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A Class of Nonlinear Elliptic Problems*

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We obtain a strict coercivity estimate, (generalizing that of T. I. Seidman [J. Differential Equations **19** (1975), 242–257] in considering spatial variation) for second order elliptic operators $\mathbf{A}: u \mapsto -\nabla \cdot \gamma(\cdot, \nabla u)$ with γ "radial in the gradient"—$\gamma(\cdot, \xi) = a(\cdot, |\xi|)\xi$ for $\xi \in \mathbb{R}^m$. The estimate is then applied to obtain existence of solutions of boundary value problems: $-\nabla \cdot \tilde{a}(\cdot, u, |\nabla u|)\nabla u = f(\cdot, u, \nabla u)$ with Dirichlet conditions.

1. INTRODUCTION

We will be considering elliptic boundary value problems involving second order differential operators of the form:

$$\mathbf{A}: u \mapsto -\nabla \cdot a(\cdot, |\nabla u|)\nabla u \tag{1.1}$$

where $a: \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is a scalar function of suitable growth. (Note: We use $|\cdot|$ to denote the euclidean norm on $\mathbb{R}^m$.) As in the case $a(\cdot, r) = r^{p-2}$ $(p > 2)$, giving $\mathbf{A}u := -\nabla \cdot |\nabla u|^{p-2} \nabla u$, which has been extensively studied, we are interested in the possibility of nonuniform ellipticity—say, with $a(\cdot, 0) = 0$. Such operators arise in a variety of physical applications (e.g., the original motivation for [2] involved induced eddy currents in a nonlinearly ferromagnetic material) and we now wish to consider spacial variation, partly to be able to treat material inhomogeneity.

It will be convenient to impose conditions not directly on $a(\cdot\,\cdot)$ but on $g: \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$g(x, r) := ra(x, r) \qquad \text{so} \qquad |a(\cdot, |\xi|)\xi| = g(\cdot, |\xi|) \quad \text{for} \quad \xi \in \mathbb{R}^m. \tag{1.2}$$

For perspective, set

$$G(x, r) := \int_0^r g(x, r')\, dr' \qquad \text{so} \quad g = \partial G/\partial r \tag{1.3}$$

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and consider the functional Γ defined by

$$\Gamma[u] := \int_\Omega G(x, |\nabla u|)\, dx. \tag{1.4}$$

(At this point the definition of Γ is purely formal since the space on which Γ can be defined must be related to the growth of G.) Continuing to proceed formally, the Gâteaux differential of Γ is given by

$$\begin{aligned}\Gamma'[u]v &:= \frac{d}{d\varepsilon}\Gamma[u+\varepsilon v]\bigg|_{\varepsilon=0} \\ &= \int_\Omega \left[\frac{\partial G}{\partial r}(\cdot, |\nabla u|)\right]\left[\frac{d\,|\xi|}{d\xi}\bigg|_{\xi=\nabla u}\right]\cdot \nabla v \\ &= \int_\Omega g(\cdot, |\nabla u|)[\nabla u/|\nabla u|]\cdot \nabla v \\ &= \int_\Omega a(\cdot, |\nabla u|)\, \nabla u \cdot \nabla v \end{aligned} \tag{1.5}$$

and, if boundary conditions are imposed which permit application of the divergence theorem here without boundary terms, this gives

$$\Gamma'[u]\colon v \mapsto \int_\Omega [\mathbf{A}u]v \qquad \text{so} \quad \Gamma'[u] = \mathbf{A}u. \tag{1.6}$$

It is well known that (strict) convexity of the functional Γ corresponds to a monotonicity condition on the operator Γ':

$$\langle \Gamma'u - \Gamma'v, u - v\rangle > 0 \qquad (u \neq v). \tag{1.7}$$

A stronger variant of (1.7) ensures continuous invertibility of Γ' which corresponds to the existence of a minimum, depending continuously on f, for the functional $(\Gamma[u] - \langle f, v\rangle)$. This variant,

$$\phi(\|\nabla u - \nabla v\|) \leqslant \int_\Omega (u-v)[\mathbf{A}u - \mathbf{A}v] \tag{1.8}$$

(where $\phi(r) \to 0$ or bounded implies $r \to 0$ or bounded, resp.), which we call a *coercivity estimate* for $\mathbf{A}$, will be the principal result of the paper and will be developed in Section 2.

This immediately gives well-posedness for elliptic boundary value problems:

$$\mathbf{A}u = f \text{ on } \Omega \qquad + \text{(suitable boundary conditions)} \tag{1.9}$$

and we also prove continuous dependence of the solution on the structure of **A**, i.e., on the (nonlinear) coefficient function $a(\cdot\,\cdot)$. This can be used to obtain existence of solutions of more complicated problems of the form:

$$-\nabla\cdot\tilde{a}(\cdot, u, |\nabla u|)\,\nabla u=\tilde{f}(x, u, \nabla u). \tag{1.10}$$

The approach is to set

$$\begin{aligned} a_v(x, r) &:= \tilde{a}(x, v(x), r),\\ f_v(x) &:= \tilde{f}(x, v(x), \nabla v(x)) \end{aligned} \tag{1.11}$$

and consider the composite map

$$\mathbf{T}\colon [v, f]\mapsto [a_v, f] \overset{*}{\mapsto} u\mapsto [u, f_u], \tag{1.12}$$

where * is defined by (1.9) with $\mathbf{A} := \mathbf{A}_v$. A fixed point of **T** gives a solution of (1.10). For expository purposes we consider a particular case of (1.10):

$$-\nabla\cdot\left(\frac{|\nabla u|^2\,\nabla u}{u^2+|\nabla u|^2}\right)=f(x, u, \nabla u) \qquad \text{on } \Omega \tag{1.13}$$

(with homogeneous Dirichlet conditions) as a model problem. This is the content of Section 3. Finally, in Section 4 we discuss some variations on these problems and generalizations.

This paper is based on the report [3] and primarily represents results obtained while the author was visiting at the Université de Nice. Grateful acknowledgement is due to that Department of Mathematics for its hospitality and stimulating atmosphere. The author is particularly indebted, for comments and conversations, to P. Grisvard, E. McCarthy, and O. Veivoda as well as to the (anonymous) referee. Acknowledgment is also due to the Air Force Office of Scientific Research for support under grant no. AFOSR-82-0271.

2. Radial Functions on $\mathbb{R}^m$

By a *radial function* (vector field) on $\mathbb{R}^m$ we mean $\gamma\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

(i) the direction of $\gamma(\xi)$ is the same as the direction of ξ and

(ii) the length $|\gamma(\xi)|$ depends only on $|\xi|$ (with $\gamma(0)=0$).

Setting

$$g(r) := |\gamma(\xi)| \qquad \text{for} \quad |\xi| = r,$$

one then has

$$\gamma(\xi) := g(|\xi|)\,\xi/|\xi| \qquad \text{for} \quad \xi \in \mathbb{R}^m,\ \xi \neq 0. \tag{2.1}$$

(For G: $\mathbb{R}^+ \to \mathbb{R}^+$ with $G' = g$, one then has

$$\gamma(\xi) = \nabla_\xi G(|\xi|).$$

We are interested in convex G with "power growth": $G(r) \sim r^p$ for some $p > 1$.)

Our basic assumption on g will be the existence of a function μ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} &\text{(i)} \quad \mu \text{ is nondecreasing with } \mu(r) > 0 \text{ for } r > 0, \\ &\text{(ii)} \quad [g(r) - g(s)] \geqslant \mu(s)\, s^{p-2}(r-s) \qquad \text{for} \quad r > s > 0 \end{aligned} \tag{2.2}$$

for some fixed $p > 1$. Note that, if g is differentiable, then (2.2)(ii) is essentially equivalent to taking

$$\begin{aligned} \mu(s) &:= \inf\{ g'(r)/r^{p-2}\colon r > s\} \\ &= (p-1)\inf\{dg/dr^{p-1}\colon r > s\}. \end{aligned} \tag{2.3}$$

We note immediately that (2.2) implies

$$g(r) \geqslant Cr^{p-1}\mu(r/2), \qquad r > 0 \tag{2.4}$$

since

$$g(r) \geqslant g(r) - g(r/2) \geqslant (r/2)^{p-2}\mu(r/2)(r - r/2)$$

so (2.4) holds with $C = C_p = 2^{1-p}$. (Here and subsequently, C stands for a generic constant—possibly depending on p but on nothing else.)

For the remainder of this section we consider γ: $\mathbb{R}^m \to \mathbb{R}^m$ given by (2.1) subject to (2.2). For $\xi, \eta \in \mathbb{R}^n$ set

$$\begin{aligned} r &:= \max\{|\xi|, |\eta|\}, \qquad s := \min\{|\xi|, |\eta|\}, \\ \theta &:= \frac{\xi \cdot \eta}{rs}, \qquad \delta := |\xi - \eta|, \end{aligned} \tag{2.5}$$

so $r \geqslant s \geqslant 0$ and $r - s \leqslant \delta \leqslant r + s$, and $|\theta| \leqslant 1$; the definition of θ is irrelevant if $s = 0$. Also, set

$$\beta = \beta(\xi, \eta) := (\xi - \eta) \cdot [\gamma(\xi) - \gamma(\eta)]. \tag{2.6}$$

(Note that β is symmetric in ξ, η so there would be no loss of generality in assuming $|\xi| \geqslant |\eta|$; i.e., in taking $r = |\xi|$, $s = |\eta|$.)

By direct calculation we obtain the fundamental identities:

$$\begin{aligned} \beta &= (r - \theta s)[g(r) - g(s)] + (1 - \theta)(r + s)\, g(s), \\ \delta^2 &= (r - \theta s)[r - s] + (1 - \theta)(r + s)s. \end{aligned} \tag{2.7}$$

Observe that (2.7)—noting that g is nonnegative and, by (2.2), nondecreasing—gives $\beta \geqslant 0$, looking separately at each term, with $\beta = 0$ only if $s = r$ and $\theta = 1$. Thus

$$\beta(\xi, \eta) > 0 \qquad \text{if} \quad \xi \neq \eta \ (\xi, \xi) = 0). \tag{2.8}$$

Our aim is to improve (2.8) quantitatively.

THEOREM 1. *Let* $\gamma\colon \mathbb{R}^m \to \mathbb{R}^m$ *be given by* (2.1) *subject to* (2.2); *let* β *be given by* (2.6)—*hence, by* (2.7). *Then for any* $\varepsilon > 0$ *there exists* $C(\varepsilon) = C_p(\varepsilon)$ *such that*

$$\beta(\xi^{xi}, \eta) \geqslant C(\varepsilon)\, \mu(r/4)\, \delta^p \qquad \textit{if} \quad \delta \geqslant \varepsilon r \tag{2.9}$$

with δ, r *as in* (2.5).

Proof. We consider two cases: $s \leqslant r/2$ and $r/2 \leqslant s$.

Case 1. $0 \leqslant s \leqslant r/2$. One always has $r - s \leqslant \delta \leqslant r + s$ so in this case $(2/3)\delta \leqslant r \leqslant 2\delta$. Using the identity (2.7) and (2.2),

$$\begin{aligned} \beta \geqslant (r - \theta s)[g(r) - g(s)] &\geqslant (r - r/2)[g(r) - g(r/2)] \\ &\geqslant (r/2)[r/2)^{p-2}\mu(r/2)(r - r/2)] \\ &= (r/2)^p \mu(r/2) \geqslant 3^{-p}\mu(r/4)\, \delta^p \end{aligned}$$

since $r/2 \geqslant \delta/3$.

Case 2. $r/2 \leqslant s \leqslant r$. One now has $\delta/2 \leqslant r \leqslant \delta/\varepsilon$. Using (2.2), (2.4), and (2.7) gives

$$\begin{aligned} \beta &\geqslant (r - \theta s)\, s^{p-2}\mu(s)(r - s) + (1 - \theta)(r + s)\, Cs^{p-1}\mu(s/2) \\ &\geqslant Cs^{p-2}\mu(s/2)[(r - \theta s)(r - s) + (1 - \theta)(r + s)s] \\ &= Cs^{p-2}\mu(s/2)\, \delta^2 \end{aligned}$$

with $C = 2^{1-p}$. For $1 < p \leqslant 2$ one has $p - 2 \leqslant 0$ so $s \leqslant r \leqslant \delta/\varepsilon$ gives

$$\beta \geqslant \tfrac{1}{2}(2\varepsilon)^{2-p}\mu(s/2)\, \delta^p \geqslant \tfrac{1}{2}(2\varepsilon)^{2-p}\mu(r/4)\, \delta^p.$$

For $p \geqslant 2$ one uses $s \geqslant r/2 \geqslant \delta/4$ to obtain

$$\beta \geqslant \tfrac{1}{2} 8^{2-p} \mu(s/2)\, \delta^p \geqslant \tfrac{1}{2} 8^{2-p} \mu(r/4)\, \delta^p.$$

Combining the two cases gives (2.9) with

$$C = C(\varepsilon) = \begin{cases} \min\{3^{-p}, \tfrac{1}{2} 8^{2-p}\} & \text{for} \quad p \geqslant 2, \\ \min\{3^{-p}, \tfrac{1}{2}(2\varepsilon)^{2-p}\} & \text{for} \quad 1 < p \leqslant 2. \end{cases}$$

(Note that $C(\varepsilon)$ is independent of ε for $p \geqslant 2$ and is $\sim \varepsilon^{2-p}$ as $\varepsilon \to 0$ for $p \leqslant 2$.) ∎

We now wish to consider "radial operators"—operators of composition (Nemytsky) type acting on vector fields as $\xi \colon \Omega \to \mathbb{R}^m$:

$$[\mathbf{g}\xi](x) := \gamma(x, \xi(x)) = g(x, |\xi(x)|)\, \xi(x)/|\xi(x)|, \tag{2.10}$$

where, for each $x \in \Omega$, $\gamma(x, \cdot)$ is as discussed above, with a function $\mu(x, \cdot)$ as in (2.2). It is convenient to introduce the inverse function for μ:

$$\sigma(x, \lambda) := 4 \inf\{s > 0 \colon \mu(x, s) \geqslant \lambda\} \tag{2.11}$$

(with $\sigma(x, \lambda) := \infty$ if $\mu(x, \cdot)$ never is as big as λ).

Our fundamental assumptions now are that (for some $p > 1$)

$$\begin{array}{ll} \text{(i)} & g \colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \text{ satisfies Carathéodory conditions,} \\ \text{(ii)} & \mu \colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing in } r, \text{ measurable in } x, \\ \text{(iii)} & \mu(x, r) > 0 \text{ for } r > 0 \text{ for a.e. } x \in \Omega, \\ \text{(iv)} & [g(x, r) - g(x, s)] \geqslant \mu(x, s)\, s^{p-2}(r - s) \text{ for } r \geqslant s \geqslant 0, \text{ a.e. } x \in \Omega, \\ \text{(v)} & \text{for some } \bar{\lambda} > 0 \text{ one has } \sigma(\cdot, \bar{\lambda}) \in L^p(\Omega) \text{ for } \sigma \text{ as in (2.11).} \end{array} \tag{2.12}$$

Condition (2.12)(v) ensures that g cannot be "too flat": one need not go too far out to bound dg/dr^{p-1} below (by $\bar{\lambda}/(p-1)$). Thus, for $\lambda < \bar{\lambda}$ one has $\sigma(\cdot, \lambda)$ finite a.e., and, clearly, $\sigma(x, \lambda) \to 0$ as $\lambda \to 0$.

Once p is fixed—as in (2.12)(iv), (v)—we use $\|\cdot\|$ to denote the $L^p(\Omega)$-norm; we will not distinguish notationally between (scalar) $L^p(\Omega)$ and $L^p(\Omega \to \mathbb{R}^m)$. Observe that

$$N(\lambda) := \|\sigma(\cdot, \lambda)\|^p := \int_\Omega \sigma(x, \lambda)^p\, dx$$

is (finite and) nondecreasing on $(0, \bar{\lambda}]$ with $\sigma(\cdot, \lambda) \to 0+$ as $\lambda \to 0+$ pointwise a.e. on Ω so, by the Dominated Convergence Theorem,

$$N(\lambda) \to 0+ \qquad \text{as} \quad \lambda \to 0+. \tag{2.13}$$

We now proceed to obtain a basic coercivity estimate for the operator g. Following (2.6) we set

$$\beta(\xi, \eta) = \beta(x; \xi, \eta)$$
$$:= (\xi - \eta) \cdot [\gamma(\cdot, \xi) - \gamma(\cdot, \eta)]|_x$$

for $x \in \Omega$ and $\xi, \eta \in P^p(\Omega)$ and then set

$$B(\xi, \eta) := \int_\Omega \beta(x; \xi, \eta)\, dx = \langle \xi - \eta, \mathbf{g}\xi - \mathbf{g}\eta \rangle. \tag{2.14}$$

(Note that since $\beta \geqslant 0$ the integral in (2.14) is well-defined, although conceivably infinite as we have not yet imposed any upper growth condition on g.)

THEOREM 2. *Let* **g** *be defined on vector fields by* (2.10) *with* g, μ *as in* (2.12). *Then, for* $M > 0$, *there is a function* $\phi_M: \mathbb{R}^+ \to \mathbb{R}^+$ *depending only on* $N(\cdot)$ *such that*

(i) ϕ_M *is nondecreasing with* $\phi_M(0+) = 0$,

(ii) *given* $\|\eta\| \leqslant M$ *one has* (2.15)

$$\|\xi - \eta\| \leqslant \phi_M\left(\frac{B(\xi, \eta)}{\|\xi - \eta\|}\right).$$

Proof. Given $\xi, \eta: \Omega \to \mathbb{R}^m$, define r, δ, β pointwise on Ω by (2.5), (2.6). For any $\varepsilon > 0$, $\bar{\lambda} \geqslant \lambda > 0$, we set

$$\mathscr{U} := \{x \in \Omega: \delta \leqslant \varepsilon r\},$$
$$\mathscr{V} := \{x \in \Omega: r \leqslant \sigma(x, \lambda)\} = \{x \in \Omega: \mu(x, r/4) \leqslant \lambda\},$$
$$\mathscr{W} := \Omega \backslash (\mathscr{U} \cup \mathscr{V}) = \{x \in \Omega: \delta > \varepsilon r, \mu(x, r/4) \geqslant \lambda\}.$$

Then, for $x \in \mathscr{U}$ one has $\delta^p \leqslant \varepsilon^p r^p \leqslant \varepsilon^p(|\xi|^p + |\eta|^p)$ so

$$\int_{\mathscr{U}} \delta^p \leqslant \varepsilon^p \int_U (|\xi|^p + |\eta|^p) \leqslant \varepsilon^p(\|\xi\|^p + \|\eta\|^p).$$

For $x \in \mathscr{V}$ one has $\delta \leqslant 2r \leqslant 2\sigma(x, \lambda)$ so

$$\int_{\mathscr{V}} \delta^p \leqslant 2^p \int_{\mathscr{V}} \sigma^p(\cdot, \lambda) \leqslant 2^p N(\lambda).$$

Finally, for $x \in \mathscr{W}$ we can apply Theorem 1. Then, by (2.9),

$$\int_{\mathscr{W}} \delta^p \leqslant \int_{\mathscr{W}} \frac{\beta}{C(\varepsilon)\, \mu(r/4)} \leqslant \frac{B(\xi, \eta)}{C(\varepsilon) \lambda}.$$

Combining these three estimates gives

$$\int_\Omega \delta^p \leqslant \varepsilon^p(\|\xi\|^p + \|\eta\|^p) + 2^p N(\lambda) + B(\xi, \eta)/\lambda C(\varepsilon) \qquad (2.16)$$

for arbitrary $\varepsilon > 0$, $\bar{\lambda} \geqslant \lambda > 0$.

Now let $D := \|\xi - \eta\| = \|\delta\|$ so, as $\|\eta\| \leqslant M$, $\|\xi\|^p \leqslant (M+D)^p \leqslant 2^{p-1}(M^p + D^p)$; set $B := B(\xi, \eta)$ and

$$\rho := \frac{B}{D} = \frac{\int_\Omega B(\xi, \eta)\, dx}{\|\xi - \eta\|}.$$

Assuming $2\varepsilon \leqslant 1$, (2.16) gives (on dividing by $[1 - \varepsilon^p 2^{p-1}] \geqslant \frac{1}{2}$)

$$D^p \leqslant (2 + 2^p)\, \varepsilon^p M^p + 2^{p+1} N(\lambda) + [2/\lambda C(\varepsilon)]\, \rho D. \qquad (2.17)$$

Clearly this gives a bound on D for any (suitable) choice of ε, λ and we may take $\phi_M(\rho)$ to be the minimum of these bounds. This proves (2.15)(ii). That ϕ_M is nondecreasing in ρ is clear so we need only show $\phi_M(\rho) \to 0$ as $\rho \to 0+$. To see this, suppose we wished $\phi_M(\rho) \leqslant \bar{\phi}$ (for arbitrarily small $\bar{\phi} > 0$). We see that we can make the first term on the right of (2.17) small by taking ε small and, by (2.13), can make the second small by taking λ small. Thus, their sum S can be made smaller than $\bar{\phi}^p/2$ by appropriate choice of ε, λ; this fixes $\lambda C(\varepsilon)$ and we choose ρ so $c := 2\rho/\lambda C(\varepsilon) < \bar{\phi}^{p-1}/2$. Then (2.17) gives

$$(D^{p-1} - c)D \leqslant S, \qquad (2D^{p-1} - \bar{\phi}^{p-1})D \leqslant \bar{\phi}^p = \bar{\phi}^{p-1}\bar{\phi}$$

so either $D \leqslant \bar{\phi}$ or else $2D^{p-1} - \bar{\phi}^{p-1} \leqslant \bar{\phi}^{p-1}$ which again gives $D \leqslant \bar{\phi}$. Since this estimate for D bounds $\phi_M(\rho)$ we have shown (2.15)(i). ∎

COROLLARY. *Let* $\mathbf{A}$ *be given by* (1.1) *as a map from* $\mathscr{V} := W_0^{1,p}(\Omega)$ *to* $\mathscr{V}^*$. *Suppose* $g: \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, *given by* (1.2), *satisfies* (2.12). *Suppose* $u, \hat{u} \in W_0^{1,p}(\Omega)$ *satisfy, respectively,*

$$\mathbf{A}u = f \quad on \quad \Omega, u|_{\partial\Omega} = 0; \qquad \mathbf{A}\hat{u} = \hat{f} \quad on\ \Omega, \hat{u}|_{\partial\Omega} = 0.$$

Then one has (1.7), *i.e.,*

$$\|\nabla u - \nabla \hat{u}\|_p \leqslant \phi_M\left(\int_\Omega (u - \hat{u})(f - \hat{f})\, dx / \|\nabla u - \nabla u\|_p\right) \qquad (2.18)$$

provided M *is a bound on* $\|\nabla \hat{u}\|_p$ (ϕ_M *as in Theorem* 2.)

Proof. Let $\xi := \nabla u \in L^p(\Omega)$, $\eta := \nabla \hat{u}$. The weak interpretation of $\mathbf{A}\tilde{u} = \tilde{f}$ is just that

$$\int_\Omega v\tilde{f} = \int_\Omega v(-\nabla \cdot a(\cdot, |\nabla \tilde{u}|)\nabla \tilde{u})$$

$$= \int_\Omega \nabla v \cdot a(\cdot, |\nabla \tilde{u}|)\nabla \tilde{u} = \int_\Omega \nabla v \cdot [\mathbf{g}(\nabla \tilde{u})].$$

Thus,

$$\int_\Omega (u - \hat{u})(f - \hat{f}) = \int_\Omega \nabla(u - \hat{u}) \cdot [\mathbf{g}(\nabla u) - \mathbf{g}(\nabla \hat{u})] = B(\xi, \eta).$$

Then (2.18) is just (2.15)(ii). ∎

Remark 1. For $p \geqslant 2$ one can take $\varepsilon \to 0+$ without affecting $C_p(\varepsilon)$ in (2.9) so ϕ_M is independent of M.

Remark 2. It may occasionally be of interest to introduce a (strictly positive a.e.) weight function W on Ω and so consider $L^p_W(\Omega)$ (i.e., $\|u\| := [\int_\Omega |u|^p W\,dx]^{1/p}$). We observe that the entire argument above for Theorem 2 ($W \equiv 1$) goes through in the more general case without essential modification provided one redefines σ to be

$$\sigma(x, \lambda) := 4 \inf\{s > 0 : \mu(x, s) \geqslant \lambda W(x)\} \tag{2.11'}$$

for use in (2.12)(v) and in defining $N(\lambda)$.

EXAMPLE 1. One may continue to consider the case $g = g(r)$, independent of $x \in \Omega$, satisfying (2.2). One now has a single nondecreasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ and, since we assume Ω bounded so $|\Omega| < \infty$, one has

$$N(\lambda) = |\Omega| \, [4\mu^{-1}(\lambda)]^p$$

for λ in the range of $\mu(\cdot)$. In particular, for the "standard" example $g(r) := r^{p-1}$ one has $\mu(s) \equiv (p-1)$ so

$$\sigma(\lambda) = \begin{cases} 0, & \lambda \leqslant \bar{\lambda} = p - 1 \\ \infty, & \lambda > p - 1 \end{cases}$$

which gives $N(\lambda) = 0$ for $0 \leqslant \lambda \leqslant p - 1$. Using this in (2.17) for $p \geqslant 2$ (so one can take $\varepsilon = 0$, $C(\varepsilon) = C_p$) gives

$$\phi(\rho) := C\rho^{1/(p-1)}$$

while for $1 < p \leqslant 2$ one obtains

$$\phi_M(\rho) := \begin{cases} C\rho & \text{for small enough } \rho \\ \mathcal{O}(\rho^{p-1}) & \text{for large } \rho. \end{cases}$$

Here C depends on p in each case and on M as well for $p < 2$.

EXAMPLE 2. We consider the case

$$\gamma(x, \xi) := \left(\frac{|\xi|^2}{c^2(x) + |\xi|^2}\right)\xi, \qquad g(x, r) := \frac{r^3}{c^2(x) + r^2} \tag{2.19}$$

with $c \in L^2_+(\Omega)$. It is convenient to set $\rho = \rho(x) := r^2/c^2(x)$ and then direct calculation shows

$$g_r(x, r) = \psi(\rho) := (\rho^2 + 3\rho)/(1 + \rho)^2.$$

Note that $\psi(0) = 0$ and ψ rises to a maximum (of $\frac{9}{8}$ at $\rho = 3$) and then decreases asymptotically to 1 as $\rho \to \infty$. We may thus take $p = 2$ and, following (2.3), one has

$$\mu(x, r) = \begin{cases} \psi(\rho) & \text{for } \rho \leqslant 1 \ (r \leqslant c(x)), \\ 1 & \text{for } \rho \geqslant 1. \end{cases}$$

For $\lambda < 1$ one inverts $\mu(x, r) = \lambda$ to get $\sigma(x, \lambda) = r$ so

$$\sigma(x, \lambda) = 4\left(\frac{\sqrt{9 - 8\lambda} - 3 + 2\lambda}{2 - 2\lambda}\right)^{1/2} c(x).$$

As $\lambda \to 1$ one has $\sigma \to 4c(x)$ so we may take $\bar{\lambda} = 1-$ with $N(1-) = 16\,\|c\|^2$. We thus have (2.12) and so could apply Theorem 2—we wish to see how $\phi(\rho)$ behaves as $\rho \to 0, \infty$; note that there is no dependence of ϕ on M as $p = 2$.

In the present case the inequality (2.17) becomes

$$D^2 \leqslant 8N(\lambda) + (2/C\lambda)\,\rho D$$

and, in minimizing the resulting bound on D over λ, one clearly would have $\lambda \to 0$ as $\rho \to 0$. As $\lambda \to 0$ one has

$$\sigma \sim 4\sqrt{\lambda/3}\,c(x), \qquad N(\lambda) \sim (16/3)\,\|c\|^2\lambda. \tag{2.20}$$

Using the latter in (2.17), we see that minimization occurs for λ such that

$$\|c\|^2\lambda^3 = 3(\rho/4C)^2.$$

This gives

$$\phi(\rho) \sim C'(\|c\|^2\rho)^{1/3} \qquad \text{as} \quad \rho \to 0+ \tag{2.21}$$

where C' is an absolute costant. For large ρ one minimizes in (2.17) with $\varepsilon = 0+$, $\lambda = 1-$ and obtains $\phi \sim C''\rho$ as $\rho \to \infty$; note that C'' is independent of $c(\cdot)$.

In defining an operator $\mathbf{g}$ in (3.10) from the function $g(\cdot)$, no mention was made of domain or codomain. With p as in (2.12)(iv, v) we wish to take the domain of $\mathbf{g}$ to be $\mathscr{V} := L^p(\Omega \to \mathbb{R}^m)$ and the codomain to be $\mathscr{V}^* = L^q(\Omega \to \mathbb{R}^m)$, where $1/p + 1/q = 1$. (Note that we take

$$\|\xi\|_{\mathscr{V}} := \left[\int_\Omega \left(\sum_1^m |\xi_j(x)|^2\right)^{p/2} dx\right]^{1/p} = \left[\int_\Omega |\xi|^p\right]^{1/p},$$

using the euclidean norm on $\mathbb{R}^m$ regardless of p.) To this end we impose the standard condition

$$0 \leqslant g(x, r) \leqslant g_0(x) + Cr^{p-1} \qquad \text{with} \quad g_0 \in L^q_+(\Omega) \tag{2.22}$$

which ensures, given (2.12)(i), that the Nemytsky operator $\mathbf{g}$ is well defined and continuous from $\mathscr{V}$ to $\mathscr{V}^*$. This choice of domain and codomain means that the integrals appearing in Theorem 2 are finite. Thus we define the set of admissible functions

$$\mathscr{G} = \mathscr{G}_p := \{g\colon (2.12), (2.22)\} \tag{2.23}$$

with $p > 1$ fixed. We define (sequential) convergence in $\mathscr{G}$ to mean: $g_k \to g$ in $\mathscr{G}$ if and only if:

(i) each g_k is in $\mathscr{G}$ with g_0, C fixed in (2.22),

(ii) for each fixed $r(\cdot) \in L^p(\Omega)$ one has

$$g_k(\cdot, r(\cdot)) \to g(\cdot, r(\cdot)) \text{ in } L^q(\Omega), \tag{2.24}$$

(iii) $N_* := \sup_k\{N_k\}$ is finite on an interval $[0, \bar{\lambda}_*]$ and $N_*(\lambda) \to 0$ as $\lambda \to 0$.

Remark 3. We note a sufficient condition for (2.24)(ii).

LEMMA 1. *Suppose* $g_k\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying Carathéodory conditions with* (2.22) *holding* (g_0 *and* C *fixed*) *for each* g_k; *suppose* $g_k(\cdot, \bar{r}) \to g(\cdot, \bar{r})$ *pointwise on* Ω *for each* $\bar{r} \in \mathbb{R}^+$. *Then* (2.24)(ii) *holds.*

Proof. Let $r \in L^p(\Omega)$ be given and set $\hat{g}_k(\cdot) := g_k(\cdot, r(\cdot)) \in L^q(\Omega)$. For each x, set $\bar{r} := r(x)$ and note that

$$\hat{g}_k(x) = g_k(x, \bar{r}) \to \hat{g}(x) := g(x, r(x)).$$

Then

$$\| g_k(\cdot, r(\cdot)) - g(\cdot, r(\cdot))\|_q = \| \hat{g}_k - \hat{g}\|_q = \left[\int_\Omega |\hat{g}_k - g|^q \right]^{1/q}.$$

The integrand goes pointwise to 0 and is dominated by the integrable function

$$(2[g_0(\cdot) + Cr(\cdot)^{p-1}])^q$$

by (2.22). Hence $\| \hat{g}_k - \hat{g}\|_q \to 0$. ∎

We also note that (2.24)(ii) implies pointwise convergence of $\mathbf{g}_k$ to $\mathbf{g}$ since, for each $\xi \in \mathscr{V}$, $x \in \Omega$, one has

$$\begin{aligned} |[\mathbf{g}_k\xi - \mathbf{g}\xi](x)| &= \left| g_k(x, |\xi(x)|) \frac{\xi(x)}{|\xi(x)|} - g(x, |\xi(x)|) \frac{\xi(x)}{|\xi(x)|} \right| \\ &= |g_k(x, r(x)) - g(x, r(x))| \end{aligned}$$

with $r(\cdot) := |\xi(\cdot)| \in L^p(\Omega)$, so $\|\mathbf{g}_k\xi - \mathbf{g}\xi\|_{\mathscr{V}^*} = \| g_k(\cdot, r(\cdot)) - g(\cdot, r(\cdot))\|_q$,

$$\|\mathbf{g}_k\xi - \mathbf{g}\xi\|_{\mathscr{V}^*} = \| g_k(\cdot, r(\cdot)) - g(\cdot, r(\cdot))\|_q \to 0. \tag{2.25}$$

Primarily as adumbration of the results on elliptic boundary value problems (Theorem 4, below), etc., we now consider inversion of $\mathbf{g} \in \mathscr{G}$.

THEOREM 3. *For $g \in \mathscr{G}$, define $\mathbf{g}: \mathscr{V} \to \mathscr{V}^*$ by* (2.10). *Then for every $\psi \in \mathscr{V}^*$ there is a unique solution $\xi \in \mathscr{V}$ of*

$$\mathbf{g}\xi = \psi. \tag{2.26}$$

The map

$$[g, \psi] \mapsto \xi: \mathscr{G} \times \mathscr{V}^* \to \mathscr{V}$$

defined by (2.26) *is* (*sequentially*) *continuous.*

Proof. Existence and uniqueness are trivial since (2.26) means that for each $x \in \Omega$ the direction of $\xi(x)$ is that of $\psi(x)$ while $|\xi(x)| =: r$ is to satisfy $g(x, r) = |\psi(x)|$. Since (2.12)(iv) implies that $g(x; \cdot)$ increases strictly monotonically from 0 to ∞ one has continuity of $h(x,\cdot) := g^{-1}(x,\cdot)$ as well as measurability in x by (2.12)(i). One has then,

$$\xi(x) = h(x, |\psi(x)|)\, \psi(x)/|\psi(x)|.$$

(Note that this is well-defined where $\psi(x) = 0$ as $g(x, 0) = 0$ gives $h(x, 0) = 0$.) Measurability of $\psi(\cdot)$ thus gives measurability of $\xi(\cdot)$ and only a suitable estimate is needed to ensure that $\psi \in \mathscr{V}^*$ gives $\xi \in \mathscr{V}$. Note that $\xi \cdot \psi \geqslant 0$ and

$$0 \leqslant B(\xi, 0) = \int_\Omega \xi \cdot (\mathbf{g}\xi) = \int_\Omega \xi \cdot \psi = \langle \xi, \psi \rangle \leqslant \|\xi\|_{\mathscr{V}} \|\psi\|_{\mathscr{V}^*}$$

so that $B(\xi, 0)/\|\xi - 0\| \leqslant \|\psi\|$. By Theorem 2 with $\eta = 0$,

$$\|\xi\|_p \leqslant \phi_0(\|\psi\|_q) =: M \tag{2.27}$$

(This could be taken as showing that $\xi \in \mathscr{V}$ but a more rigorous argument might involve truncating ψ, replacing $\psi(x)$ by

$$\psi_K(x) := \begin{cases} \psi(x) & \text{if } \ |\psi(x)| \leqslant K, \\ K\psi(x)/|\psi(x)| & \text{if } \ |\psi(x)| > K, \end{cases}$$

and letting $K \to \infty$ with (2.26) giving a uniform estimate for the resulting solutions ξ_K, necessarily converging pointwise on Ω to ξ.) Next, let $\hat{\mathbf{g}}\hat{\xi} = \hat{\psi}$. Then

$$\begin{aligned}
\text{(i)} \qquad \hat{B}(\hat{\xi}, \xi) &= \langle \hat{\xi} - \xi, \hat{\mathbf{g}}\hat{\xi} - \mathbf{g}\xi \rangle \\
&= \langle \hat{\xi} - \xi, \hat{\psi} - \psi \rangle + \langle \hat{\xi} - \xi \rangle, \mathbf{g}\xi - \hat{\mathbf{g}}\xi \rangle \\
&\leqslant \|\hat{\xi} - \xi\| \, (\|\hat{\psi} - \psi\| + \|\mathbf{g}\xi - \hat{\mathbf{g}}\xi\|),
\end{aligned} \tag{2.28}$$

$$\text{(ii)} \qquad \|\hat{\xi} - \xi\| \leqslant \hat{\phi}_M(\|\hat{\psi} - \psi\| + \|\mathbf{g}\xi - \hat{\mathbf{g}}\xi\|).$$

If $\hat{\psi} = \hat{\psi}_k \to \psi$ in $\mathscr{V}^*$ and $\hat{g} = \hat{g}_k \to g$ in $\mathscr{G}$, then the argument of $\hat{\phi}_M$ in (2.28)(ii) goes to 0 and, defining ϕ_M^* by using N_* (as in (2.24)(iii)) for $N(\cdot)$ in (2.17), we see

$$\|\hat{\xi}_k - \xi\| \leqslant \phi_M^*(\|\hat{\psi}_k - \psi\| + \|\hat{\mathbf{g}}_k \xi - \mathbf{g}\xi\|) \to 0$$

proving continuity of the map: $[g, \psi] \mapsto \xi$. ∎

3. The Elliptic Dirichlet Problem

We begin this section with a well-posedness result for the problem

$$\begin{aligned} &\text{(i)} \qquad -\nabla\cdot a(\cdot, |\nabla u|)\,\nabla u = f \text{ on } \Omega, \\ &\text{(ii)} \qquad u = \bar{u} \text{ on } \partial\Omega \end{aligned} \tag{3.1}$$

and then apply this to obtain an existence result for problems such as (1.13). The result for (3.1) is much like Theorem 3 and we point out here the significance of the consideration of *structural stability* (= continuous dependence of the solution on $g \in \mathscr{G}$). Not only does this make the application to (1.13) possible but, directly in consideration of (3.1), we note that in applications the nonlinear diffusion coefficient $a(\cdot\,\cdot)$ is typically not known exactly (e.g., by theory) but approximately (by measurement or by inference).

The problem (3.1) will be interpreted weakly. Assuming g (given by (1.2) from the diffusion coefficient) is in $\mathscr{G} = \mathscr{G}_p$, we assume that we have $f \in \mathscr{W}_0^* := [W_0^{1,p}(\Omega)]^*$ and that we seek $u \in \mathscr{W} := W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\text{(i)} \quad \int_\Omega a(\cdot, |\nabla u|)\,\nabla u\cdot\nabla v = \int_\Omega fv \qquad \text{for} \quad v \in \mathscr{W}_0 := W_0^{1,p}(\Omega), \\ &\text{(ii)} \quad (u - \bar{u}) \in \mathscr{W}_0. \end{aligned} \tag{3.2}$$

(Note that (3.2)(ii), interpreting (3.1)(ii), assumes $\bar{u}$ is known—extended to $\bar{\Omega}$—as an element of $\mathscr{W}$.) Observe, also, that (3.2)(i) can be viewed as

$$\langle \mathbf{A}u, v\rangle := \langle \mathbf{g}\xi, \nabla v\rangle = \langle f, v\rangle, \qquad \xi := \nabla u \in L^p(\Omega \to \mathbb{R}^m) =: \mathscr{V}, \tag{3.3}$$

where the duality product on the left is between $\mathscr{V}^* = L^q(\Omega \to \mathbb{R}^m)$ and $\mathscr{V}$ while that on the right is for $\mathscr{W}_0^* - \mathscr{W}_0$; as noted in the previous section, the assumptions that $g \in \mathscr{G}$ and $f \in \mathscr{W}_0^*$ ensure that (3.3) makes sense.

THEOREM 4. *Let $g \in \mathscr{G}$, i.e., satisfying* (2.12), (2.22), *be related to the nonlinear diffusion coefficient $a(\cdot\,\cdot)$ by* (1.2). *Set $\mathscr{W} := W^{1,p}(\Omega)$, $\mathscr{W}_0 := W_0^{1,p}(\Omega) := \{v \in \mathscr{W}: v|_{\partial\Omega} = 0\}$; assume Ω is a bounded region in $\mathbb{R}^m$ with sufficiently smooth boundary $\partial\Omega$. Let f and $\bar{u}$ be in $\mathscr{W}_0^*$ and $\mathscr{W}$, respectively. Then there is a unique weak soluion $u \in \mathscr{W}$ for* (3.1)*—taken in the sense of* (3.2). *The map*

$$[g, f, \bar{u}] \mapsto u\colon \mathscr{G} \times \mathscr{W}_0^* \times \mathscr{W} \to \mathscr{W}$$

is continuous.

Proof. As indicated in (3.3), the map

$$\mathbf{A}\colon u \mapsto [v \mapsto \langle \mathbf{g}\xi, \nabla v\rangle : \xi := \nabla u] \colon \mathscr{W} \to \mathscr{W}_0^*$$

is well defined and continuous. Restricting ourselves to the affine subspace $[\bar{u} + \mathscr{W}_0]$ as given by (3.2)(ii), we note that $(u - \bar{u}) \mapsto u \mapsto \mathbf{A}u$ is a well-defined operator from $\mathcal{W}_0$ to $\mathcal{W}_0^*$. By Theorem 2, this operator is coercive and (strictly) monotone so a standard argument gives existence of $(u-\bar{u}) \in \mathcal{W}_0$ such that (3.3) holds whence u is a weak solution of (3.1).

The argument for continuity is much as in the proof of Theorem 3 but complicated somewhat by the inhomogeneous boundary condition. Suppose, then, one has

$$[g_k, f_k, \bar{u}_k] \to [g, f, u] \qquad \text{in} \quad \mathcal{G} \times \mathcal{W}_0^* \times \mathcal{W} \tag{3.4}$$

with u_k satisfying $(3.1)_k$, i.e., setting $\xi_k := \nabla u_k$, one has

$$\langle \mathbf{g}_k \xi_k, \nabla v \rangle = \langle f_k, v \rangle \qquad \text{for} \quad v \in \mathcal{W}_0,\ (u_k - \bar{u}_k) \in \mathcal{W}_0.$$

It is convenient to set $w_k := u_k - \bar{u}_k$, $\bar{\xi}_k := \nabla \bar{u}_k$. Then, with $(w_k - w) \in \mathcal{W}_0$, one has

$$\begin{aligned} \langle f_k - f, w_k - w \rangle_{\mathcal{V}} &= \langle \mathbf{g}_k \xi_k - \mathbf{g}\xi, (\xi_k - \bar{\xi}_k) - (\xi - \bar{\xi}) \rangle_r \\ &= \langle \mathbf{g}_k \xi_k - \mathbf{g}_k \xi, \xi_k - \xi \rangle_r \\ &\quad + \langle \mathbf{g}_k \xi - \mathbf{g}\xi, \xi_k - \xi \rangle_r - \langle \mathbf{g}_k \xi_k - \mathbf{g}\xi, \bar{\xi}_k - \bar{\xi} \rangle_r. \end{aligned} \tag{3.5}$$

Since $\|\nabla v\|_p$ may be taken as an equivalent $\mathcal{W}_0$-norm for $v \in \mathcal{W}_0$, we have

$$\begin{aligned} \langle f_k - f, w_k - w \rangle &\leqslant \|f_k - f\|_* \, \|\nabla(w_k - w)\|_p \\ &\leqslant \|f_k - f\|_* (\|\xi_k - \xi\|_p + \|\bar{\xi}_k - \bar{\xi}\|_p). \end{aligned}$$

Noting that the first term on the right of (3.5) is $B_k(\xi_k, \xi)$, this gives

$$\begin{aligned} \text{(i)} \quad B_k(\xi_k, \xi) &\leqslant \|f_k - f\|_* \, \|\xi_k - \xi\|_p + \|f_k - f\|_* \, \|\bar{\xi}_k - \bar{\xi}\|_p \\ &\quad + \|\mathbf{g}_k \xi - \mathbf{g}\xi\|_q \, \|\xi_k - \xi\|_p + \|\mathbf{g}_k \xi_k - \mathbf{g}\xi\| \, \|\bar{\xi}_k - \bar{\xi}\|_p, \\ \text{(ii)} \quad B_k(\xi_k, \bar{\xi}_k) &= \langle f_k, w_k \rangle_{\mathcal{V}} - \langle \mathbf{g}_k \bar{\xi}_k, \xi_k - \bar{\xi}_k \rangle \\ &\leqslant (\|f_k\|_* + \|\mathbf{g}_k \bar{\xi}_k\|_q) \, \|\xi_k - \bar{\xi}_k\|_p. \end{aligned} \tag{3.6}$$

Since $\{\|f_k\|_*\}$, $\{\|\mathbf{g}_k \bar{\xi}_k\|_q\}$ are uniformly bounded (the latter by (2.24)(i) as $\bar{\xi}_k \to \bar{\xi}$) in view of (3.4), one has from (3.6)(ii) a bound on $\{B_k(\xi_k, \bar{\xi}_k)/\|\xi_k - \bar{\xi}_k\|\}$ and so, as in (2.26), a bound on $\{\xi_k\}$ since $\bar{\xi}_k \to \bar{\xi}$; thus $\|\xi_k\| \leqslant M$ for some M by Theorem 2.

We wish to show $\xi_k \to \xi$—suppose, to the contrary, one had a subsequence (denoted as above, without re-indexing) for which $\|\xi_k - \xi\| \geqslant \varepsilon > 0$. Then, from (3.6)(i),

$$\frac{B_k(\xi_k, \xi)}{\|\xi_k - \xi\|} \leqslant \|f_k - f\|_* + \|\mathbf{g}_k \xi - \mathbf{g}\xi\|_q + \|\bar{\xi}_k - \bar{\xi}\| \, \frac{\|f_k - f\|_* + \|\mathbf{g}_k \xi_k - \mathbf{g}\xi\|_q}{\varepsilon}.$$

The first term on the right goes to 0 by assumption, the second as in (2.25), the third as $\|\bar{\xi}_k - \bar{\xi}\| \to 0$ by assumption (as $\bar{u}_k \to \bar{u}$ in $\mathscr{V} = W^{1,p}(\Omega)$) and $\|\mathbf{g}_k \xi_k - \mathbf{g}\xi\|_q$ must be bounded as $\{\xi_k\}$ is bounded and $\mathbf{g}_k \to \mathbf{g}$. Applying Theorem 2 as for (2.27)(ii) gives

$$\|\xi_k - \xi\| \leqslant \phi_M^*([\to 0]) \to 0$$

(taking ϕ_M^* uniform in k by (2.24)(iii) as in the proof of Theorem 3), contradicting the assumption. Thus (3.4) implies $\xi_k \to \xi$.

Since $\bar{\xi}_k \to \bar{\xi}$, this shows $\nabla w_k \to \nabla w$ in $\mathscr{V}$ so $w_k \to w$ in $\mathscr{W}_0$. Hence, $u_k = (w_k + \bar{u}_k) \to (w + \bar{u}) = u$ in $\mathscr{W}$, proving the desired continuity. ▌

(We remark that if the Dirichlet data were fixed (i.e., independent of k—say, homogeneous), then the proof would have been almost identical to that for Theorem 3. Alternatively, we could have generalized slightly the definition of "radial operator"—admitting a "center" other than 0, permitting translation—and then could have proceeded to work entirely in $\mathscr{W}_0$, absorbing variable boundary data in the specification of the operator and right-hand side of translation.)

We now consider more complicated equations of the form

$$-\nabla \cdot \tilde{a}(\cdot, u, |\nabla u|)\, \nabla u = \tilde{f}(\cdot, u, \nabla u). \tag{3.7}$$

Although it should become clear that this is an inessential restriction, we will simplify the arguments somewhat by considering (3.7) only with the *homogeneous* Dirichlet condition: $u|_{\partial\Omega} = 0$. The assumptions, essentially, are that: for admissible v the problem

$$-\nabla \cdot \tilde{a}(\cdot, v(\cdot), |\nabla u|)\, \nabla u = f, \qquad u|_{\partial\Omega} = 0 \tag{3.8}$$

will be of the sort considered in Theorem 4 and

$$f = f_v(\cdot) := \tilde{f}(\cdot, v(\cdot), \nabla v(\cdot)) \tag{3.9}$$

will be a suitable right-hand side. The hypotheses will suffice to ensure existence but there will be no assurance of uniqueness of solutions for (3.7).

Given a nonlinear diffusion coefficient $\tilde{a}\colon \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ in (3.7) and a function $v\colon \Omega \to \mathbb{R}$, we define $a_v, g_v\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$a_v(x, r) := \tilde{a}(x, v(x), r), \qquad g_v(x, r) := r a_v(x, r). \tag{3.10}$$

We assume Ω is bounded in $\mathbb{R}^m$ with "sufficiently smooth" $\partial\Omega$. The assumptions on $\tilde{a}$ will be

(i) $\tilde{a}$ satisfies Carathéodory conditions: measurability on Ω for every $(s, r) \in \mathbb{R} \times \mathbb{R}^+$ and continuity on $\mathbb{R} \times \mathbb{R}^+$ a.e. on Ω.

(ii) $0 \leqslant \tilde{a}(x, s, r)r \leqslant g_0(x) + C[|s| + r]^{p-1}$ with $g_0 \in L^q_+(\Omega)$ $(1/p + 1/q = 1)$.

(iii) For every $v \in \mathcal{W}_0 := W_0^{1,p}(\Omega)$, one has g_v—as defined by (3.10)—in $\mathcal{G}$ (satisfying (2.12); write μ_v, σ_v, N_v for the corresponding functions).

(iv) If $v \rightharpoonup \hat{v}$ in $\mathcal{W}_0$ (weak convergence), then

$$N_*(\lambda) := \sup_v \{N_v(\lambda)\} \to 0 \qquad \text{as} \quad \lambda \to 0.$$

(v) There is a function $\Lambda: \mathbb{R}^+ \to \mathbb{R}^+$ with $\Lambda(v) = o(v^{p-1-\gamma})$ as $v \to \infty$ (γ as in (3.12)(ii)) such that

$$\hat{N}(v) := \sup\{N_v(1/\Lambda(v)) \colon \|v\|_{\mathcal{W}} \leqslant v\} = o(v^p). \tag{3.11}$$

The assumptions on $\bar{f}: \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ will be

(i) $\bar{f}$ satisfies Carathéodory conditions.

(ii) $|\bar{f}(x, s, \xi)| \leqslant f_0(x) + C(|s| + |\xi|)^\gamma$ (3.12)
with $f_0 \in L^{\bar{q}}(\Omega)$, where $1/p + 1/\bar{q} < 1 + 1/m$ and with $\gamma \leqslant p/\bar{q}$.

Lemma 2. *Given* (3.11)(i)–(iv) *and* (3.12), *the map*: $v \mapsto [g_v, f_v]$ *is well defined, continuous, and compact from* $\mathcal{W}_0$ *to* $\mathcal{G} \times \mathcal{W}_0^*$ *and the map* $\mathbf{T}: v \mapsto u$ *is well defined for* $v \in \mathcal{W}$ *by weak solution of*

$$-\nabla \cdot a_v(\cdot, |\nabla u|)\, \nabla u = f_v, \qquad u \in \mathcal{V}_0. \tag{3.13}$$

The map $\mathbf{T}: \mathcal{W}_0 \to \mathcal{W}_0$ *is continuous and compact.*

Proof. It is standard that the Nemytsky operator

$$\{[v, \xi]\} \mapsto \bar{f}(\cdot, v(\cdot), \xi(\cdot)) \colon L^p(\Omega \to \mathbb{R}^{1+m}) \to L^{\bar{q}}(\Omega)$$

is continuous, given (3.12). Since $v \mapsto [v, \nabla v]$ is continuous from $\mathcal{W}_0$ to $L^p(\Omega \times \mathbb{R}^{1+m})$ one has $v \mapsto f_v$ continuous from $\mathcal{W}_0$ to $L^{\bar{q}}(\Omega)$. The condition on $\bar{q}$ ensures compactness of the embedding $\mathcal{W}_0 \hookrightarrow L^{\bar{p}}(\Omega)$ $(1/\bar{p} + 1/\bar{q} = 1)$ by the Rellich–Kondrachov theorem (cf., e.g., [1]) and so of $L^{\bar{q}}(\Omega) \hookrightarrow \mathcal{W}_0^*$ by duality. This shows that

$$v \mapsto f_v \colon \mathcal{W}_0 \to \mathcal{W}_0^* \qquad \text{is continuous and compact.} \tag{3.14}$$

Next, for $v \in \mathcal{W}_0$ we have $g_v \in \mathcal{G}$ by assumption. If $v = v_k \rightharpoonup \hat{v}$ (weakly) in

$\mathscr{W}_0$, then compactness of the embedding gives $v_k \to \tilde{v}$ (strongly) in $L^p(\Omega)$. For each fixed $r(\cdot) \in L^p(\Omega)$ one then has

$$g_k(\cdot, r(\cdot)) := g_{v_k}(\cdot, r(\cdot)) := g(\cdot, v_k(\cdot), r(\cdot))$$

converging in $L^q(\Omega)$ to $g_{\tilde{v}}(\cdot, r(\cdot)) := g(\cdot, \tilde{v}(\cdot), r(\cdot))$ by the continuity (ensured by (3.11)(i), (ii)) of the Nemytsky operator

$$[v, r] \mapsto g(\cdot, v(\cdot), r(\cdot)): L^p(\Omega \to \mathbb{R}^2) \to L^q(\Omega).$$

This gives (2.24)(i), (ii) and (3.11)(iv) gives (2.24)(iii). Thus $g_v \to g_{\tilde{v}}$ in $\mathscr{G}$.

By (3.14) and since bounded sets in $\mathscr{W}_0$ are weakly sequentially precompact, every bounded set in $\mathscr{W}_0$ contains a subsequence for which $\{[g_v, f_v]\}$ is convergent in $\mathscr{G} \times \mathscr{W}_0^*$ and, if one starts with a convergent sequence: $v_k \to \tilde{v}$, then one need not extract a subsequence. This shows $v \mapsto [g_v, f_v]$ is well-defined continuous and sequentially compact. By Theorem 4, if $\{[g_v, f_v]\}$ converges in $\mathscr{G} \times \mathscr{W}_0^*$ to $[g_{\tilde{v}}, f_{\tilde{v}}]$, then $u := \mathbf{T}v$ converges in $\mathscr{W}_0$ to $\tilde{u} := \mathbf{T}\tilde{v}$. Thus, by composition, $\mathbf{T}$ is well-defined, continuous and compact. ∎

THEOREM 5. *Assume Ω is a bounded region in $\mathbb{R}^m$ with sufficiently smooth boundary $\partial\Omega$. Given* (3.11) *and* (3.12) *there is a weak solution u of the problem*

$$-\nabla \cdot \tilde{a}(\cdot, u, |\nabla u|) \nabla u = \tilde{f}(\cdot, u, \nabla u), \qquad u \in \mathscr{W}_0 := W_0^{1,p}(\Omega). \qquad (3.15)$$

Proof. For $v \in \mathscr{W}_0$, define a_v, g_v, etc., by (3.10) and $f_v \in \mathscr{W}_0^*$ by (3.9). By a weak solution of (3.15) we mean $u \in \mathscr{W}_0$ such that

$$\langle \mathbf{g}_u \xi, \nabla w \rangle_{\mathscr{V}} = \langle f_u, w \rangle_{\mathscr{W}_0} \qquad \text{for every } w \in \mathscr{W}_0$$

where $\xi := \nabla u \in \mathscr{V} := L^p(\Omega \to \mathbb{R}^m)$. In Lemma 2 we have already defined $\mathbf{T}: v \mapsto u$ by

$$\langle \mathbf{g}_v \xi, \nabla w \rangle_{\mathscr{V}} = \langle f_v, w \rangle_{\mathscr{W}_0} \qquad \text{for} \quad w \in \mathscr{W}_0 \text{ with } \xi = \nabla u,\ u \in \mathscr{W}_0. \qquad (3.16)$$

Clearly, any $u \in \mathscr{W}_0$ is a weak solution of (3.15) if and only if u is a fixed point of $\mathbf{T}$.

Since Lemma 2 shows $\mathbf{T}$ is continuous and compact on $\mathscr{W}_0$, given (3.11) and (3.12), the Schauder fixed point theorem is applicable to complete the existence proof once we can find a ball $\mathfrak{B}_{\bar{v}} := \{v \in \mathscr{W}_0 : \|v\| \leqslant \bar{v}\}$ invariant under $\mathbf{T}$. We obtain the required estimate from (2.17) and (3.11)(v). Thus, suppose $v \in \mathscr{W}_0$ with $\|v\| \leqslant \bar{v}$ and let $u := \mathbf{T}v$ and, using the fact that the L^p-norm for $\xi := \nabla u$ gives an equivalent norm on $\mathscr{W}_0$, set

$$D := \|u\|_{\mathscr{W}_0} := \|\nabla u\|_p := \left[\int_\Omega |\xi|^p\right]^{1/p}.$$

Then, for suitable ε, $\lambda > 0$, D satisfies (2.17) with $N = N_v$, $\eta = 0$ and (applying (3.16) with $w = u$)

$$\begin{aligned}\rho D := B_v(\xi, 0) &:= \langle \mathbf{g}_v \xi - \mathbf{g}_v 0, \xi - 0\rangle \\ &= \langle f_v, u\rangle \leqslant \|f_v\|_{\mathscr{W}_0^*} D\end{aligned}$$

so, using (3.12)(ii),

$$\rho \leqslant C \|f_v\|_{\tilde{q}} = \mathcal{O}(v^\gamma)$$

for large v. Fixing $\varepsilon > 0$ (noting $M = 0$ in (2.17) as $\eta = 0$) and so $C(\varepsilon)$ and taking $\lambda := 1/\Lambda(v)$, we have

$$D^p \leqslant 2^{p+1}\hat{N}(v) + D\mathcal{O}(v^\gamma)\,\Lambda(v) = o(v^p) + D o(v^{p-1}), \tag{3.17}$$

uniformly for $v \in \mathfrak{B}_v$, as $v \to \infty$. It follows from (3.17) that

$$\sup\{D = \|\nabla u\|_p = \|\mathbf{T}v\|_{\mathscr{W}_0} : \|v\|_{\mathscr{W}_0} \leqslant v\} = o(v)$$

as $v \to \infty$ so, for large enough v $(v > \bar{v})$, one has

$$\|v\|_{\mathscr{W}_0} \leqslant v \qquad \text{implies } \|\mathbf{T}v\|_{\mathscr{W}_0} < v.$$

This both implies the invariance under $\mathbf{T}$ of $\mathfrak{B}_{\bar{v}}$—ensuring existence of a fixed point u of $\mathbf{T}$ and so of a weak solution u of (3.15) in $\mathfrak{B}_{\bar{v}}$ by application of the Schauder theorem—and gives the a priori bound $\bar{v}$ for $\|u\|_{\mathscr{W}_0}$ for any such solution (fixed point). ∎

EXAMPLE 2 (revisited). As an example of the applicability of Theorem 5, we consider (1.13) with homogeneous Dirichlet conditions. We need only verify (3.11) (with $p = 2$). In this case we have

$$\tilde{a}(x, s, r) := \frac{r^2}{s^2 + r^2},$$

so

$$g_v(x, r) := \tilde{a}(x, v(x), r)r = \frac{r^3}{v^2(x) + r^2} \tag{3.18}$$

which is of the form (2.19) with $c(x) := |v(x)|$.

One obviously has (3.11)(i) and, as $0 \leqslant \tilde{a} \leqslant 1$, one has (3.11)(ii). The previous treatment of (2.19) shows that $g_v \in \mathscr{G} = \mathscr{G}_2$ for $v \in L^2(\Omega) \subset W_0^{1,2}(\Omega) = H_0^1(\Omega)$. For $v_k \to v$ in $\mathscr{W}_0$ one certainly has $\{v_k\}$ bounded in $L^2(\Omega)$—say, $\|v_k\|_2 \leqslant v$—so $N_*(\lambda) \leqslant Cv^2\lambda \to 0$ as $\lambda \to 0$. Similarly, choosing $\zeta < p-1-\gamma$ one can set $\Lambda(v) := v^\zeta = o(v^{p-1-\gamma})$ and have

$$\begin{aligned} \hat{N}_v &:= \sup\{N_v(v^{-\zeta}) \colon \|v\|_{\mathscr{W}} \leqslant v \text{ so } \|v\|_2 \leqslant C_v\} \\ &\leqslant Cv^2 v^{-\zeta} = o(v^2). \end{aligned}$$

Thus (3.11)(iv), (v) are also verified. From Theorem 5 the following is then immediate.

COROLLARY. *Let Ω be bounded in $\mathbb{R}^m$ with sufficiently smooth $\partial\Omega$ and let $\tilde{f} \colon \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ satisfy* (3.12) *with $p = 2$. Then the equation*

$$-\nabla \cdot \left(\frac{|\nabla u|^2 \nabla u}{u^2 + |\nabla u|^2} \right) = \tilde{f}(\cdot, u, \nabla u)$$

has at least one (weak) solution $u \in H_0^1(\Omega)$.

4. FURTHER REMARKS

The basic hypothesis (2.2) provides local strict convexity and global coercivity in an L^p context—essentially making $\xi \cdot \mathbf{g}\xi$ behave like $|\xi|^p$ for large $|\xi|$. This combination of power growth at infinity while permitting flatter behavior locally proved adequate to give well-posedness results—including structural stability: continuous dependence on the form of the nonlinearity. This was extended to a variable context pemitting consideration of material inhomogeneity and problems with more complicated nonlinearities, including various forms of degeneracy. In this section we note briefly some further examples and extensions of the material presented.

EXAMPLE 3. We consider g of the form

$$g(\cdot, r) := \frac{r^\alpha}{c(\cdot) + r^\beta} \qquad \text{on} \quad \Omega \times \mathbb{R}^+ \tag{4.1}$$

with $c \geqslant 0$. (Note that Example 2 was the special case $\alpha = 3$, $\beta = 2$.) We also follow Remark 2 in introducing a positive weight function W on Ω.

The growth of $rg(\cdot, r)$ is now like r^p with $p = 1 + \alpha - \beta$ and, to have $p > 1$, we require $\alpha > \beta$. Setting $\rho = \rho(\cdot) := r^\beta / c(\cdot)$, elementary calculus gives

$$\psi(\rho) := r^{-(p-2)} g_r = \frac{(\alpha - \beta)\,\rho^2 + \alpha\rho}{(1+\rho)^2}.$$

For $\beta < \alpha < 2\beta$ one has strict increase of ψ from 0 at 0 to a global maximum of $\alpha^2/4\beta$ at $\rho = (\alpha - \beta)/(2\beta - \alpha)$ and then decrease to the asymptotic limit $(\alpha - \beta)$. For $\alpha \geqslant 2\beta$ the situation is simpler: ψ increases on $\mathbb{R}^+$ from 0 toward $(\alpha - \beta)$. Thus, using (2.3), one has

$$\mu(\cdot, s) = \min\{\alpha - \beta, \psi(s^\beta/c)\}$$

(with $(\alpha - \beta)$ giving the minimum only for $s^\beta/c > (\alpha - \beta)/(2\beta - \alpha)$, provided $\beta < \alpha < 2\beta$).

Using (2.11′), we see that we must require W to be bounded since $\sigma(\cdot, \lambda)$ is infinite where $\lambda W > (\alpha - \beta)$. When finite $(\lambda W < \alpha - \beta)$, one has $\sigma(\cdot, \lambda) = 4(\rho c)^{1/\beta}$ where ρ is the (smaller) positive root of the (quadratic) equation

$$(\alpha - \beta - \lambda W)\rho^2 + (\alpha - 2\lambda W)\rho - \lambda W = 0 \tag{4.2}$$

obtained by setting $\psi(\rho) = \lambda W$. It is then easy to see that $\rho/\lambda \to 2W(\alpha - \beta)/\alpha$ uniformly as $\lambda \to 0$ so

$$N(\lambda) \sim C\left[\int_\Omega (cW)^{p/\beta} W\right] \lambda^{p/\beta} \qquad \text{as} \quad \lambda \to 0 \tag{4.3}$$

which gives (2.12)(v) and (2.13) provided $(cW)^{1/\beta} W$ is integrable, i.e., provided $(cW)^{1/\beta}$ is in the W-weighted L^p space. (This may be viewed as determining an appropriate weight function if $c(\cdot)$ is given in (4.1); in particular, if Ω is bounded and $c^{p/\beta}$ is integrable one can take $W \equiv 1$.)

A variant of this is to consider

$$\hat{g}(\cdot, r) := \frac{b(\cdot) r^\alpha}{c(\cdot) + r^\beta}, \tag{4.1′}$$

with $b, c \geqslant 0$. Clearly, this gives $\hat{\psi}(\rho) = b(\cdot)\, \psi(\rho)$ so $\hat{\mu}(\cdot, s) = b(\cdot)\, \mu(\cdot, s)$. Thus $\hat{\sigma}$ is infinite where $\lambda W > (\alpha - \beta)b$ so (2.12)(v) requires boundedness of $\hat{W} := W/b$. Where finite, one again has $\hat{\sigma}(\cdot, \lambda) = 4(\hat{\rho} c)^{1/\beta}$ with $\hat{\rho}$ given by (4.2) using $\hat{W}$. Thus, in this more general setting one has

$$N(\lambda) \sim C\left[\int_\Omega (cW/b)^{p/\beta} W\right] \lambda^{p/\beta} \qquad \text{as} \quad \lambda \to 0 \tag{4.3′}$$

and (2.12)(v), (2.13) hold if W/b is bounded and $(cW/b)^{1/\beta}$ is in the W-weighted L^p space. Another variant on Example 2 would be to consider

$$\tilde{g}(\cdot, r) = \left(\frac{b(\cdot) + r^\beta}{c(\cdot) + r^\beta}\right) r$$

with $b, c \geqslant 0$. One must impose on b, c the condition: $0 < b/c < \bar{\rho}^2$ with $\bar{\rho} := (\beta+1)/(\beta-1)$ in order to keep $\tilde{g}$ increasing in r,

The asymptotic form (4.3′) can then be used in (2.17) for small λ to obtain ϕ more explicitly as in obtaining (2.21) in Example 2. Assuming $p \geqslant 2$ for simplicity (so we may take $\varepsilon = 0$) and again letting $\rho := B(\xi, \eta)/D$ with $D := \|\xi - \eta\|_p$, we use (4.3′) in (2.17) to obtain

$$D^p \leqslant C\Gamma\lambda^{p/\beta} + C\rho\, D/\lambda$$

with $\Gamma := \int (cW/b)^{p/\beta} W$ and λ small enough. Minimizing the right-hand side over λ gives $\Gamma\lambda^{p/\beta} = C\rho\, D/\lambda$ and using this value of λ on the right of (2.17)—valid for small ρ, giving small λ—one obtains $D \leqslant \phi(\rho)$ with

$$\phi(\rho) = C\Gamma^{\beta/p}\rho^{1/\alpha} \qquad \text{for small } \rho. \tag{4.4}$$

In the context of elliptic boundary value problems, this with Theorem 4 ensures, for example, that the solution of the problem

$$-\nabla \cdot \left(\frac{b(\cdot)\,|\nabla u|^4}{c(\cdot) + |\nabla u|^2}\right) \nabla u = f, \qquad u|_{\partial\Omega} = 0 \tag{4.5}$$

depends Hölder continuously in $W^{1,4}(\Omega)$ on f (taken with the $W^{-1,4/3}(\Omega)$-norm) with Hölder exponent $\frac{1}{5}$—provided $b(\cdot)$ is bounded away from 0 and $[c/b]$ is in $L^2(\Omega)$. Note that we have, here, $\beta = 2$ and $\alpha = 5$ so $p = 4$.

Remark 4. We note that one could consider for Theorem 4—and so, similarly, for Theorem 5—first order (Robin) boundary conditions, i.e., replacing (3.1) by

$$\begin{aligned} &\text{(i)} \quad -\nabla \cdot a(\cdot, |\nabla u|)\, \nabla u = f \text{ on } \Omega, \\ &\text{(ii)} \quad -a\, \nabla u \cdot \hat{\mathbf{n}} = \lambda u - \phi \text{ on } \partial\Omega \end{aligned} \tag{4.6}$$

with $\lambda > 0$ on $\partial\Omega$. The weak form of (4.6) is then

$$\int_\Omega a(\cdot, |\nabla u|)\, \nabla u \cdot \nabla v + \int_{\partial\Omega} \lambda uv = \int_\Omega fv + \int_{\partial\Omega} \phi v \tag{4.7}$$

for $v \in \mathscr{W} := W^{1,p}(\Omega)$. For $u, v \in \mathscr{W}$ one has from trace theory (cf., [1]) that $u, v \in L^r(\partial\Omega)$ with $r := (m-1)p/(m-p)$ for suitably smooth $\partial\Omega$. Thus, the right-hand side is continuous in $v \in \mathscr{W}$ if

$$f \in \mathscr{W}^* \qquad \text{and} \qquad \phi \in L^q(\partial\Omega) \left(q := \frac{p(m-1)}{m(p-1)}\right) \tag{4.8}$$

and the left-hand side is continuous in $v \in \mathscr{W}$ for each $u \in \mathscr{W}$ if $a(\cdot\,\cdot)$ is as in Theorem 4 and

$$\lambda \in L^{q'}(\partial\Omega) \qquad \left(q' := \frac{p(m-1)}{(m+1)\,p-2m}\right),$$

assuming $p > 2m/(m+1)$. In this case one can apply the arguments for Theorem 4 to the problem with the Robin boundary conditions as in (4.6) and obtain corresponding results as to existence and continuous dependence on $[g, f, \phi]$ with $[f, \phi]$ as in (4.8).

Only slightly more difficult is the case of pure Neumann conditions ($\lambda = 0$ in (4.6)(ii)). One now must impose the consistency condition

$$\int_\Omega f + \int_{\partial\Omega} \phi = 0, \tag{4.9}$$

corresponding to taking $v = 1$ in (4.7) with $\lambda = 0$. One also (as is of course, standard) considers $u, v \in \mathscr{W}_0$ where, now,

$$\mathscr{W}_0 := \left\{ v \in \mathscr{W} := W^{1,p}(\Omega) \colon \int_\Omega v = 0 \right\}.$$

Again one obtains results comparable to Theorem 4. Now the generalization to obtain results like Theorem 5 is complicated by the condition (4.9). The attendant technical difficulties for this situation will be addressed in a subsequent paper in the context of solution of a parabolic problem:

$$\begin{aligned} &\text{(i)} \quad \dot{u} - \nabla \cdot \tilde{a}(\cdot, u, |\nabla u|)\, \nabla u = \tilde{f}(\cdot, u, \nabla u) \qquad \text{on} \quad \mathscr{Q} := \mathbb{R} \times \Omega, \\ &\text{(ii)} \quad -\tilde{a}\, \nabla u \cdot \mathbf{n} = \phi(\cdot) \qquad \text{on} \quad \Sigma := \mathbb{R} \times \partial\Omega, \\ &\text{(iii)} \quad \text{periodicity in } t, \end{aligned} \tag{4.10}$$

generalizing the considerations of [2].

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