\mathbb{Z}}^d} |\boldsymbol{k}|^{2s}|\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2 \quad \text{for } \boldsymbol{u} \in \mathcal{D}(A^{s/2}) = \dot{L}^2(\Omega) \cap H^s(\Omega).$$

It is well known that $\|A^{s/2}\cdot\| \sim \|\cdot\|_{H^s(\Omega)}$ on $\mathcal{D}(A^{s/2})$, and the Poincaré inequality holds, i.e.,

$$\|A^{s/2}\boldsymbol{u}\| \sim \|\boldsymbol{u}\|_{H^s(\Omega)} \text{ and } \|A^{s/2}\boldsymbol{u}\| \geq \kappa_0^s\|\boldsymbol{u}\|, \quad \boldsymbol{u} \in \dot{L}^2(\Omega) \cap H^s(\Omega). \tag{2.1}$$

Using the Sobolev and interpolation inequalities, we also have

$$\|\boldsymbol{u}\|_{L^p} \lesssim \|A^{s/2}\boldsymbol{u}\| \leq \|\boldsymbol{u}\|^{1-s}\|A^{1/2}\boldsymbol{u}\|^s,\ p = \frac{2d}{d-2s}. \tag{2.2}$$

We will find it useful to define the so-called Wiener algebra,

$$\mathcal{W} := \{\boldsymbol{u} \in H : \|\boldsymbol{u}\|_{\mathcal{W}} := \sum_{\boldsymbol{k}} |\widehat{\boldsymbol{u}}(\boldsymbol{k})| < \infty\}. \tag{2.3}$$

Clearly, from the expression of $\boldsymbol{u}$ in terms of its Fourier series, $\boldsymbol{u}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\dot{\mathbb{Z}}^d} \widehat{\boldsymbol{u}}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, it immediately follows that

$$\|\boldsymbol{u}\|_{L^\infty} \leq \|\boldsymbol{u}\|_{\mathcal{W}}.$$

In addition, we have the elementary inequality

$$\|\boldsymbol{u}\|_{L^\infty} \leq \|\boldsymbol{u}\|_{\mathcal{W}} \leq \tfrac{2\pi^{d-1}}{l^d}\tfrac{2s-d+1}{2s-d}\|A^{s/2}\boldsymbol{u}\|,\ s > \frac{d}{2}. \tag{2.4}$$

We will be using (2.4) with $s = r - \frac{1}{2}$ for a number $r > \frac{d}{2} + \frac{1}{2}$. For readability, let

$$c_r = \tfrac{2\pi^{d-1}}{l^d}\tfrac{2(r-\frac{1}{2})-d+1}{2(r-\frac{1}{2})-d} = \tfrac{1}{\pi 2^{d-1}}\tfrac{2r-d}{2r-1-d}.$$

Note that for the SQG and the Boussinesq, we will be interested in the evolution of a scalar valued field, η. We make no distinction in the notation for these cases, but it should be clear through the context that $d = 1$ in the above definitions when considering a scalar field. Furthermore, the phase space for an evolving scalar field is simply $\dot{L}^2(\Omega)$.

2.1. **Analytic Gevrey Classes.** For the remainder of this paper, let $r > \frac{d+1}{2}$ be arbitrary but fixed. For any $0 < \beta < \infty$ denote the Gevrey norm by

$$\|\mathbf{f}\|_\beta = \|A^{r/2}e^{\beta A^{1/2}}\mathbf{f}\|.$$

Although the Gevrey norm depends on r and β, we are mainly concerned with the β dependence (hence we let r be fixed).

The Gevrey norm controls the decay rate of higher order derivatives, namely, if $\|\mathbf{f}\|_\beta < \infty$ for some $\beta > 0$, then we have the higher derivative estimates

$$\|\mathbf{f}\|_{H^{r+n}(\Omega)} \leq \left(\frac{n!}{\beta^n}\right)\|\mathbf{f}\|_\beta \text{ where } n \in \mathbb{N}. \tag{2.5}$$

In particular, $\mathbf{f}$ in (2.5) is analytic with (uniform) analyticity radius β. See Theorem 4 in [38] and Theorem 5 in [42] for these and other facts about the Gevrey norm.

2.2. **Complexification.** In order to extend our evolution equations to complex times we will need to complexify the codomains of our function spaces and extend the definitions of our operators to these complexified spaces. Accordingly, let $\mathcal{L}$ be an arbitrary, real, separable Hilbert space with (real) inner-product $\langle \cdot\,, \cdot \rangle$. The complexified Hilbert space $\mathcal{L}_{\mathbb{C}}$ and the associated inner-product is given by

$$\mathcal{L}_{\mathbb{C}} = \{\boldsymbol{u} = \boldsymbol{u}_1 + i\boldsymbol{u}_2 : \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{L}\},$$

and for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{L}_{\mathbb{C}}$ with $\boldsymbol{u} = \boldsymbol{u}_1 + i\boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{v}_1 + i\boldsymbol{v}_2$, the complex inner-product is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbb{C}} = \langle \boldsymbol{u}_1, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{u}_2, \boldsymbol{v}_2 \rangle + i[\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle - \langle \boldsymbol{u}_1, \boldsymbol{v}_2 \rangle].$$

Observe that the complex inner-product $\langle \cdot\,, \cdot \rangle_{\mathbb{C}}$ is linear in the first argument while it is conjugate linear in the second argument. If A is a linear operator on $\mathcal{L}$ with domain $\mathcal{D}(A)$, we extend it to a linear operator $A_{\mathbb{C}}$ with domain $\mathcal{D}(A_{\mathbb{C}}) = \mathcal{D}(A) + i\mathcal{D}(A)$ by

$$A_{\mathbb{C}}(\boldsymbol{u}_1 + i\boldsymbol{u}_2) = A\boldsymbol{u}_1 + iA\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{D}(A).$$

Henceforth, we will drop the subscript notation from the complexified operators and inner-products and denote $A_{\mathbb{C}}$ and $\langle \cdot\,, \cdot \rangle_C$ respectively as A and $\langle \cdot\,, \cdot \rangle$, but will retain the subscript in the notation of the complexified Hilbert spaces.

3. Incompressible Euler Equations

The incompressible Euler equations, on a spatial domain $\Omega = [0, 2\pi]^d,\ d = 2 \,\text{or}\, 3$, are given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{3.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{3.1b}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \text{in } \Omega, \tag{3.1c}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ denotes the fluid velocity at a location $\boldsymbol{x} \in \Omega$ and time $t \in \mathbb{R}_+ := [0, \infty)$ and $p = p(\boldsymbol{x}, t)$ is the fluid pressure. Since its introduction in [23], it has been the subject of extensive research both in analysis and mathematical physics; see [6, 40] for a survey of recent results. As discussed in Section 2, we supplement (3.1) with the space periodic boundary condition with space period 2π, i.e., we require that the functions $\boldsymbol{u}$ and p be periodic with period 2π in all spatial directions.

We will apply the Leray-Helmholtz orthogonal projection operator, $\mathbb{P} : \dot{L}^2(\Omega) \to H$ (which maps $\dot{L}^2(\Omega)$ onto the closed subspace H of $\dot{L}^2(\Omega)$), to the Euler equations. As $\nabla \cdot \boldsymbol{u} = 0$, formally we have $\nabla \cdot \partial_t \boldsymbol{u} = \partial_t \nabla \cdot \boldsymbol{u} = 0$, so $\mathbb{P}\, \partial_t \boldsymbol{u} = \partial_t \boldsymbol{u}$. Also, $\mathbb{P}\nabla p = 0$. For the quadratic term, we define the operator

$$B(\boldsymbol{u}, \boldsymbol{v}) = \mathbb{P}\,(\boldsymbol{u} \cdot \nabla \boldsymbol{v}) = \mathbb{P}\,\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v}), \tag{3.2}$$

From (2.2), it readily follows that if $\boldsymbol{u}, \boldsymbol{v} \in V$, then $\|\boldsymbol{u} \otimes \boldsymbol{v}\| < \infty$ and consequently, $B(\boldsymbol{u}, \boldsymbol{v}) \in V'$.

After applying the Leray-Helmholtz projection, we obtain the functional form of the incompressible Euler equations:

$$\frac{d}{dt}\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u}) = 0. \tag{3.3}$$

We will be considering the complexified version of the functional form, which is given by

$$\frac{d\boldsymbol{u}}{d\zeta} + B_{\mathbb{C}}(\boldsymbol{u},\boldsymbol{u}) = 0, \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3.4}$$

where, for $\boldsymbol{u} = \boldsymbol{u}_1 + i\boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{v}_1 + i\boldsymbol{v}_2 \in H_{\mathbb{C}}$, the complexified nonlinear term is given by

$$B_{\mathbb{C}}(\boldsymbol{u},\boldsymbol{v}) := B(\boldsymbol{u}_1,\boldsymbol{v}_1) - B(\boldsymbol{u}_2,\boldsymbol{v}_2) + i[B(\boldsymbol{u}_1,\boldsymbol{v}_2) + B(\boldsymbol{u}_2,\boldsymbol{v}_1)].$$

As before, we will drop the subscript and write $B = B_{\mathbb{C}}$.

Theorem 3.1. *Let $\beta_0 > 0$ be fixed, and let $\boldsymbol{u}_0$ be such that $\|\boldsymbol{u}_0\|_{\beta_0} < \infty$. Then the complexified Euler equation* (3.4) *admits a unique classical solution $\boldsymbol{u} \in Hol(\mathcal{R}; H_{\mathbb{C}})$, with*

$$\mathcal{R} = \left\{ \zeta = se^{i\theta} : \theta \in [0,2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|\boldsymbol{u}_0\|_{\beta_0}} \right\}. \tag{3.5}$$

Furthermore, $\boldsymbol{u}(\zeta)$ is Gevrey regular for all $\zeta \in \mathcal{R}$ (i.e. $\|\boldsymbol{u}(\zeta)\|_\beta < \infty$ for some $\beta > 0$).

For the Euler equations in the real setting, it is well known (see [3, 7, 34, 35, 38]) that if the Beale-Kato-Majda condition, $\int_0^T \|\nabla \times \boldsymbol{u}\|_{L^\infty} < \infty$, is satisfied, and there exists β_0 such that $\|A^{r/2}e^{\beta_0 A^{1/2}}\boldsymbol{u}_0\| < \infty$, then there exists a positive continuous function $\beta : [0,T] \to (0,\infty)$ such that

$$\sup_{[0,T]} \|A^{r/2}e^{\beta(t)A^{1/2}}\boldsymbol{u}(t)\| < \infty.$$

In this case, $\boldsymbol{u}$ extends as a holomorphic function solving (3.4) in a neighborhood of $[0,T]$ in $\mathbb{C}$. More precisely we have the following:

Corollary 3.2. *Let $\boldsymbol{u} \in C^1([0,T]; H)$ be a classical solution of* (3.3) *and suppose that there exists a continuous function $\beta(\cdot) > 0$ on $[0,T]$ such that*

$$M := \sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|_{\beta(t)} < \infty. \tag{3.6}$$

Then $\boldsymbol{u}(\cdot)$ extends as a holomorphic function $\boldsymbol{u}_{\mathbb{C}} \in Hol(\mathcal{R}, H_{\mathbb{C}})$ solving (3.4)*, where $\mathcal{R}$ is a complex neighborhood of $(0,T)$.*

Proof. Let $\beta_0 = \min_{t\in[0,T]} \beta(t) > 0$. Then by Theorem 3.1, $\boldsymbol{u}$ extends as a holomorphic function in a complex neighborhood of $(0,\varepsilon)$, with $\varepsilon = \frac{C\beta_0}{2^r c_r M}$. The proof follows by reapplying Theorem 3.1 with $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$, for each $t_0 \in \{\frac{\varepsilon}{2}, \frac{2\varepsilon}{2}, \frac{3\varepsilon}{2}, \dots\} \cap [0,T]$. $\square$

Remark 3.3. The domain of analyticity given in Theorem 3.1 may not be optimal. One may be able to obtain a larger domain by letting s depend on θ and then optimizing for s with respect to θ. Moreover, it would also be of interest to connect the shape of the domain dynamically with the space analyticity radius of the solution at time t, and possibly express it explicitly in terms of $\|\nabla \times u\|_{L^\infty}$, as has been done in case of the space analyticity radius in [34, 38]. We leave these issues for future consideration.

Before proceeding with the proof of the theorem, we will need the following estimate of the nonlinear term.

Proposition 3.4. *Let $\boldsymbol{u} \in H_{\mathbb{C}}$ with $\|A^{1/4}\boldsymbol{u}\|_\beta < \infty$. Then,*

$$|\langle B(\boldsymbol{u},\boldsymbol{u}), A^r e^{2\beta A^{1/2}}\boldsymbol{u}\rangle| \lesssim 2^r c_r \|\boldsymbol{u}\|_\beta \|A^{1/4}\boldsymbol{u}\|_\beta^2. \tag{3.7}$$

Proof. Observe that for $\boldsymbol{k} = \boldsymbol{h} + \boldsymbol{j}$, $\boldsymbol{h}, \boldsymbol{j}, \boldsymbol{k} \in \dot{\mathbb{Z}}^d$, we have

$$|\boldsymbol{k}|^r \le 2^{r-1}(|\boldsymbol{h}|^r + |\boldsymbol{j}|^r).$$

Thus,

$$\begin{aligned} &|\langle B(\boldsymbol{u},\boldsymbol{v}), A^r e^{2\beta A^{1/2}}\boldsymbol{w}\rangle| \\ &\quad\le \sum_{\boldsymbol{h}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{0}} |\widehat{\boldsymbol{u}}(\boldsymbol{h})||\boldsymbol{j}||\widehat{\boldsymbol{v}}(\boldsymbol{j})||\boldsymbol{k}|^{2r}|\widehat{\boldsymbol{w}}(\boldsymbol{k})|e^{2\beta|\boldsymbol{k}|} \\ &\quad\le 2^{r-1}\sum_{\boldsymbol{h}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{0}} |\boldsymbol{h}|^r|\widehat{\boldsymbol{u}}(\boldsymbol{h})||\boldsymbol{j}||\widehat{\boldsymbol{v}}(\boldsymbol{j})||\boldsymbol{k}|^r|\widehat{\boldsymbol{w}}(\boldsymbol{k})|e^{2\beta|\boldsymbol{k}|} \\ &\qquad + 2^{r-1}\sum_{\boldsymbol{h}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{0}} |\widehat{\boldsymbol{u}}(\boldsymbol{h})||\boldsymbol{j}||\boldsymbol{j}|^r|\widehat{\boldsymbol{v}}(\boldsymbol{j})||\boldsymbol{k}|^r|\widehat{\boldsymbol{w}}(\boldsymbol{k})|e^{2\beta|\boldsymbol{k}|}. \end{aligned} \tag{3.8}$$

Because $\boldsymbol{j}, \boldsymbol{h}, \boldsymbol{k} \neq \boldsymbol{0}$, we have $\min\{|\boldsymbol{j}|, |\boldsymbol{h}|, |\boldsymbol{k}|\} \ge 1$ and therefore,

$$|\boldsymbol{j}| \le |\boldsymbol{h}| + |\boldsymbol{k}| \le 2|\boldsymbol{h}||\boldsymbol{k}| \text{ which implies } |\boldsymbol{j}|^{\frac{1}{2}} \lesssim |\boldsymbol{h}|^{\frac{1}{2}}|\boldsymbol{k}|^{\frac{1}{2}}. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} &|\langle B(\boldsymbol{u},\boldsymbol{v}), A^r e^{2\beta A^{1/2}}\boldsymbol{w}\rangle| \\ &\quad\le 2^{r-1/2}\sum_{\boldsymbol{h}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{0}} e^{\beta|\boldsymbol{h}|}|\boldsymbol{h}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{u}}(\boldsymbol{h})|e^{\beta|\boldsymbol{j}|}|\boldsymbol{j}|^{\frac{1}{2}}|\widehat{\boldsymbol{v}}(\boldsymbol{j})||\boldsymbol{k}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{w}}(\boldsymbol{k})|e^{\beta|\boldsymbol{k}|} \\ &\qquad + 2^{r-1/2}\sum_{\boldsymbol{h}+\boldsymbol{j}-\boldsymbol{k}=\boldsymbol{0}} e^{\beta|\boldsymbol{h}|}|\boldsymbol{h}|^{\frac{1}{2}}|\widehat{\boldsymbol{u}}(\boldsymbol{h})|e^{\beta|\boldsymbol{j}|}|\boldsymbol{j}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{v}}(\boldsymbol{j})||\boldsymbol{k}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{w}}(\boldsymbol{k})|e^{\beta|\boldsymbol{k}|}, \\ &\quad\lesssim 2^r\left(\|A^{\frac{1}{4}}e^{\beta A^{1/2}}\boldsymbol{v}\|_{\mathcal{W}}\|A^{\frac{1}{4}}\boldsymbol{u}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{w}\|_\beta + \|A^{\frac{1}{4}}e^{\beta A^{1/2}}\boldsymbol{u}\|_{\mathcal{W}}\|A^{\frac{1}{4}}\boldsymbol{v}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{w}\|_\beta\right), \end{aligned}$$

$$\lesssim 2^r c_r\left(\|\boldsymbol{v}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{u}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{w}\|_\beta + \|\boldsymbol{u}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{v}\|_\beta\|A^{\frac{1}{4}}\boldsymbol{w}\|_\beta\right), \tag{3.10}$$

where to obtain (3.10) we used (2.4) with $s := r - \frac{1}{2} > \frac{d}{2}$. We readily obtain

$$|\langle B(\boldsymbol{u},\boldsymbol{u}), A^r e^{2\beta A^{1/2}}\boldsymbol{u}\rangle| \lesssim 2^r c_r \|\boldsymbol{u}\|_\beta \|A^{1/4}\boldsymbol{u}\|_\beta^2. \tag{3.11}$$

□

3.1. **Proof of Theorem 3.1.**

Proof. Recall that for each $N \in \mathbb{N}$, $H_N = \operatorname{span}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset H_{\mathbb{C}}$, where $\{\boldsymbol{e}_i\}_{i=1}^\infty$ is the complete, orthonormal system (in $H_{\mathbb{C}}$) of eigenvectors of A. Denote the orthogonal projection on H_N by P_N. The Galerkin system corresponding to (3.4) is given by

$$\frac{d\boldsymbol{v}}{d\zeta} + P_N B(\boldsymbol{v},\boldsymbol{v}) = 0,\ \boldsymbol{v}(0) = P_N\boldsymbol{u}_0,\ \boldsymbol{v}(\zeta) \in H_N. \tag{3.12}$$

The Galerkin system is an ODE with a quadratic nonlinerity. Therefore it admits a unique classical solution in a neighborhood of the origin in $\mathbb{C}$. We will obtain *a priori* estimates on the Galerkin system in $\mathcal{R}$ (defined in (3.5)) independent of N. This will show that the for each N, the solution of the Galerkin system exists and is holomorphic on all of $\mathcal{R}$, and the set of Galerkin solutions forms a normal family on the domain $\mathcal{R}$. We can then pass to the limit through a subsequence by Montel's theorem to obtain a solution of (3.4) in $\mathcal{R}$.

Let $N \in \mathbb{N}$. Fix $\theta \in [0, 2\pi)$, and let

$$\zeta = se^{i\theta}, \ s > 0.$$

We assume that the initial data $\boldsymbol{u}_0$ satisfies $\|\boldsymbol{u}_0\|_{\beta_0} < \infty$ for some $\beta_0 > 0$. Fix $\delta > 0$, to be chosen later and define the time-varying norm

$$|\boldsymbol{u}(\zeta)| = \|\boldsymbol{u}(\zeta)\|_{\beta_0 - \delta s}.$$

The corresponding (time-varying) inner product will be denoted by $((,))$, i.e.,

$$\begin{aligned}&((\boldsymbol{u}, \boldsymbol{v}))\\ &\quad= \langle A^{r/2} e^{(\beta_0 - \delta s)A^{1/2}} \boldsymbol{u}, A^{r/2} e^{(\beta_0 - \delta s)A^{1/2}} \boldsymbol{v}\rangle\\ &\quad= \langle \boldsymbol{u}, A^r e^{2(\beta_0 - \delta s)A^{1/2}} \boldsymbol{v}\rangle.\end{aligned}$$

Taking the inner-product of (3.12) (in $H_{\mathbb{C}}$) with $A^r e^{2(\beta_0-\delta s)A^{1/2}} \boldsymbol{v}$, then multiplying by $e^{i\theta}$, and finally taking the real part of the resulting equation, we readily obtain

$$\frac{1}{2}\frac{d}{ds}|\boldsymbol{v}(\zeta)|^2 + \delta\, |A^{1/4}\boldsymbol{v}(\zeta)|^2 = -Re\left(e^{i\theta}((B(\boldsymbol{v}(\zeta), \boldsymbol{v}(\zeta)), \boldsymbol{v}(\zeta)))\right) \le |((B(\boldsymbol{v}(\zeta), \boldsymbol{v}(\zeta)), \boldsymbol{v}(\zeta)))|.$$

For $s < \frac{\beta_0}{\delta}$, using Proposition 3.4, we obtain

$$\frac{1}{2}\frac{d}{ds}|\boldsymbol{v}|^2 + \delta\, |A^{1/4}\boldsymbol{v}|^2 \lesssim 2^r c_r |\boldsymbol{v}| |A^{1/4}\boldsymbol{v}|^2. \tag{3.13}$$

Now choose

$$\delta = C 2^r c_r \|\boldsymbol{u}_0\|_{\beta_0}.$$

From (3.13), we see that $|\boldsymbol{v}|$ is non-increasing and

$$|\boldsymbol{v}(\zeta)| \le \|\boldsymbol{u}_0\|_{\beta_0} \ \forall\ \zeta = se^{i\theta}, \ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{\zeta \in \mathcal{R}} \|A^{r/2}\boldsymbol{v}(\zeta)\| \le \|\boldsymbol{u}_0\|_{\beta_0}.$$

As remarked above, the proof is now complete by invoking Montel's theorem. □

4. Surface Quasi-geostrophic Equations

We consider the inviscid, two-dimensional (surface) quasi-gesotrophic equation (henceforth SQG) on $\Omega = [0, 2\pi]^2$, given by

$$\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{4.1}$$

$$\boldsymbol{u} = [-R_2\eta, R_1\eta]^T, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{4.2}$$

$$\eta(0) = \eta_0, \quad \text{in } \Omega. \tag{4.3}$$

Here $\boldsymbol{u}$ is the velocity field, η is the temperature, the operator $\Lambda = (-\Delta)^{1/2}$ (with Δ denoting the Laplacian) and the operators $R_i = \partial_i \Lambda^{-1}, i = 1, 2$, are the usual Riesz transforms.

Observe that by the definition of u, it is divergence-free. Also, without loss of generality, we will take u and η to be mean-free, i.e.,

$$\int_\Omega \boldsymbol{u} = \boldsymbol{0}, \int_\Omega \eta = 0.$$

The SQG was introduced in [16] and variants of it arises in geophysics and meteorology (see, for instance [43]). Moreover, the critical SQG is the dimensional analogue of the three dimensional Navier–Stokes equations. Existence and regularity issues for the viscous and inviscid cases were first extensively examined in [44]. This equation, particularly the dissipative case with various fractional orders of dissipation, has received considerable attention recently; see [11, 18, 30, 31], and the references therein. As in section 3, our focus here is time analyticity of the inviscid SQG, with values in an appropriate analytic Gevrey class.

As before, for $r > \frac{3}{2}, \beta > 0$, we define

$$\|\eta\|_\beta = \|\Lambda^r e^{\beta\Lambda}\eta\| \text{ and } \|\boldsymbol{u}\|_\beta = \|A^{r/2} e^{\beta A^{1/2}} \boldsymbol{u}\|.$$

Note that because $\boldsymbol{u}$ is the Riesz transform of η, we have $\|\eta\|_\beta \sim \|\boldsymbol{u}\|_\beta$.

Theorem 4.1. *Let η_0 be such that $\|\eta_0\|_{\beta_0} < \infty$ for some $\beta_0 > 0$. Then the complexified inviscid SQG equation*

$$\frac{d\eta}{d\zeta} + B(\boldsymbol{u}, \eta) = 0, \eta(0) = \eta_0, \text{ where } B(\boldsymbol{u}, \eta) = \boldsymbol{u} \cdot \nabla\eta, \tag{4.4}$$

admits a unique classical solution in $Hol(\mathcal{R}, \dot{L}^2_{\mathbb{C}}(\Omega))$, with

$$\mathcal{R} = \left\{ \zeta = se^{i\theta} : \theta \in [0, 2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|\eta_0\|_{\beta_0}} \right\}. \tag{4.5}$$

Proof. Proceeding in a similar manner as in Proposition 3.4, we obtain

$$\begin{aligned} |\langle B(\boldsymbol{u}, \eta), \Lambda^{2r} e^{2\beta\Lambda}\eta\rangle| &\lesssim 2^r c_r \left(\|\eta\|_\beta \|A^{1/4}\boldsymbol{u}\|_\beta \|\Lambda^{1/2}\eta\|_\beta + \|\boldsymbol{u}\|_\beta \|\Lambda^{1/2}\eta\|_\beta^2 \right) \\ &\lesssim 2^r c_r \|\eta\|_\beta \|\Lambda^{1/2}\eta\|_\beta^2, \end{aligned} \tag{4.6}$$

where the last inequality follows by noting that u is the Riesz transform of η.

Fix $\theta \in [0, 2\pi)$. Let

$$\zeta = se^{i\theta}, s > 0.$$

The initial data η_0 satisfies $\|\eta_0\|_{\beta_0} < \infty$ for some $\beta_0 > 0$. Fix $\delta > 0$, to be specified later, and define the time-varying norm

$$|\!|\!|\eta|\!|\!| = \|\eta(\zeta)\|_{\beta_0 - \delta s},$$

and the corresponding (time-varying) inner product, $((,))$, as we did in the proof of Theorem 3.1.

Taking the inner-product of (4.4) (in $H_{\mathbb{C}}$) with $\Lambda^{2r} e^{2(\beta_0 - \delta s)\Lambda}\eta$, multiplying by $e^{i\theta}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{ds}|\!|\!|\eta(\zeta)|\!|\!|^2 + \delta\, |\!|\!|\Lambda^{1/2}\eta(\zeta)|\!|\!|^2 = Re\left(-e^{i\theta}((B(\boldsymbol{u}(\zeta), \eta(\zeta)), \eta(\zeta))) \right).$$

Using (4.6), we deduce

$$\frac{1}{2}\frac{d}{ds}|\eta|^2 + \delta\, |\Lambda^{1/2}\eta|^2 \lesssim 2^r c_r |\eta|\, |\Lambda^{1/2}\eta|^2. \tag{4.7}$$

Now choose

$$\delta = C 2^r c_r \|\eta_0\|_{\beta_0}.$$

From (4.7), we see that $|\eta|$ is non-increasing and

$$|\eta(\zeta)| \le \|\eta_0\|_{\beta_0} \ \forall\ \zeta = s e^{i\theta},\ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z\in\mathcal{R}} \|\eta(z)\| \le \|\eta_0\|_{\beta_0}.$$

This establishes a uniform bound on the Galerkin system and the proof is complete by invoking Montel's theorem as before. □

5. Inviscid Boussinesq Equations

The inviscid Boussinesq system (without rotation) in the periodic domain $\Omega := [0, 2\pi]^d, d = 2, 3$, for time $t \ge 0$ is given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \eta\, g\mathbf{e}, \quad \text{in } \Omega\times\mathbb{R}_+, \tag{5.1a}$$

$$\partial_t \eta + (\boldsymbol{u}\cdot\nabla)\eta = 0, \quad \text{in } \Omega\times\mathbb{R}_+, \tag{5.1b}$$

$$\nabla\cdot\boldsymbol{u} = 0, \quad \text{in } \Omega\times\mathbb{R}_+, \tag{5.1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \eta(0) = \eta_0, \quad \text{in } \Omega, \tag{5.1d}$$

equipped with periodic boundary conditions in space. Here $\mathbf{e}$ denotes the unit vector in $\mathbb{R}^d$ pointing upward and g denotes the (scalar) acceleration due to gravity. The unknowns are the fluid velocity field $\boldsymbol{u}$, the fluid pressure p, and the function η, which may be interpreted physically as the temperature. The Boussinesq system arises in the study of atmospheric, oceanic and astrophysical turbulence, particularly where rotation and stratification play a dominant role [43, 45]. We will follow the notation for the norms as in Section 3 and Section 4

Theorem 5.1. *Let $(\boldsymbol{u}_0, \eta_0)$ be such that $\|(\boldsymbol{u}_0, \eta_0)\|_{\beta_0} < \infty$ for some $\beta_0 > 0$, where $\|(\boldsymbol{u}_0, \eta_0)\|^2_{\beta_0} = \|\boldsymbol{u}_0\|^2_{\beta_0} + \|\eta_0\|^2_{\beta_0}$. The complexified inviscid Boussinesq equations* (5.1) *admit a unique solution $(\boldsymbol{u}, \eta) \in Hol(\mathcal{R}, H_C) \times Hol(\mathcal{R}, \dot{L}^2_{\mathbb{C}}(\Omega))$, where*

$$\mathcal{R} = \left\{ \zeta = s e^{i\theta} : \theta \in [0, 2\pi),\ 0 < s < \min\left\{ \frac{C\beta_0}{2^r c_r \|(\boldsymbol{u}_0, \eta_0)\|_{\beta_0}}, \frac{2\ln 2}{g} \right\} \right\}. \tag{5.2}$$

Proof. We proceed as in Section 3 and Section 4 by taking the inner product of the complexified versions of (5.1a) and (5.1b) with $A^r e^{2(\beta_0 - \delta s)A^{1/2}} u$ and $\Lambda^{2r} e^{2(\beta_0-\delta s)\Lambda}\eta$ respectively, then multiplying by $e^{i\theta}$ and taking the real part. Using (3.7) and (4.6) and adding the results, for $(\boldsymbol{u}(\zeta), \eta(\zeta)), \zeta = s e^{i\theta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}(|\boldsymbol{u}|^2 + |\eta|^2) + \delta(|\Lambda^{1/2}\boldsymbol{u}|^2 + |\Lambda^{1/2}\eta|^2) \\
&\lesssim 2^r c_r (|\boldsymbol{u}| + |\eta|)(|\Lambda^{1/2}\boldsymbol{u}|^2 + |\Lambda^{1/2}\eta|^2) + g|\eta||\boldsymbol{u}| \\
&\le 2^r c_r (|\boldsymbol{u}| + |\eta|)(|\Lambda^{1/2}\boldsymbol{u}|^2 + |\Lambda^{1/2}\eta|^2) + \frac{g}{2}(|\boldsymbol{u}|^2 + |\eta|^2).
\end{aligned} \tag{5.3}$$

Thus, as long as

$$(\lvert \boldsymbol{u} \rvert + \lvert \eta \rvert) \lesssim \frac{\delta}{2^r c_r}, \tag{5.4}$$

by the Gronwall inequality, we have

$$\lvert \boldsymbol{u} \rvert^2 + \lvert \eta \rvert^2 \leq e^{Tg}(\|\boldsymbol{u}_0\|_{\beta_0}^2 + \|\eta_0\|_{\beta_0}^2), 0 < s \leq T. \tag{5.5}$$

Using the fact that $(a+b) \leq \sqrt{2(a^2+b^2)}$, as long as (5.4) holds, from (5.5) we have

$$(\lvert \boldsymbol{u} \rvert + \lvert \eta \rvert) \leq e^{\frac{Tg}{2}} \sqrt{2} \|(\boldsymbol{u}_0, \eta_0)\|_{\beta_0}. \tag{5.6}$$

Now choose

$$\delta = C 2^r c_r \|(\boldsymbol{u}_0, \eta_0)\|_{\beta_0}.$$

For all $0 < s < T = \min\{\frac{\beta_0}{\delta}, \frac{2 \ln 2}{g}\}$, (5.4) is satisfied and consequently, (5.6) holds. □

6. Inviscid Magnetohydrodynamic Equations

The inviscid incompressible magnetohydrodynamic system in the periodic domain $\Omega := [0, 2\pi]^d, d = 2, 3$, for time $t \geq 0$ is given by the following system

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \tfrac{1}{S}(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + \nabla(\tfrac{1}{\rho_0} p + \tfrac{|\boldsymbol{b}|^2}{2S}) = \boldsymbol{0}, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.1a}$$

$$\partial_t \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.1b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{b} = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{b}(0) = \boldsymbol{b}_0, \quad \text{in } \Omega, \tag{6.1d}$$

equipped with periodic boundary conditions in space. Here, $\boldsymbol{u}$ represents the fluid velocity field, $\boldsymbol{b}$ the magnetic field and p the fluid pressure. The constant ρ_0 is the fluid density, and $S = \rho_0 \mu_0$, where μ_0 is the permeability of free space.

The magnetohydrodynamic equations (MHD) govern the evolution of an electrically conductive fluid under the influence of a magnetic field, and so are useful in the design of fusion reactors, or the study of solar storms and other natural phenomenon. See [19] for more on the derivation of (6.1), and [39,49] for some applications of the magnetohydrodynamic equations. The existence and uniqueness of solutions to the incompressible MHD has been studied for the viscous case in [36, 47], for example, and in [5,12] for the inviscid case (which we consider in this paper). The space analyticity of solutions of (6.1) is discussed in [13], whereas in the present work we give criteria for solutions to be holomorphic functions of both the time and space variables.

By rewriting the equations in terms of the Elsässer variables (which are defined via the transformations $\boldsymbol{v} = \boldsymbol{u} + \frac{1}{\sqrt{S}}\boldsymbol{b}$, $\boldsymbol{w} = \boldsymbol{u} - \frac{1}{\sqrt{S}}\boldsymbol{b}$), we obtain the equivalent system

$$\partial_t \boldsymbol{v} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} + \nabla \mathcal{P} = \boldsymbol{0}, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.2a}$$

$$\partial_t \boldsymbol{w} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{w} + \nabla \mathcal{P} = \boldsymbol{0}, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.2b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{w} = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{6.2c}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \quad \boldsymbol{w}(0) = \boldsymbol{w}_0, \quad \text{in } \Omega, \tag{6.2d}$$

where $\mathcal{P} = \frac{1}{\rho_0}p + \frac{|\boldsymbol{v}-\boldsymbol{w}|^2}{8}$.

Theorem 6.1. *Let* $(\boldsymbol{v}_0, \boldsymbol{w}_0)$ *be such that* $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\beta_0} < \infty$ *for some* $\beta_0 > 0$. *The complexified inviscid magnetohydrodynamic equations* (6.2) *admit a unique classical solution* $(\boldsymbol{v}, \boldsymbol{w}) \in Hol(\mathcal{R}, H_{\mathbb{C}}) \times Hol(\mathcal{R}, H_{\mathbb{C}})$ *with*

$$\mathcal{R} = \left\{ \zeta = se^{i\theta} : \theta \in [0, 2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\beta_0}} \right\}.$$

Proof. Proceeding as in the previous sections and using (3.7), for a fixed $\theta \in [0, 2\pi)$, for $(\boldsymbol{v}(\zeta), \boldsymbol{w}(\zeta)), \zeta = se^{i\theta}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}\{|\boldsymbol{v}|^2 + |\boldsymbol{w}|^2\} + \delta(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2) \\ &\lesssim 2^r c_r(|\boldsymbol{v}| + |\boldsymbol{w}|)(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2) \\ &\lesssim 2^r c_r|(\boldsymbol{v}, \boldsymbol{w})|(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2). \end{aligned} \tag{6.3}$$

Now choose

$$\delta = C2^r c_r \|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\beta_0}.$$

From (6.3), we see that $|(\boldsymbol{v}, \boldsymbol{w})|^2$ is non-increasing and

$$|(\boldsymbol{v}(\zeta), \boldsymbol{w}(\zeta))| \le \|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\beta_0} \ \forall \ \zeta = se^{i\theta}, \ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z \in \mathcal{R}} \|(\boldsymbol{v}(z), \boldsymbol{w}(z))\| \le \|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\beta_0}.$$

This finishes the proof. □

7. Analytic Nonlinearity

In this section, we consider the more general case of an analytic nonlinearity on our basic spatial domain $\Omega := [0, 2\pi]^d$. Again, we consider an equation without viscous effects (see [24] for the dissipative version). For simplicity of exposition, we only consider the case of a scalar equation here. A vector-valued version, i.e. the case of a system, can be handled in precisely the same way, although notationally it becomes more cumbersome. Let

$$F(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a real analytic function in a neighborhood of the origin. The "majorizing function" for F is defined to be

$$F_M(s) = \sum_{n=1}^{\infty} |a_n| s^n, \quad s < \infty. \tag{7.1}$$

The functions F and F_M are clearly analytic in the open balls (in $\mathbb{R}^d$ and $\mathbb{R}$ respectively) with center zero and radius

$$R_M = \sup\{s : F_M(s) < \infty\}. \tag{7.2}$$

We assume that $R_M > 0$. The derivative of the function F_M, denoted by F'_M, is also analytic in the ball of radius R_M. Therefore, for any fixed $r > 0$, the function $\widetilde{F}$, defined by

$$\widetilde{F}(s) = \sum_{n=1}^{\infty} |a_n| n^{r+\frac{3}{2}} (c_r)^{n-1} s^{n-1}, s \in \mathbb{R}, \tag{7.3}$$

is analytic in the ball of radius R_M/c_r. Moreover,

$$\widetilde{F}(s) \geq 0 \text{ for } \ s \geq 0 \text{ and } \widetilde{F}(s_1) < \widetilde{F}(s_2) \text{ for } 0 \leq s_1 < s_2. \tag{7.4}$$

We will consider an inviscid equation of the form

$$\partial_t u = TF(u), u(0) = u_0, \tag{7.5}$$

where T is given by

$$\widehat{Tu}(\boldsymbol{k}) = m_T(\boldsymbol{k})\widehat{u}(\boldsymbol{k}), |m_T(\boldsymbol{k})| \leq C|\boldsymbol{k}|, \boldsymbol{k} \in \dot{\mathbb{Z}}^d.$$

We will assume that (7.5) preserves the mean free condition under evolution. Here, $d = 1$ and the phase space $H = \dot{L}^2(\Omega)$ and $A = (-\Delta)|_H$. As before, we fix $r > \frac{d+1}{2}$ and define

$$\|u\|_\beta = \|A^{\frac{r}{2}} e^{\beta A^{1/2}} u\|.$$

The following proposition is elementary.

Proposition 7.1. *For $x_1, \ldots, x_n \in \mathbb{R}_+$ and any $r > 0$, we have*

$$(x_1 + \cdots + x_n)^r \leq n^r (x_1^r + \cdots + x_n^r).$$

Proof. Without loss of generality, assume $x_1 = \max\{x_1, \ldots, x_n\} > 0$. Let $\xi_i = \frac{x_i}{x_1}$ and note that $0 \leq \xi_i \leq 1$. Then,

$$(\sum_{i=1}^n x_i)^r = x_1^r (\sum_{i=1}^n \xi_i)^r \leq x_1^r (\sum_{i=1}^n 1)^r = n^r x_1^r \leq n^r \sum_{i=1}^n x_i^r.$$

$\square$

We will need the following estimate of the nonlinear term to proceed.

Proposition 7.2. *Let $u \in H_{\mathbb{C}}$ with $\|A^{1/4}u\|_\beta < \infty$. Then*

$$|\langle TF(u), A^r e^{2\beta A^{1/2}} u \rangle| \lesssim \widetilde{F}(\|u\|_\beta) \|A^{1/4} u\|_\beta^2. \tag{7.6}$$

Proof. Observe that for $\boldsymbol{h}_1 + \cdots + \boldsymbol{h}_n + \boldsymbol{k} = \boldsymbol{0}$, $\boldsymbol{h}_i, \boldsymbol{k} \in \dot{\mathbb{Z}}^d$, by triangle inequality and Proposition 7.1, we have

$$|\boldsymbol{k}|^r \leq n^r (|\boldsymbol{h}_1|^r + \cdots + |\boldsymbol{h}_n|^r). \tag{7.7}$$

Denote

$$I \subset \dot{\mathbb{Z}}^{d+1}, I = \{(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_n, \boldsymbol{k}) : \boldsymbol{h}_1 + \cdots + \boldsymbol{h}_n + \boldsymbol{k} = \boldsymbol{0}, \boldsymbol{h}_i, \boldsymbol{k} \in \dot{\mathbb{Z}}^d\}.$$

Thus,

$$
\begin{aligned}
&|\langle Tu^n, A^r e^{2\beta A^{1/2}}u\rangle| \\
&\quad\lesssim \sum_I |u(\boldsymbol{h}_1)|\dots|u(\boldsymbol{h}_n)||u(\boldsymbol{k})||\boldsymbol{k}|^{2r+1}e^{2\beta|\boldsymbol{k}|} \\
&\quad\lesssim n^r\left(\sum_I |\boldsymbol{h}_1|^r e^{\beta|\boldsymbol{h}_1|}|u(\boldsymbol{h}_1)|\dots e^{\beta|\boldsymbol{h}_n|}|u(\boldsymbol{h}_n)||u(\boldsymbol{k})||\boldsymbol{k}|^{r+1}e^{\beta|\boldsymbol{k}|}\right. \\
&\qquad\qquad \left.+\dots+\sum_I e^{\beta|\boldsymbol{h}_1|}|u(\boldsymbol{h}_1)|\dots|\boldsymbol{h}_n|^r e^{\beta|\boldsymbol{h}_n|}|u(\boldsymbol{h}_n)||u(\boldsymbol{k})||\boldsymbol{k}|^{r+1}\boldsymbol{e}^{\beta|\boldsymbol{k}|}\right), \qquad (7.8)
\end{aligned}
$$

where to obtain (7.8), we used (7.7) as well as the triangle inequality $|\boldsymbol{k}| \le \sum_i |\boldsymbol{h}_i|$. Because $\min\{|\boldsymbol{h}_1|,\dots,|\boldsymbol{h}_n|,|\boldsymbol{k}|\} \ge 1$, we have

$$
|\boldsymbol{k}| \le \sum_i |\boldsymbol{h}_i| \le n|\boldsymbol{h}_1|\dots|\boldsymbol{h}_n|, \text{ which implies } |\boldsymbol{k}|^{\frac12} \lesssim n^{1/2}|\boldsymbol{h}_1|^{\frac12}\dots|\boldsymbol{h}_n|^{\frac12}.
$$

Consequently, from (7.8), we conclude

$$
\begin{aligned}
&|\langle Tu^n, A^r e^{2\beta A^{1/2}}u\rangle| \\
&\quad\lesssim n^{r+\frac12}\left(\sum_I |\boldsymbol{h}_1|^{r+\frac12} e^{\beta|\boldsymbol{h}_1|}|u(\boldsymbol{h}_1)|\dots e^{\beta|\boldsymbol{h}_n|}|\boldsymbol{h}_n|^{\frac12}|u(\boldsymbol{h}_n)||u(\boldsymbol{k})||\boldsymbol{k}|^{r+\frac12}e^{\beta|\boldsymbol{k}|}\right. \\
&\qquad \left.+\dots+\sum_I e^{\beta|\boldsymbol{h}_1|}|\boldsymbol{h}_1|^{\frac12}|u(\boldsymbol{h}_1)|\dots e^{\beta|\boldsymbol{h}_n|}|\boldsymbol{h}_n|^{r+\frac12}|u(\boldsymbol{h}_n)||u(\boldsymbol{k})||\boldsymbol{k}|^{r+\frac12}e^{\beta|\boldsymbol{k}|}\right) \\
&\quad\lesssim n^{r+\frac32}(c_r)^{n-1}\|A^{\frac14}u\|_\beta^2\|u\|_\beta^{n-1}, \qquad (7.9)
\end{aligned}
$$

where the last inequality follows exactly as in the proof of (3.10). This immediately yields (7.6). ☐

Theorem 7.3. *Let $r > \frac{d+1}{2}$ and $\beta_0 > 0$ be fixed and u_0 be such that $\|u_0\|_{\beta_0} < \infty$. Then, the complexified equation* (7.5) *admits a unique classical solution in $Hol(\mathcal{R}, \dot{L}^2_{\mathbb{C}}(\Omega))$ with*

$$
\mathcal{R} = \left\{ z = se^{i\theta} : \theta \in [0,2\pi), 0 < s < \frac{C\beta_0}{\widetilde{F}(\|u_0\|_{\beta_0})} \right\}.
$$

Proof. Fix $\delta > 0$, to be chosen later and, as before, define the time-varying norm

$$
|\!|u(\zeta)|\!| = \|u(\zeta)\|_{\beta_0-\delta s}.
$$

Recall that the corresponding (time-varying) inner product is denoted by $((,))$, i.e.,

$$
\begin{aligned}
((u,v)) &= \langle A^{r/2}e^{(\beta_0-\delta s)A^{1/2}}u, A^{r/2}e^{(\beta_0-\delta s)A^{1/2}}v\rangle \\
&= \langle u, A^r e^{2(\beta_0-\delta s)A^{1/2}}v\rangle.
\end{aligned}
$$

Multiplying (7.5) by $e^{i\theta}$, taking the real part and then the inner-product with $A^r e^{2(\beta_0-\delta s)A^{1/2}}u$, we readily obtain

$$
\frac12\frac{d}{ds}|\!|u(\zeta)|\!|^2 + \delta\,|\!|A^{1/4}u(\zeta)|\!|^2 = -((\ Re(e^{i\theta}F(u(\zeta))), u(\zeta)\ )),\ \zeta = se^{i\theta}.
$$

Using Proposition 7.2, we obtain

$$\frac{1}{2}\frac{d}{ds}|u|^2 + \delta\,|A^{1/4}u|^2 \lesssim \widetilde{F}(|u|)|A^{1/4}u|^2. \tag{7.10}$$

Now choose

$$\delta = C\widetilde{F}(\|\boldsymbol{u}_0\|_{\beta_0}).$$

From (7.10), and the fact that $\widetilde{F}(\cdot)$ is strictly increasing (7.4), we see that $|u|$ is non-increasing and

$$|u(\zeta)| \le \|u_0\|_{\beta_0}\ \forall\ \zeta = se^{i\theta},\ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z\in\mathcal{R}} \|u(z)\| \le \|u_0\|_{\beta_0}.$$

As before, the proof is now complete by invoking Montel's theorem.

□

Remark 7.4. *One can extend the method of this section to handle a nonlinearity of the form*

$$F(u) = T_0G(T_1u, \ldots, T_nu),$$

where G is an analytic function of n-variables and T_i are Fourier multipliers with symbol m_i satisfying

$$|m_i(\boldsymbol{k})| \lesssim |\boldsymbol{k}|^{\alpha_i}\ \forall\ \boldsymbol{k} \in \dot{\mathbb{Z}}^d, 0 \le i \le n, \sum_{i=0}^{n} \alpha_i \le 1.$$

Using the exact same technique, one can in fact also consider the case of systems, in which case Theorem 3.1 becomes a special case.

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