

L^∞ Bounds for Solutions of Parametrized Elliptic Equations*

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This generalizes earlier results (T. I. Seidman, *Indiana Univ. Math. J.* **30** (1981), 305–311) for $-Au = \lambda f(u)$. For the family of equations (*) $Au = g(u, \lambda)$ with appropriate boundary conditions the object is to construct from g and the boundary conditions a function $\eta(\lambda, r)$ such that a bound $y(\lambda)$ on $\|u\|_\infty$ can be obtained by solving the ODE: $y'(\lambda) = \eta(\lambda, y)$ with $y(\lambda_0) = B(\lambda_0) = \text{bound at } \lambda = \lambda_0$. © 1985 Academic Press, Inc.

1. INTRODUCTION

For a bounded region $\Omega \subset \mathbb{R}^m$, consider a second order elliptic (linear) operator

$$Au := -\nabla \cdot a \nabla u + b \cdot \nabla u \quad (1.1)$$

where $b = b(x)$ is \mathbb{R}^m valued on Ω and $a = a(x)$ is either scalar or non-negative matrix valued with $a(x) \geq \alpha > 0$. Let $g = g(x, r, \lambda) > 0$ be defined on $\bar{\Omega} \times \mathbb{R}^+ \times [\lambda_0, \bar{\lambda}]$ with $g_r := \partial g / \partial r$ and $g_\lambda := \partial g / \partial \lambda$ satisfying

$$g_r < 0 \leq g_\lambda. \quad (1.2)$$

We shall consider the parametrized (family of) elliptic partial differential equation(s)

$$Au = g(\cdot, u, \lambda) \quad (1.3)$$

with boundary conditions of either of the forms:

$$\begin{aligned} \text{(i)} \quad u &= \gamma(\cdot, \lambda) \quad \text{or} \\ \text{(ii)} \quad -u_\nu &= \gamma(\cdot, u, \lambda) \end{aligned} \quad (1.4)$$

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on $\partial\Omega$. (In (ii), u_ν is the outward conormal derivative $a\nabla u \cdot \mathbf{n}$ on $\partial\Omega$.) We require, corresponding to (1.4), that

$$\begin{aligned} \text{(i)} \quad & 0 \leq \gamma, 0 \leq \gamma_\lambda && \text{on } \partial\Omega \times [\lambda_0, \bar{\lambda}] && \text{or} \\ \text{(ii)} \quad & \gamma(\cdot, 0, \cdot) \leq 0, \gamma_\lambda \leq 0 < \gamma_r && \text{on } \partial\Omega \times \mathbb{R}^+ \times [\lambda_0, \bar{\lambda}]. \end{aligned} \quad (1.5)$$

A solution u will be interpreted as an element of $H^1(\Omega)$ satisfying (1.3), (1.4) in the usual weak formulation. Our object will be to show that, in addition, one has

$$0 \leq u \leq \gamma(\lambda) \quad \text{a.e. on } \Omega \quad (1.6)$$

with $\gamma(\lambda)$ suitably constructed on $[\lambda_0, \bar{\lambda}]$. The argument will proceed by the application of (pointwise) maximum principle (MP) arguments under the temporary assumption that u is a smooth (classical) solution of (1.3), (1.4) with smooth g, γ after which a limit argument provides the corresponding estimate for the more general case.

In the earlier paper [4] the case $\mathbf{A} = -\Delta$, $g(x, r, \lambda) = \lambda \exp[-h(r)]$ ($-h$ convex and nondecreasing, $\lambda > 0$) was considered with homogeneous Dirichlet conditions and it was shown that, with $f(r) := \exp[-h(r)]$,

$$B(\lambda) = \mathcal{O}(f^{-1}(C/\lambda)) \quad \text{as } \lambda \rightarrow \infty \quad (1.7)$$

where we use $B(\lambda)$ to denote $\|u\|_\infty$ for the solution u of the equation with a certain λ . The argument used an MP argument to reduce this to the case $\Omega =$ (a ball) and then made essential use of the invariance properties of $-\Delta$ and of the spatial independence of $g = \lambda f$ to ensure that u attained its maximum at a point, the center of the ball, which did not depend on λ .

The point of the present paper is to remove these restrictions on the operator and on the nonlinearity. (We note also a debt to the (anonymous) referee who considered an earlier version of this paper—under the (provisional) title [5]—and recommended inclusion of the first order term $b \cdot \nabla u$ in the operator \mathbf{A} .)

As in the original case [4], differentiate (1.3), (1.4) with respect to λ . This gives a corresponding linear problem for $w := \partial u / \partial \lambda$:

$$\mathbf{A}w = g_r w + g_\lambda \quad \text{on } \Omega \quad (1.8)$$

$$\text{(i)} \quad w = \gamma_\lambda \quad \text{or} \quad \text{at } \partial\Omega \quad (1.9)$$

$$\text{(ii)} \quad -w_\nu = \gamma_r w + \gamma_\lambda$$

(with $g_r, g_\lambda, \gamma_\lambda, \gamma_r$ evaluated for $u = u(\cdot, \lambda)$). We can obtain a differential inequality in λ for $B := \|u\|_\infty$ by estimating w from application of MP to (1.8), (1.9).

2. ESTIMATION

Throughout this paper we assume that Ω is a bounded (connected) open set in \mathbb{R}^m with "sufficiently smooth" boundary $\partial\Omega$. (This last is merely an announcement that our concerns are only with those technical difficulties arising from considerations other than regularity of $\partial\Omega$ —e.g., [3] imposes C^∞ regularity on $\partial\Omega$ but notes that this is excessive and, without further precision, we invoke the same spirit.) In this section we at first assume sufficient regularity of $a(\cdot)$, $b(\cdot)$ in (1.1), of g in (1.3) and of γ in (1.4) to ensure that the solution is a classical solution:

$$u(\cdot, \lambda) \in C^2(\Omega) \cap C^1(\bar{\Omega})$$

and to permit continuous differentiation with respect to λ as well. This is certainly the case if a, b, g, γ are C^∞ , for example: use standard regularity results (e.g., [3] with "bootstrapping" and the Sobolev Embedding Theorem). The Maximum Principle arguments to be employed will give $u \geq 0$ so the hypotheses need only be considered there. Also, although the thrust of the paper is toward estimates as $\lambda \rightarrow \infty$, our results will be given in a comparison form which is meaningful if consideration is restricted to λ in an interval $[\lambda_0, \bar{\lambda}]$ in which case the hypotheses need only hold there.

Our principal hypothesis on $g = g(x, r, \lambda)$ is (1.2): that it is increasing in λ and decreasing in r . We introduce the function η_0 given on $[\lambda_0, \bar{\lambda}] \times \mathbb{R}^+$ by

$$\eta_0(\lambda, \bar{r}) := \max \left\{ -\frac{g_\lambda(x, r, \lambda)}{g_r(x, r, \lambda)}; \begin{array}{l} x \in \Omega \\ 0 \leq r \leq \bar{r} \end{array} \right\}. \quad (2.1)$$

(This is well defined, positive and finite for smooth g satisfying (1.2).) To estimate the effect of the boundary conditions we introduce the function η_1 given (depending on the choice of (i), (ii) in (1.4)) by

$$\begin{aligned} \text{(i)} \quad \eta_1(\lambda, \bar{r}) = \eta_1(\lambda) &:= \max \{ \gamma_\lambda(x, \lambda); x \in \partial\Omega \} \quad \text{or} \\ \text{(ii)} \quad \eta_1(\lambda, \bar{r}) &:= \max \left\{ -\frac{\gamma_\lambda(x, r, \lambda)}{\gamma_r(x, r, \lambda)}; \begin{array}{l} x \in \partial\Omega \\ 0 \leq r \leq \bar{r} \end{array} \right\}. \end{aligned} \quad (2.2)$$

(Again, η_1 is well defined, positive and finite for smooth γ satisfying (1.5).) We now set

$$\eta(\lambda, \bar{r}) := \max \{ \eta_0(\lambda, \bar{r}), \eta_1(\lambda, \bar{r}) \}. \quad (2.3)$$

We will only treat the case (1.4ii) in detail. The Dirichlet case (1.4i) could be treated in a corresponding fashion, with minor modifications. Alternatively, given (1.4i) one could set

$$\hat{\gamma}^e(x, r, \lambda) := [r - \gamma(x, \lambda)]/\varepsilon$$

for $\varepsilon > 0$, obtain u^ε from (1.3) with $\hat{\gamma}^\varepsilon$ used for γ in (1.4ii), apply our results to estimate u^ε , and finally note that $u^\varepsilon \rightarrow u$ (given by (1.3), (1.4i)) as $\varepsilon \rightarrow 0+$. (Observe that η_1 given by (2.2ii) with this $\gamma = \hat{\gamma}^\varepsilon$ is independent of ε and is the same η_1 as would be given by (2.2i).)

THEOREM 1. *Let $\partial\Omega$ and all the functions involved be smooth; let \mathbf{A} be given by (1.1) and let g, γ satisfy (1.2), (1.5), respectively. For $\lambda_0 \leq \lambda < \bar{\lambda}$, let u be the solution of (1.3), (1.4) and set $B(\lambda) := \|u\|_\infty$. Define η on $[\lambda_0, \bar{\lambda}] \times \mathbb{R}^+$ by (2.1)–(2.3) and, with $b_0 \geq B(\lambda_0)$, let y be the (maximal) solution of*

$$y'(\lambda) = \eta(\lambda, y(\lambda)), \quad y(\lambda_0) = b_0 \quad (2.4)$$

on $[\lambda_0, \bar{\lambda}]$. Then $B(\lambda) \leq y(\lambda)$ on $[\lambda_0, \bar{\lambda}]$, i.e.,

$$0 \leq u(x; \lambda) \leq y(\lambda) \quad \text{on } \bar{\Omega} \times [\lambda_0, \bar{\lambda}]. \quad (2.5)$$

Proof. As noted above, we treat only the case (1.4ii).

Since, g is decreasing in u for fixed λ , a standard argument shows existence of a (weak) solution $u = u(\cdot, \lambda)$ of (1.3), (1.4ii) and (with a, b, g, λ smooth) regularity results show this is a smooth classical solution. If u were to attain its minimum (on $\bar{\Omega}$) at a boundary point $\bar{x} \in \partial\Omega$, then $u_v(\bar{x}) \leq 0$ so (1.4ii) gives $\gamma(\bar{x}, u(\bar{x}), \lambda) \geq 0$ which, by (1.5ii), is impossible unless $u(\bar{x}) \geq 0$. If u attains its minimum at $\bar{x} \in \Omega$ then $[Au](\bar{x}) \leq 0$ which is impossible since $g \geq 0$. Hence $u \geq 0$ on Ω .

Since $g_\lambda \geq 0$, $\gamma_\lambda \leq 0$, a comparison argument along similar lines shows that u is isototonically dependent on λ . Setting

$$w^h(x) := [u(x; \lambda + h) - u(x; \lambda)]/h$$

one obtains from (1.3)

$$\begin{aligned} \mathbf{A}w^h &= [g(\cdot, u(\cdot; \lambda + h), \lambda + h) - g(\cdot, u(\cdot; \lambda), \lambda)]/h \\ &= g_r(\cdot, \hat{u}, \hat{\lambda}) w^h + g_\lambda(\cdot, \tilde{u}, \tilde{\lambda}) \end{aligned} \quad (2.6)$$

and, similarly, one obtains from (1.4ii)

$$\begin{aligned} -w_v^h &= [\gamma(\cdot, u(\cdot; \lambda + h), \lambda + h) - \gamma(\cdot, u(\cdot; \lambda), \lambda)]/h \\ &= \gamma_r(\cdot, \hat{u}, \hat{\lambda}) w^h + g_\lambda(\cdot, \tilde{u}, \tilde{\lambda}) \end{aligned} \quad (2.7)$$

with, in each case,

$$u(\cdot; \lambda) \leq \hat{u}, \tilde{u} \leq u(\cdot; \lambda + h) \leq B(\lambda + h), \quad \lambda \leq \hat{\lambda}, \tilde{\lambda} \leq \lambda + h. \quad (2.8)$$

By the isotonic dependence of u on λ (or by considering the point at which w^h attains its minimum on $\bar{\Omega}$ and using an argument much as for u above) one sees that $w^h \geq 0$ on $\bar{\Omega}$. Now consider the point \bar{x} at which w^h attains its maximum. If $\bar{x} \in \partial\Omega$, then $w_v^h(\bar{x}) \geq 0$ so, from (2.7),

$$w^h(\bar{x}) \leq -\frac{\gamma_\lambda(\bar{x}, \tilde{u}, \tilde{\lambda})}{\gamma_r(\bar{x}, \hat{u}, \hat{\lambda})} \quad (2.9)$$

while $\bar{x} \in \Omega$ gives $[Aw^h](\bar{x}) \geq 0$ so then, as $g_r < 0$,

$$w^h(\bar{x}) \leq -\frac{g_\lambda(\bar{x}, \tilde{u}, \tilde{\lambda})}{g_r(\bar{x}, \hat{u}, \hat{\lambda})}. \quad (2.10)$$

Using the smoothness of γ , g and (2.8), one has uniform bounds on the right hand sides of (2.9), (2.10) for, say, $0 < h < \bar{h}$. Since this bounds the right hand sides of (2.6), (2.7), elliptic theory gives a uniform $H^1(\Omega)$ bound for w^h so we may assume (extracting a subsequence if necessary) that w^h converges as $h \rightarrow 0$. Since (2.8) gives $\hat{u}, \tilde{u} \rightarrow u(\cdot, \lambda)$ and $\hat{\lambda}, \tilde{\lambda} \rightarrow \lambda$ as $h \rightarrow 0$, the limit must be the unique solution w of (1.8), (1.9)—indeed, bootstrapping the regularity shows that the convergence can be taken to be uniform on $\bar{\Omega}$ so (observing that, as the limit for the convergent subsequences is unique, there was no need to extract subsequences at all) we have justified the formal differentiation of (1.3), (1.4ii) to show that $w := \partial u / \partial \lambda := \lim w^h$ exists and satisfies (1.8), (1.9). Further, taking the limits as $h \rightarrow 0$ on the right-hand sides of (2.9), (2.10)—or arguing directly and similarly for w —one obtains the combined estimate

$$0 \leq w \leq \max\{-\gamma_\lambda/\gamma_r, -g_\lambda/g_r\} \leq \eta(\lambda, B(\lambda)) \quad (2.11)$$

on $\bar{\Omega}$ for each λ in $[\lambda_0, \bar{\lambda}]$.

Now, returning to u , suppose $u = u(\cdot, \lambda)$ attains its maximum on $\bar{\Omega}$ at $\bar{x} = \bar{x}(\lambda)$ —not necessarily unique—so one has

$$B(\lambda) = u(\bar{x}(\lambda); \lambda)$$

for each λ . For $h > 0$, then, one has from (2.11) that

$$\begin{aligned} B(\lambda + h) &= u(\bar{x}(\lambda + h); \lambda + h) \\ &= u(\bar{x}(\lambda + h); \lambda) + hw(\bar{x}(\lambda + h); \lambda + \delta) \\ &\leq B(\lambda) + h\eta(\lambda + \delta, B(\lambda + \delta)) \end{aligned}$$

(using the Mean Value Theorem) with $0 < \delta < h$. The assumed smoothness implies continuity of η , B so we may let $h \rightarrow 0+$ to obtain the desired differential inequality

$$\frac{d^+ B}{d\lambda} \leq \eta(\lambda, B(\lambda)). \quad (2.12)$$

A Gronwall-type argument then gives the comparison $B(\lambda) \leq y(\lambda)$ with y given by (2.4) provided one has initially that $y(\lambda_0) = b_0 \geq B(\lambda_0)$.

As noted earlier, the case (1.4i) could be treated in essentially the same way or, alternatively, derived from this. ■

We now seek to relax the regularity assumptions. Without seeking the best possible result, we merely indicate an approach which permits a substantial weakening of the smoothness requirements imposed above, permitting the consideration of weak solutions.

For continuous coefficients a, b in (1.1) there is no difficulty in finding C^∞ functions a^m, b^m such that

$$\begin{aligned} a^m &\rightarrow a && \text{uniformly on } \bar{\Omega}, && a^m \geq \alpha \\ b^m &\rightarrow b && \text{uniformly on } \bar{\Omega}. \end{aligned} \quad (2.13)$$

Denote by A_m the operator defined as in (1.1) using a^m, b^m .

Looking at the derivation of the key estimate (2.11) from (2.6), (2.7), we redefine η_0, η_1 as follows:

$$\eta_0(\lambda, \bar{r}) := \inf \left\{ \begin{array}{l} \text{for } h \text{ small enough one has} \\ \eta: g(x, r + \delta, \lambda + h) < g(x, r, \lambda) \\ \text{for all } x \in \Omega, r \leq \bar{r}, \delta > \eta h \end{array} \right\} \quad (2.1')$$

$$(i) \quad \eta_1(\lambda, \bar{r}) = \eta_1(\lambda) := \limsup_{h \rightarrow 0+} \sup \left\{ \frac{\gamma(x, \lambda + h) - \gamma(x, \lambda)}{h} : x \in \partial\Omega \right\} \quad (2.2')$$

$$(ii) \quad \eta_1(\lambda, \bar{r}) := \inf \left\{ \begin{array}{l} \text{for } h \text{ small enough one has} \\ \eta: \gamma(x, r + \delta, \lambda + h) > \gamma(x, r, \lambda) \\ \text{for all } x \in \partial\Omega, r \leq \bar{r}, \delta > \eta h \end{array} \right\}.$$

These reduce to (2.1), (2.2) for smooth g, γ ; we continue to take $\eta := \max\{\eta_0, \eta_1\}$.

Now assume one can approximate g by a sequence of C^∞ functions (g^m) in such a way that

$$\begin{aligned} -g_r^m &\geq \varepsilon > 0 \text{ uniformly on } \bar{\Omega} \times [0, \hat{y}] \times [\lambda_0, \hat{\lambda}], \quad m = 1, 2, \dots, \\ z^m &\rightarrow z \text{ in } L^2(\Omega \rightarrow [0, y]) \text{ implies } g^m(\cdot, z^m, \lambda) \rightarrow g(\cdot, z, \lambda) \text{ in } L^2(\Omega), \\ \eta_0(\cdot; g^m) &\rightarrow \eta_0(\cdot; g) \quad \text{uniformly on } [\lambda_0, \hat{\lambda}] \times [0, \hat{y}] \end{aligned} \quad (2.14)$$

where $\hat{y} := y(\hat{\lambda}) + \varepsilon$ with $y(\cdot)$ defined by (2.4) using $\eta = \eta(\cdot; g)$. It is not clear exactly what would be the minimal regularity for g needed to justify existence of such an approximating sequence but it would certainly be sufficient to have g, g_r, g_λ continuous with $g_r \leq -\varepsilon$ and η well defined (finite)

from (2.1). Similarly, assume γ can be suitably approximated by C^∞ functions γ^m such that (1.5) holds for each r^m and

- (i) $\gamma^m(\cdot, \lambda) \rightarrow \gamma(\cdot, \lambda)$ in $L^2(\partial\Omega)$
 $\eta_1(\cdot; g^m) \rightarrow \eta_1(\cdot; g)$ uniformly on $[\lambda_0, \hat{\lambda}]$
(ii) $z^m \rightarrow z$ in $L^2(\partial\Omega \rightarrow [0, \hat{y}])$ implies $\gamma^m(\cdot, z_m, \lambda) \rightarrow \gamma(\cdot, z, \lambda)$ in $L^2(\partial\Omega)$,
 $\eta_1(\cdot; \gamma^m) \rightarrow \eta_1(\cdot; \gamma)$ uniformly on $[\lambda_0, \hat{\lambda}] \times [0, \hat{y}]$. (2.15)

We now let u^m be the unique classical solution of the problem (1.3)^m, (1.4)^m using the approximating A_m, g^m, γ^m .

The conditions (2.13), (2.14), (2.15) ensure that the sequence (u^m) is bounded in $H^1(\Omega)$ and so, extracting a subsequence, can be taken weakly convergent in $H^1(\Omega)$ to a function \hat{u} and strongly convergent to \hat{u} in $L^2(\Omega)$ and $L^2(\partial\Omega)$. Using the standard weak formulation of the problems, one easily sees that \hat{u} satisfies (1.3), (1.4) and so, by uniqueness, coincides with u .

Applying the theorem to each u^m gives $0 \leq u^m \leq y^m(\lambda)$ for $\lambda_0 \leq \lambda \leq \hat{\lambda}$. Since $\eta^m \rightarrow \eta$ uniformly by (2.14), (2.15), one has $y^m \rightarrow y$. Hence, for any $\varepsilon > 0$ one has each $0 \leq u^m \leq y(\lambda) + \varepsilon$ for large enough m and so $0 \leq u = u^m \leq y(\lambda) + \varepsilon$. Thus, the conclusion of Theorem 1 holds under the weakened regularity hypotheses.

3. EXTENSION AND DISCUSSION

Remark 1. Observe that an essentially similar analysis can be made for parabolic equations

$$\dot{u} + Au = g(t, x, u, \lambda) \quad \text{on } Q := (0, T) \times \Omega$$

etc., with A as in (1.1). Besides the boundary conditions at $(0, T) \times \partial\Omega$ one must now include consideration of initial conditions

$$u(0, x) = u_0(x, \lambda) \quad \text{on } \Omega \quad (3.1)$$

which introduce a term

$$\eta_2(\lambda, \bar{r}) = \eta_2(\bar{r}) := \sup \{ \partial u_0 / \partial \lambda : x \in \Omega \} \quad (3.2)$$

now giving $\eta := \max \{ \eta_0, \eta_1, \eta_2 \}$. This assumes $0 \leq u_0 \in L^\infty(\Omega)$ and $b_0 \geq \sup_x u_0$. (One first, as above, considers the situation with u_0 smooth as well—and also imposing the standard parabolic consistency conditions at $\{t=0\} \times \partial\Omega$ to work with smooth, classical solutions and a pointwise

maximum principle, followed by a weakening of the regularity hypotheses through approximation and a limit argument.) The conclusions are then the same as for the elliptic case treated in the previous section.

Remark 2. The arguments of Theorem 1 cannot be applied *directly* when the operator A is nonlinear or depends on λ —say, $a = a(x, r, \lambda)$ with $b \equiv 0$ now for simplicity so (1.3) becomes

$$-\nabla \cdot a(\cdot, u, \lambda) \nabla u = g(\cdot, u, \lambda) \quad \text{on } \Omega \quad (3.3)$$

with g as earlier. One may nevertheless be able to use Theorem 1 to bound solutions of (3.3).

Fixing $\lambda = \hat{\lambda}$, let \hat{u} satisfy (3.3) with appropriate boundary conditions as in (1.4) and set

$$\hat{a}(x) := a(x, \hat{u}(x), \hat{\lambda}).$$

Clearly \hat{u} is also the (unique) solution of (1.3), (1.4) with $\hat{A}: u \mapsto -\nabla \cdot \hat{a} \nabla u$ replacing A in (1.3).

Now consider the new parametrized family of elliptic problems ($\lambda_0 \leq \lambda \leq \hat{\lambda}$)

$$\hat{A}u = g(\cdot, u, \lambda), \quad (1.4) \quad (3.4)$$

where, now, \hat{A} is independent of λ . If one were able to obtain a bound b_0 on the solution $\hat{u}_0 = \hat{u}(\cdot; \lambda_0)$ of (3.4) with $\lambda = \lambda_0$, then Theorem 1 provides the estimate $0 \leq \hat{u} \leq y(\hat{\lambda})$ with y given by (2.4).

Note that (2.4) depends on the operator—and so on the (presumably unknown) function \hat{u} —only through the choice of b_0 bounding \hat{u}_0 . If one could obtain a *uniform* estimate b_0 for the solutions of (3.4) as \hat{a} ranges over all the functions $a(\cdot, u(\cdot; \hat{\lambda}), \hat{\lambda})$ for $\lambda_0 \leq \hat{\lambda} < \hat{\lambda}$, then the (maximal) solution y of (2.4) with this b_0 would bound $\|u(\cdot; \lambda)\|_\infty$ on $[\lambda_0, \hat{\lambda}]$. The possible advantage of this would depend on g being easy to work with at $\lambda = \lambda_0$. The difficulty in estimating b_0 is that $\hat{u} = u(\cdot; \hat{\lambda})$ is unknown: otherwise there would be no point in using such an indirect procedure. This need not be impossible, however, and we indicate an approach in the Dirichlet case.

Consider the problem of bounding the solution of

$$\hat{A}u := -\nabla \cdot \hat{a} \nabla u = g(\cdot, u) \quad \text{on } \Omega, \quad u|_{\partial\Omega} = \gamma(\cdot) \quad (3.5)$$

with $\gamma \geq 0$ and with $g > 0$ and decreasing in u . Define

$$\bar{g}(r) := \sup\{g(x, r): x \in \bar{\Omega}\}$$

$$\bar{\gamma} := \sup\{\gamma(x): x \in \partial\Omega\}$$

and note that a (weak) MP argument gives $0 \leq u \leq \bar{u}$, where \bar{u} is the solution of

$$\hat{A}\bar{u} = \bar{g}(\bar{u}) \quad \text{on } \Omega, \quad \bar{u}|_{\partial\Omega} = \bar{\gamma}. \quad (3.6)$$

Now set $v := \bar{g}(\bar{u})$. Assuming \bar{g} is differentiable with $\bar{g}' < 0$, we have $\nabla v = \bar{g}'(\bar{u}) \nabla \bar{u}$ and can set

$$\theta(x) := -\hat{a}(x)/\bar{g}'(\bar{u}(x)) \quad \text{so} \quad \theta \nabla v = \hat{A} \nabla \bar{u}$$

obtaining

$$-\nabla \cdot \theta \nabla v + v = 0 \quad \text{on } \Omega, \quad v|_{\partial\Omega} = \bar{g}(\bar{\gamma}) > 0.$$

The maximum principle then gives $0 \leq v \leq \bar{g}(\bar{\gamma})$ on $\bar{\Omega}$ so $-\nabla \cdot \theta \nabla v \geq -\bar{g}(\bar{\gamma})$ and another application of MP gives $v \geq \bar{g}(\bar{\gamma})[1 - w]$ where w is the solution of

$$-\nabla \cdot \theta \nabla w = 1 \quad \text{on } \Omega, \quad w|_{\partial\Omega} = 0.$$

Combining these gives

$$0 \leq u \leq \bar{g}^{-1}(v_{\min}) \leq \bar{g}^{-1}(\bar{g}(\bar{\gamma})[1 - w_{\max}]). \quad (3.7)$$

The function \bar{g}^{-1} and the number $\bar{g}(\bar{\gamma})$ are, in principle, known. To estimate w_{\max} we appeal to results on "radially decreasing rearrangements" (cf., e.g., [1, 6]) which give existence of a constant C_Ω , depending only on Ω , such that

$$w_{\max} \leq C_\Omega \sup\{1/\theta(x) : x \in \Omega\}.$$

We now add the assumption that \bar{g} is convex ($-\bar{g}'$ nonincreasing) and $\hat{a} \geq \alpha > 0$ so

$$\begin{aligned} \theta &\geq \alpha / \sup\{-\bar{g}'(\bar{u})\} \geq -\alpha / \bar{g}'(\bar{\gamma}) \\ w_{\max} &\leq C_\Omega [-\bar{g}'(\bar{\gamma})] / \alpha. \end{aligned}$$

This estimate is useless to us unless

$$-\bar{g}'(\bar{\gamma}) < \alpha / C_\Omega$$

but if that is the case (3.7) provides the desired bound

$$b_0 := \bar{g}^{-1}(\bar{g}(\bar{\gamma})[1 + C_\Omega \bar{g}'(\bar{\gamma})/\alpha]) \quad (3.8)$$

in a form depending only on the right-hand sides g, γ of (2.5), on the domain Ω , and on the uniform ellipticity constant α .

Remark 3. For comparative purposes we now reconsider the case $g(x, r, \lambda) := \lambda f(r)$ discussed in [4], taking linear boundary conditions independent of λ :

$$\begin{aligned} \text{(i)} \quad & u = \gamma(\cdot) \quad \text{or} \\ \text{(ii)} \quad & -[u_\nu + cu] = \gamma(\cdot) \end{aligned} \quad (3.9)$$

with $\gamma \geq 0$ on $\partial\Omega$. This gives $\eta_1 = 0$ in each case so $\eta = \eta_0$. From (1.2) we ask that $\lambda > 0$ and that $f(r)$ be positive and strictly decreasing in r . Setting

$$\Phi(z) := \int^z \inf\{-f'(s)/f(s); 0 \leq s \leq r\} dr$$

one has from (2.1) that $\eta = \eta_0(\lambda, \bar{r}) = 1/\lambda \Phi'(\bar{r})$. Thus, (2.4) gives

$$\frac{d\Phi(y)}{dy} \frac{dy}{d\lambda} = \frac{1}{\lambda}, \quad y(\lambda_0) = b_0$$

so

$$\lambda e^{-\Phi(y(\lambda))} = \text{const.} := \lambda_0 e^{-\Phi(b_0)}. \quad (3.10)$$

In the light of Theorem 1 and noting that Φ is increasing, this can also be interpreted as asserting that

$$\lambda e^{-\Phi(B(\lambda))} \text{ is a nondecreasing function of } \lambda. \quad (3.11)$$

In the logarithmically convex case (i.e., $\log f$ convex so f'/f is non-decreasing) considered in [4] one has $f = e^{-\Phi}$ on making the proper choice of the arbitrary additive constant in the definition of Φ . In this case (3.10) gives

$$y(\lambda) = f^{-1}(C/\lambda) \quad \text{for } \lambda \geq \lambda_0 \quad (3.12)$$

with $C := \lambda_0 f(b_0)$. (If $C \leq 1$ this implies directly that $y = \mathcal{O}(f^{-1}(1/\lambda))$ while otherwise that would follow from the existence of $M = M_c$ such that $\Phi(Mz) + c \leq \Phi(z)$ for some $c > 0$ and large z .) This recovers the estimate of [4].

Now consider the very special case

$$Au = \lambda e^{-cu} \quad \text{on } \Omega, \quad u|_{\partial\Omega} = 0. \quad (3.13)$$

From (3.12) and Theorem 1 one gets

$$B(\lambda) \leq y(\lambda) = [c^{-1} \log \lambda + \text{const.}].$$

To see that this is sharp, we consider $\Omega := [\text{unit ball in } \mathbb{R}^2]$ and note the explicit solution

$$u(x; \lambda) = \frac{2}{c} \log \left(\frac{\xi - |x|^2}{\xi - 1} \right) \quad (3.14)$$

where ξ is defined by: $\lambda c(\xi - 1)^2 = 8\xi$. This gives

$$\begin{aligned} B(\lambda) = u(0; \lambda) &= \frac{2}{c} \log[\sqrt{2\lambda/8 + 1/4} + 1/2] \\ &= [c^{-1} \log \lambda + c^{-1} \log(c/8) + \mathcal{O}(1/\sqrt{\lambda})] \end{aligned}$$

as $\lambda \rightarrow \infty$ which shows—at least in this case for which an explicit solution is available—that we not only have $B(\lambda) \leq y(\lambda)$ but actually

$$B(\lambda) \sim y(\lambda) \quad \text{as } \lambda \rightarrow \infty. \quad (3.15)$$

It is quite easy to obtain a lower bound for $B(\lambda)$, as in [4], for the general case considered in Theorem 1. For simplicity we consider only Dirichlet boundary conditions (1.4i) and set

$$\begin{aligned} \bar{g}(r, \lambda) &:= \inf \{ g(x, r, \lambda) : x \in \Omega \} \\ \bar{\gamma}(\lambda) &:= \inf \{ \gamma(x, \lambda) : x \in \partial\Omega \} \end{aligned}$$

so $u \leq B = B(\lambda)$ gives $\bar{g}(B, \lambda) \leq g(x, B, \lambda) \leq g(x, u, \lambda)$ for each $x \in \Omega$, noting that $g_r < 0$. Let u be the solution of (1.3), (1.4i) and let v satisfy

$$Av = \bar{g}(B, \lambda) \quad \text{on } \Omega, \quad v|_{\partial\Omega} = \bar{\gamma}(\lambda).$$

This makes $A(v - u) \geq 0$, $(v - u)|_{\partial\Omega} \geq 0$ so, considering the point \bar{x} at which $(v - u)$ attains its maximum (as in the proof of Theorem 1), we have $0 \leq u \leq v$. Letting v_* be the solution of

$$Av_* = 1 \quad \text{on } \Omega, \quad v_*|_{\partial\Omega} = 0$$

we see from the form of A that $v = \bar{g}(B, \lambda) v_* + \bar{\gamma}(\lambda)$ so

$$\bar{g}(B, \lambda) K_* + \bar{\gamma}(\lambda) \leq B = B(\lambda)$$

with $K_* := \|v_*\|_\infty = \max v_*$. This may be rewritten as

$$B - K_* \bar{g}(B, \lambda) \geq \bar{\gamma}(\lambda) \quad (3.16)$$

and, since \bar{g} is decreasing in B , the left-hand side of (3.16) is strictly increasing in B so (3.16) can be used to get a lower bound for $B = B(\lambda)$.

For the case $g(x, r, \lambda) = e^{-cr}$ considered above (with γ independent of λ so $\bar{\gamma}$ is a constant) one gets from (3.16)

$$B - K_* \lambda e^{-cB} \geq \bar{\gamma}.$$

Setting $\mu = \log[cK_* \lambda e^{-c\bar{\gamma}}]$, this gives $c[B - \bar{\gamma}] \geq \beta$ where β is defined by the transcendental equation $\beta e^\beta = e^\mu$ which gives

$$\begin{aligned} \beta &= \mu - \log \mu + (\log \mu)/\mu \cdots \\ &= \log \lambda - \log \log \lambda + \mathcal{O}(1). \end{aligned}$$

This, in turn gives

$$B(\lambda) \geq \beta/c + \bar{\gamma} = c^{-1} \log \lambda + o(\log \lambda)$$

so that one obtains (3.15) even without the 2-dimensional explicit solution (3.14).

Somewhat more generally, one can consider the logarithmically convex case $g(x, r, \lambda) = \lambda e^{-\Phi(r)}$ with $-\Phi$ decreasing and convex in r . For simplicity we take homogeneous Dirichlet conditions so $\bar{\gamma} = 0$. We have then $\beta \leq B \leq y$ with

$$e^{\Phi(y)} = c\lambda \quad \text{from (3.10)}$$

$$\beta e^{\Phi(\beta)} = K_* \lambda \quad \text{from (3.16)}.$$

Hence

$$\Phi(\beta) + \log \beta = \Phi(y) + \text{const.} \quad (3.17)$$

and, setting $\rho := y/\beta$, the Mean Value Theorem gives

$$(\rho - 1) \Phi'(\zeta) = \mathcal{O}([\log \beta]/\beta)$$

with $\beta \rightarrow \infty$ as $\lambda \rightarrow \infty$ and $\beta < \zeta < y$ so $\Phi'(\zeta) \geq \Phi'(\rho\beta)$. If $\Phi'(r)$ does not fall off too rapidly as $r \rightarrow \infty$, this will give $\rho \rightarrow 1$ and so one again obtains (3.15). (On the other hand, (3.17) gives

$$(y - \beta) \Phi'(\beta) \geq (y - \beta) \Phi'(\zeta_*) = \log \beta + \text{const.} \rightarrow \infty$$

so the difference $(y - \beta)$ cannot remain bounded as $\lambda \rightarrow \infty$.)

Remark 4. We are indebted to the (anonymous) referee of an earlier version [5] of this paper for suggesting a possible alternative to the use of (2.3) in (2.11). In the case of Dirichlet boundary conditions (1.4i) one can

apply a somewhat different MP estimate [2, p. 35] to the problem (1.8), (1.9i) to bound w by

$$\eta_1(\lambda) + C_* \sup\{g_\lambda(x, r, \lambda): x \in \Omega, r \leq \bar{r}\} =: \eta_*(\lambda, \bar{r}) \quad (3.18)$$

where C_* is a constant depending only on \mathbf{A} (specifically, on $\sup_x\{|b|/\alpha\}$) and on the diameter of Ω . The proof of Theorem 1 would then continue from (2.11) as before.

Looking once again at the test problem $g := \lambda f = \lambda e^{-\phi(r)}$ (with ϕ non-decreasing so $g_r \leq 0$) and taking γ independent of λ in (1.4i) for simplicity (so $\eta_1 = 0$), we see that (2.3) gives

$$\eta(\lambda, \bar{r}) = \sup\{1/\phi'(r): r \leq \bar{r}\}/\lambda = 1/\lambda\phi'(\bar{r})$$

while $\eta_* = C_* e^{-\phi(0)} = \text{const.}$ Note that the use of (3.18) gives the same linear (in λ) estimate for B as if one directly estimated u with g replaced by its maximum value $g(0, \lambda) = \lambda e^{-\phi(0)}$ whereas (2.3) gives a better estimate.

On the other hand, for an example such as $g := e^{-r/\lambda}$ one would have $\eta_* = C_*/\lambda^2$, where (2.3) would give $\eta = 1/\lambda$ so (3.18) would be better for large λ . Clearly one does best by taking η (for each λ, \bar{r}) to be the smaller of η_* given by (3.18) and the original η as given by (2.3). This minimum works in (2.11) and so, by the proof of Theorem 1, can be used in (2.4).

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